

The Determination of Parabolic Points in Modular and Extended Modular Groups by Continued Fractions

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Abstract. The modular group $\Gamma = PSL(2, \mathbb{Z})$ is generated by two fractional transformations $T(z) = -1/z$ and $S(z) = -1/(z+1)$. The extended modular group $\bar{\Gamma} = PGL(2, \mathbb{Z})$ is the group obtained by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group Γ . In this study, we calculate the parabolic points of any given element of the modular and the extended modular groups by simple continued fractions and blocks.

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1. Introduction

Hecke groups $H(\lambda)$ are the subgroups $PSL(2, \mathbb{R})$ generated by two fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

which take the upper half of the complex plane into itself. Let $S = TU$, i.e.

$$S(z) = -\frac{1}{z + \lambda}.$$

In [2], Hecke proved that $H(\lambda)$ are discrete only when $\lambda = \lambda_q = 2 \cos \pi/q$, q integer and $q \geq 3$, or $\lambda \geq 2$. For $q = 3$, the resulting Hecke group $H(\lambda_3)$ is the modular group $\Gamma = PSL(2, \mathbb{Z})$, i.e.

$$PSL(2, \mathbb{Z}) = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

This group is equal to $SL(2, \mathbb{Z})/\{\pm I\}$.

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Then the modular group Γ is isomorphic to the free product of two finite cyclic groups of orders 2 and 3 and it has a presentation

$$(1.1) \quad \Gamma = \langle T, S \mid T^2 = S^3 = I \rangle = C_2 * C_3.$$

The modular group Γ and its normal subgroups have especially been of great interest in many fields of Mathematics, for example number theory, automorphic function theory, group theory and graph theory (see [6, 7, 9, 10]).

The extended modular group $\bar{\Gamma} = PGL(2, \mathbb{Z})$ is defined by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group Γ (see [3, 12, 14]). Thus the extended modular group $\bar{\Gamma}$ has the representation

$$(1.2) \quad \bar{\Gamma} = \langle T, S, R \mid T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I \rangle = D_2 *_{z_2} D_3.$$

The extended modular group $\bar{\Gamma}$ is also known to be a free product with amalgamation of two dihedral groups of orders 4 and 6 with a cyclic group of orders 2. The extended modular group $\bar{\Gamma}$ has been intensively studied. For examples of these studies see, [3, 8, 12, 14].

Parabolic points (cusp points) are basically the images of infinity under the group elements. All coefficients of the elements of the modular group Γ are rational integers. This implies that the parabolic points of Γ are just rational number and the set of parabolic points of Γ is equal to $Q \cup \{\infty\}$. In the literature, there have been several attempts to find these points. In [13], Schmidt and Sheingorn give the relationship between cusp points and fundamental domain of Hecke groups. In [15], Özgür and Cangül determine all parabolic points of $H(\lambda), \lambda \geq 2$. In [11], Rosen, shows $V(\infty) = a/c = [r_0\lambda, -1/r_1\lambda, \dots, -1/r_{n-1}\lambda]$ for $V(z) = (az + b)/(cz + d) = S^{r_0}TS^{r_1}T\dots S^{r_n}$ in Hecke groups, but he excludes the modular group $\Gamma = PSL(2, \mathbb{Z})$ and a subgroup of the modular group, i.e. Hecke group $H(2)$. His calculations are done by $\lambda = \lambda_q$ continued fractions. In this study, unlike him, we use simple continued fractions and blocks and thus we get a shorter way in the modular group Γ and the extended modular group $\bar{\Gamma}$.

2. Parabolic points of the modular group Γ

In this section, some properties of blocks and parabolic points in the modular group Γ have been provided. We begin by giving the definition of a simple continued fraction.

Definition 2.1. *A finite continued fraction is an expression of the form*

$$r_0 + \frac{1}{r_1 + \frac{1}{r_2 + \frac{1}{r_3 + \dots + \frac{1}{r_n}}}}$$

where r_1, r_2, \dots, r_n are positive real numbers and r_0 is any real number. The real numbers r_1, r_2, \dots, r_n are called the partial quotients of the continued fraction. Moreover, if $r_0, r_1, r_2, \dots, r_n$ are all integers, then it is called simple.

$$(2.1) \quad [r_0; r_1, r_2, \dots, r_n]$$

is the notation for a finite continued fraction.

An element of Γ can be expressed, in terms of the generator T and S , as

$$(2.2) \quad V(z) = \frac{az + b}{cz + d} = S^{r_0} T S^{r_1} T \dots S^{r_n},$$

where r_i is 0, 1 or 2 ($0 \leq i \leq n$). To obtain a parabolic point of any given element in the modular and the extended modular groups, the following transformations are needed.

$$TS : z \mapsto z + 1, \quad TS^2 : z \mapsto \frac{z}{z + 1}.$$

These transformations are called blocks [1, 4].

Lemma 2.1. *For positive integers m and n , we get the following:*

$$(TS)^m : z \mapsto z + m \quad \text{and} \quad (TS^2)^n : z \mapsto \frac{z}{nz + 1}.$$

Proof. It is proved by utilizing presentations of the matrices. ■

A reduced word $W(T, S)$ within Γ is written in a *block form*. For example,

$$W(T, S) = TSTS^2TS^2TS^2TSTSTSTS^2TSTSTS$$

is a word but it is not easy to get parabolic point for this word. This word is written in a block form, i.e.

$$W(T, S) = (TS)(TS^2)^3(TS)^3(TS^2)(TS)^3.$$

That it is a parabolic point can be computed by the simple continued fraction.

Remark 2.1. Conjugate elements may not have the same parabolic point. For example, TS and ST are conjugate, but their parabolic points are different.

From (1.1), we know that the modular group Γ is the free product of two finite cyclic groups of orders 2 and 3 and therefore any word in (2.2) can be expressed by using blocks as follows.

$$(2.3) \quad W(T, S) = S^i(TS)^{m_0}(TS^2)^{n_0} \dots (TS)^{m_k}(TS^2)^{n_k} T^j$$

for $i = 0, 1, 2$ and $j = 0, 1$. The exponents of blocks are positive integers but m_0 and n_k may be zero. This representation is general in the modular group Γ . In the following four theorems, we will try to compute parabolic points of these words.

Representations of continued fraction, given in (2.1), will be used for the following theorems and proofs. We will separate the general representation of the word in four cases. We will make the four cases with initial blocks and final blocks.

Theorem 2.1. *For a given word in Γ , if the first block is (TS) and the final block is (TS^2) , the form of the word is*

$$W(T, S) = S^i(TS)^{m_0}(TS^2)^{n_0} \dots (TS)^{m_k}(TS^2)^{n_k} T^j.$$

Then, the form of the parabolic points is

<i>Conditions</i>	<i>Parabolic points</i>
$i = 0, j = 0$	$[m_0; n_0, m_1, n_1, \dots, m_k, n_k]$
$i = 0, j = 1$	$[m_0; n_0, m_1, n_1, \dots, m_k]$
$i = 1, j = 0$	$-\frac{1}{[m_0; n_0, m_1, n_1, \dots, m_k, n_k] + 1}$
$i = 1, j = 1$	$-\frac{1}{[m_0; n_0, m_1, n_1, \dots, m_k] + 1}$
$i = 2, j = 0$	$-1 - \frac{1}{[m_0; n_0, m_1, n_1, \dots, m_k, n_k]}$
$i = 2, j = 1$	$-1 - \frac{1}{[m_0; n_0, m_1, n_1, \dots, m_k]}$

Proof. Let $i = 0$ and $j = 0$, hence, the form of the word is

$$(TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k}.$$

The proof is done by induction. It is clear that

$$(TS)^{m_k} (TS^2)^{n_k} (\infty) = [m_k; n_k].$$

Suppose that

$$(TS)^{m_1} (TS^2)^{n_1} \dots (TS)^{m_k} (TS^2)^{n_k} (\infty) = [m_1; n_1, \dots, m_k, n_k] = K.$$

Hence, from Lemma 2.1,

$$(TS)^{m_0} (TS^2)^{n_0} (K) = m_0 + \frac{1}{n_0 + \frac{1}{K}} = [m_0; n_0, m_1, n_1, \dots, m_k, n_k].$$

Let $i = 0$ and $j = 1$. Therefore, the form of the word is

$$(TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} T.$$

We get $T(\infty) = 0$, $(TS^2)^{n_k}(0) = 0$ and $(TS)^{m_k}(0) = m_k$. Other steps are found by induction and it follows as

$$(TS)^{m_0} (TS^2)^{n_0} \dots (TS^2)^{n_{k-1}}(m_k) = [m_0; n_0, m_1, n_1, \dots, m_k].$$

In $W(T, S) = S^i (TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} T^j$ for the cases $1 \leq i \leq 2$, $0 \leq j \leq 1$. The desired results can be obtained by $T(\infty) = 0$, $S(z) = -1/(z+1)$ and $S^2(z) = -1 - 1/z$. In the following three theorems and proofs only case $i = 0, j = 0$ will be given. Using the above mentioned equations for all words $W(T, S)$ in the cases $0 \leq i \leq 2, 0 \leq j \leq 1$, parabolic points can be easily calculated by the help of simple continued fractions. ■

Theorem 2.2. *If the first block is (TS) and the final block is (TS) , the form of the word is*

$$W(T, S) = S^i (TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} (TS)^{m_{k+1}} T^j.$$

Therefore, the form of the parabolic points of the word

$$W^*(T, S) = (TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} (TS)^{m_{k+1}}$$

is

$$[m_0; n_0, m_1, n_1, \dots, m_k, n_k].$$

Proof. Similarly, it is proved as Theorem 2.1. ■

Theorem 2.3. *If the first block is (TS^2) and the final block is (TS^2) , the form of the word is*

$$W(T, S) = S^i(TS^2)^{m_0}(TS)^{n_0} \dots (TS^2)^{m_k}(TS)^{n_k}(TS^2)^{m_{k+1}}T^j.$$

Thus, the form of the parabolic points of the word

$$W^*(T, S) = (TS^2)^{m_0}(TS)^{n_0} \dots (TS^2)^{m_k}(TS)^{n_k}(TS^2)^{m_{k+1}}$$

is

$$\frac{1}{[m_0; n_0, m_1, n_1, \dots, m_k, n_k, m_{k+1}]}.$$

Proof. The form of the word is

$$W^*(T, S) = (TS^2)^{m_0}(TS)^{n_0} \dots (TS^2)^{m_k}(TS)^{n_k}(TS^2)^{m_{k+1}}.$$

Afterwards,

$$(TS^2)^{m_{k+1}}(\infty) = \frac{1}{m_{k+1}}$$

can be found out. The rest of the proof is done by induction. It is clear that

$$(TS^2)^{m_k}(TS)^{n_k} \left(\frac{1}{m_{k+1}} \right) = \frac{1}{[m_k; n_k, m_{k+1}]}.$$

Suppose that

$$(TS^2)^{m_1}(TS)^{n_1} \dots (TS^2)^{m_k}(TS)^{n_k} \left(\frac{1}{m_{k+1}} \right) = \frac{1}{[m_1; n_1, m_2, n_2, \dots, m_k, n_k, m_{k+1}]}.$$

Finally, we write this result into $(TS^2)^{m_0}(TS)^{n_0}$ and get

$$\begin{aligned} & (TS^2)^{m_0}(TS)^{n_0} \dots (TS^2)^{m_k}(TS)^{n_k}(TS^2)^{m_{k+1}}(\infty) \\ &= \frac{1}{[m_0; n_0, m_1, n_1, \dots, m_k, n_k, m_{k+1}]} \end{aligned}$$

Theorem 2.4. *If the first block is (TS^2) and the final block is (TS) , the form of the word is*

$$W(T, S) = S^i(TS^2)^{m_0}(TS)^{n_0} \dots (TS^2)^{m_k}(TS)^{n_k}T^j.$$

Therefore, the form of the parabolic points of the word

$$W^*(T, S) = (TS^2)^{m_0}(TS)^{n_0} \dots (TS^2)^{m_k}(TS)^{n_k}$$

is

$$\frac{1}{[m_0; n_0, m_1, n_1, \dots, m_k]}.$$

Proof. The proof is similar to that used in Theorem 2.3. ■

Let us give an example using one of the four theorems given above for the modular group Γ .

Example 2.1. Let the word

$$W(T, S) = STSTS^2TS^2TSTS^2TS$$

in the modular group Γ . This word is in block form

$$W(T, S) = S(TS)(TS^2)^2(TS)(TS^2)(TS)$$

in the modular group Γ . Now, let us extract the word

$$W^*(T, S) = (TS)(TS^2)^2(TS)(TS^2)(TS).$$

Thus, from Theorem 2.2, the parabolic point of this word $W^*(T, S)$ is $[1; 2, 1, 1] = 7/5$.

Finally, the parabolic point of $W(T, S)$ is $S(7/5) = -\frac{1}{\frac{7}{5}+1} = -5/12$.

3. Parabolic points of the extended modular group $\bar{\Gamma}$

In this section, some properties necessary to calculate parabolic points of the extended modular group $\bar{\Gamma}$ are given. There are several studies in the literature to determine parabolic points of Hecke groups $H(\lambda)$ and the modular group Γ . These groups contain automorphism. Therefore, Hecke groups $H(\lambda)$ are Fuchsian groups. Since the extended modular group $\bar{\Gamma}$ contains automorphism and anti-automorphisms, it is a *NEC* (non-Euclidean crystallographic) group. From (1.2), we know that the extended modular group $\bar{\Gamma}$ is a free product with amalgamation as $\bar{\Gamma} = D_2 *_{\mathbb{Z}_2} D_3$. Due to the relations in the presentation of the extended modular group $\bar{\Gamma}$, any word $W(T, S, R)$ of $\bar{\Gamma}$ can be expressed, in terms of the generator T, S and R as

$$(3.1) \quad W(T, S, R) = R^k S^{r_0} T S^{r_1} T \dots S^{r_n}$$

where r_i is 0, 1 or 2 ($0 \leq i \leq n$) and $k = 0$ or 1.

There are two block forms of the word in the extended modular group $\bar{\Gamma}$. These are

$$(3.2) \quad W(T, S, R) = W(T, S) = S^i (TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} T^j$$

and

$$(3.3) \quad W(T, S, R) = RS^i (TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} T^j.$$

for $i = 0, 1, 2$ and $j = 0, 1$. The exponents of blocks are positive integers but m_0 and n_k may be zero. If the sum of exponent of R in $W(T, S, R)$ is even (i.e. $W(T, S)$), we get the block form of these words as (3.2) and the parabolic points of $W(T, S)$ can be computed as Theorem 2.1-2.4. Now, we give the following theorem for the block form in (3.3).

Theorem 3.1. *If the sum of exponent of R in $W(T, S, R)$ is odd, the form of that word is*

$$W(T, S, R) = RS^i (TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} T^j.$$

Now, let us extract the word

$$W^*(T, S) = S^i (TS)^{m_0} (TS^2)^{n_0} \dots (TS)^{m_k} (TS^2)^{n_k} T^j$$

from the whole word. Let the parabolic point of this extract be K . In this case, the parabolic point of the word $W(T, S, R)$ is $R(K) = 1/K$.

Proof. The proof is done by relations of the extended modular group $\bar{\Gamma}$ and being $R(z) = 1/\bar{z}$. ■

Let us give an example for the results found so far.

Example 3.1. Let the word

$$W(T, S, R) = RTS^2TS^2RSRT$$

in the extended modular group $\bar{\Gamma}$. This word is in block form

$$W(T, S, R) = R(TS^2)(TS)T$$

due to the relations in the presentation of the extended modular group $\bar{\Gamma}$. As a result, the conditions of $i = 0$ and $j = 1$ of Theorem 2.4 are realized. Consequently, the parabolic point of this word is

$$R\left(\frac{1}{2}\right) = 2.$$

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