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Weakly Stable Rings and Related Comparability

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Abstract. Let A be a right R-module having the finite exchange property, and let $A = \bigoplus_{i\in I} A_i$. Suppose that each A_i is fully invariant, equal to a direct sum of isomorphic indecomposable submodules. Then $End_R(A)$ satisfies related comparability. As an application, we prove that the regular endomorphism ring of every reduced torsion abelian group satisfies related comparability.

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1. Introduction

A ring R is said to be weakly stable provided that $aR + bR = R$ implies that there exists a $y \in R$ such that $a + by \in R$ is right or left invertible. Many authors investigated weakly stable rings (cf. [4–5, 8, 10]). Following Goodearl, a regular ring R satisfies general comparability, provided that, for any $x, y \in R$, there exists a $u \in B(R)$ such that $uxR \lesssim uyR$ and $(1-u)yR \lesssim (1-u)xR$. This concept evolved from operator algebras and Baer rings, where it is one of the objectives of the axiomatic development (see [9]). As a generalization of weakly stable ring and general comparability, the author introduced related comparability over exchange rings (cf. [6]). We say that a ring R satisfies related comparability provided that for any idempotents $e, f \in R$ with $e = 1 + ab$ and $f = 1 + ba$ for some $a, b \in R$, there exists a $u \in B(R)$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$. The class of rings satisfying related comparability includes exchange rings satisfying general comparability, weakly stable rings, partially unit-regular rings (cf. [6]).

Recall that a right R-module has the finite exchange property if for every right R-module A and two decompositions $K = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong A$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $K = M \oplus \big(\bigoplus_{i\in I} A_i'\big)$. A ring R is an exchange ring provided that R has the finite exchange property as a right R-module. A ring R is an exchange ring if and only if for any $a \in R$,

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there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$ (cf. [13, Theorem 29.2]). The class of exchange rings includes regular rings, π-regular rings, strongly π-regular rings, semiperfect rings, left or right continuous rings, clean rings, and unit C^*-algebras of real rank zero. It is well known that a right R-module has the finite exchange property if and only if $End_R(A)$ is an exchange ring. Such rings have been extensively studied by many authors (cf. [1, 13]).

A submodule B of a right R-module A is fully invariant in case for any $f \in End_R(A)$, $f(B) \subseteq B$. Let A be a right R-module having the finite exchange property, and let $A = \bigoplus_{i \in I} A_i$. Suppose that each A_i is fully invariant, equal to a direct sum of isomorphic indecomposable submodules. It is shown that $End_R(A)$ satisfies related comparability. As an application, we prove that if G is a reduced torsion abelian group such that $End(G)$ is regular, then $End(G)$ satisfies related comparability.

Throughout, all rings are associative with identity and all right R-modules are unital. $B(R)$ denotes the Boolean algebra of all central idempotents in R. $A \lesssim^{\oplus} B$ means that A is isomorphic to a direct summand of B and $A \subseteq^{\oplus} B$ means that A is a direct summand of B. $FP(R)$ stands for the category of all finitely generated projective right R-modules.

2. Comparability of modules

Many elementary element-wise characterizations of weakly stable rings have been studied by Wei (cf. [10]). The main purpose of this section is to investigate comparability of modules over a weakly stable ring. We begin with an extension of [5, Theorem 5].

Lemma 2.1. *Let A be a right R-module such that $End_R(A)$ is weakly stable. Then $End_R(nA)$ is weakly stable for all $n \in \mathbb{N}$.*

Proof. Given $M = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong nA \cong A_2$, we have $M = A_{11} \oplus \cdots \oplus A_{1n} \oplus B = A_{21} \oplus \cdots \oplus A_{2n} \oplus C$ with $A_{1i} \cong A \cong A_{2i}$ for all i. By virtue of [5, Proposition 2], we can find some $D_1, E_1 \subseteq M$ such that $M = D_1 \oplus E_1 \oplus \big(A_{12} \oplus \cdots \oplus A_{1n} \oplus B\big) = D_1 \oplus \big(A_{22} \oplus \cdots \oplus A_{2n} \oplus C\big)$ or $M = D_1 \oplus \big(A_{12} \oplus \cdots \oplus A_{1n} \oplus B\big) = D_1 \oplus E_1 \oplus \big(A_{22} \oplus \cdots \oplus A_{2n} \oplus C\big)$. Thus we get $M = \big(E_1 \oplus A_{12}\big) \oplus \big(A_{13} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1\big) = A_{22} \oplus \big(A_{23} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1\big)$ or $M = A_{12} \oplus \big(A_{13} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1\big) = \big(E_1 \oplus A_{22}\big) \oplus \big(A_{23} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1\big)$. As a result, $M = A'_{12} \oplus \big(A_{13} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1\big) = A'_{22} \oplus \big(A_{23} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1\big)$, where $A'_{12} = E_1 \oplus A_{12}, A'_{22} = A_{22}$ or $A'_{12} = A_{12}, A'_{22} = E_1 \oplus A_{22}$. Clearly, $A'_{12} \cong A \cong A'_{22}$. By [5, Proposition 2] again, we can find $D_2 \subseteq M$ such that $M = A'_{13} \oplus \big(A_{14} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1 \oplus D_2\big) = A'_{23} \oplus \big(A_{24} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1 \oplus D_2\big)$ with $A'_{13} \cong A \cong A'_{23}$. By iteration of this process, we get $D_3, \cdots, D_{n-1} \subseteq M$ such that $M = A'_{1n} \oplus \big(D_1 \oplus D_2 \oplus \cdots \oplus D_{n-1} \oplus B\big) = A'_{2n} \oplus \big(D_1 \oplus D_2 \oplus \cdots \oplus D_{n-1} \oplus C\big)$ with $A'_{1n} \cong A \cong A'_{2n}$. Thus we can find $D_n, E \subseteq M$ such that $M = \big(D_1 \oplus D_2 \oplus \cdots \oplus D_n\big) \oplus E \oplus B = \big(D_1 \oplus D_2 \oplus \cdots \oplus D_n\big) \oplus C$ or $M = \big(D_1 \oplus D_2 \oplus \cdots \oplus D_n\big) \oplus B = \big(D_1 \oplus D_2 \oplus \cdots \oplus D_n\big) \oplus E \oplus C$. By [5, Proposition 2] again, $End_R(nA)$ is weakly stable. ∎

Theorem 2.1. *Weakly stable property is Morita invariant.*

Proof. Let R be weakly stable and S is Morita equivalent to R. Then there is a positive integer n and an idempotent matrix $e \in M_n(R)$ such that $S \cong eM_n(R)e$.

Clearly, $M_n(R) \cong End_R(nR)$ is weakly stable by Lemma 2.1. According to [5, Proposition 3], S is weakly stable, as desired. ▮

Corollary 2.1. *Let A be a finitely generated projective right R-module over a weakly stable ring R. If B and C are any right R-modules such that $A \oplus B \cong A \oplus C$, then $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$.*

Proof. Since $\psi : A \oplus B \cong A \oplus C$, we have $A \oplus C = \psi(A) \oplus \psi(B)$ with $A \cong \psi(A)$. By virtue of Theorem 2.1, $End_R(A)$ is weakly stable. According to [5, Proposition 2], there are some right R-modules D and E such that $A \oplus C = D \oplus E \oplus C = D \oplus \psi(B)$ or $A \oplus C = D \oplus C = D \oplus E \oplus \psi(B)$. Thus, $E \oplus C \cong \psi(B) \cong B$ or $C \cong E \oplus \psi(B) \cong E \oplus B$. Consequently, $C \lesssim^{\oplus} B$ or $B \lesssim^{\oplus} C$. ▮

Corollary 2.2. *Let A be a right R-module having the finite exchange property, and let $E = End_R(A)$. Then the following are equivalent:*

(1) *E is weakly stable.*
(2) *For any right R-modules B and C, $A \oplus B \cong A \oplus C$ implies that $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$.*
(3) *$A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$ implies that $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$.*
(4) *For any idempotents $e, f \in E$, $eA \cong fA$ implies that $(1-e)A \lesssim^{\oplus} (1-f)A$ or $(1-f)A \lesssim^{\oplus} (1-e)A$.*

Proof.
(1) $\Longrightarrow$ (2) is clear by [5, Proposition 2].
(2) $\Longrightarrow$ (3) Given $A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$, then $A \oplus B \cong A \oplus C$. By hypothesis, $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$.
(3) $\Longrightarrow$ (4) For any idempotents $e, f \in E$, we see that $A = eA \oplus (1-e)A = fA \oplus (1-f)A$. Thus, $eA \cong fA$ implies that $(1-e)A \lesssim^{\oplus} (1-f)A$ or $(1-f)A \lesssim^{\oplus} (1-e)A$.
(4) $\Longrightarrow$ (1) Given any regular $x \in E$, there exists a $y \in E$ such that $x = xyx$ and $y = yxy$. Clearly, $\varphi : xyA \cong yxA$ given by $\varphi(xya) = yxya$ for any $a \in A$. By hypothesis, we get $(1-xy)A \lesssim^{\oplus} (1-yx)A$ or $(1-yx)A \lesssim^{\oplus} (1-xy)A$. Thus, we have a split R-monomorphism $\psi : (1-xy)A \to (1-yx)A$ or a split R-epimorphism $\psi : (1-xy)A \to (1-yx)A$. Construct a R-morphism $\phi : A = xyA \oplus (1-xy)A \to yxA \oplus (1-yx)A = A$ given by $\phi(a) = \varphi(xya) + \psi\big((1-xy)a\big)$ for any $a \in A$. One easily checks that $\phi \in E$ is left or right invertible. Furthermore, $x = x\phi x$. In view of [13, Theorem 28.7], E is an exchange ring. By [10, Theorem 3.4], we complete the proof. ▮

Let A be a right R-module having the finite exchange property. It follows by Corollary 2.2 and [5, Proposition 2] that $End_R(A)$ is weakly stable if and only if $A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$ implies that there exist $D, E \subseteq A$ such that $A = D \oplus E \oplus B = D \oplus C$ or $A = D \oplus B = D \oplus E \oplus C$.

Many classes of exchange rings belong to weakly stable rings. But there exist exchange rings which are not weakly stable as the following shows.

Example 2.1. Let V be an infinite-dimensional vector space over a division ring D and set

$$R = \begin{pmatrix} End_D(V) & End_D(V) \\ 0 & End_D(V) \end{pmatrix}.$$

Then R is an exchange ring, while it is not weakly stable.

Proof. Obviously, R is an exchange ring. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a basis of V. Define $\sigma : V \to V$ given by $\sigma(x_i) = x_{i+1} (i = 1, 2, \cdots)$ and $\tau : V \to V$ given by $\tau(x_1) = 0$ and $\tau(x_i) = x_{i-1} (i = 2, 3, \cdots)$. Then $\tau\sigma = 1_V$ and $\sigma\tau \neq 1_V$. Assume that R is weakly stable. Since

$$\begin{pmatrix} \tau & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1_V - \sigma\tau \end{pmatrix} = \mathrm{diag}(1_V, 1_V),$$

we have some $\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} \in R$ such that

$$\begin{pmatrix} \tau & 0 \\ 0 & \sigma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1_V - \sigma\tau \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} \in R$$

is right or left invertible. This implies that $\tau \in End_D(V)$ is left invertible or $\sigma + (1_V - \sigma\tau)\beta \in End_D(V)$ is right invertible. Clearly, $\tau\big(\sigma + (1_V - \sigma\tau)\beta\big) = 1_V$. Thus, $\tau \in Aut_D(V)$ or $\sigma + (1_V - \sigma\tau)\beta \in Aut_D(V)$. If $\sigma + (1_V - \sigma\tau)\beta \in Aut_D(V)$, then $\tau \in Aut_D(V)$. In any case, $\tau \in Aut_D(V)$, a contradiction. Therefore R is not weakly stable. ∎

In the proceeding example, we choose $e = \mathrm{diag}(1_V, 0)$. Then $eRe \cong End_D(V) \cong (1_R - e)R(1_R - e)$. Thus, it is possible to have a ring R, with an idempotent e, such that both eRe and $(1 - e)R(1 - e)R$ are weakly stable, but R is not.

Recall that a right R-module A is directly finite if A is not isomorphic to any proper direct summand of itself. Equivalently, A is directly finite if and only if $B = 0$ is the only module for which $A \oplus B \cong A$. A module which is not directly finite is said to be directly infinite.

Lemma 2.2. *Let A be a right R-module having the finite exchange property, and let $E = End_R(A)$. Suppose that A is expressible as a direct sum of isomorphic indecomposable submodules. Then:*

(1) *E is weakly stable.*
(2) *E has stable range one if and only if A is a direct sum of finite many isomorphic indecomposable submodules.*

Proof.

(1) Assume that $A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$, then $A = A_1 \oplus B = \bigoplus_{i \in I} Y_i$, where each Y_i is isomorphic to an indecomposable submodule Y of A. In view of [13, Lemma 28.1], A_1 has the finite exchange property. Thus, we have some $Y_i' \subseteq Y_i$ such that $A = A_1 \oplus \big(\bigoplus_{i \in I} Y_i'\big)$. It is easy to verify that $Y_i' \subseteq^{\oplus} Y_i$ for all $i \in I$. As each Y_i is indecomposable, we see that either $Y_i' = 0$ or $Y_i' = Y_i$. Thus, there is a subset H_1 of I such that $B \cong \bigoplus_{i \in H_1} Y_i$. Likewise, there is a subset H_2 of I such that $C \cong \bigoplus_{i \in H_2} Y_i$. Clearly, $|H_1| \leq |H_2|$ or $|H_2| \leq |H_1|$, whence either $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$. According to Corollary 2.2, E is weakly stable.
(2) If E has stable range one, then A is directly finite. Hence, A is not isomorphic to a proper submodule of itself. But then the index set I is finite. Conversely, assume that $A = \bigoplus_{i=1}^{n} Y_i$ where each Y_i is isomorphic to a indecomposable module Y. Since A has the finite exchange property, so has each Y_i by [13,

Lemma 28.1]. In view of [13, Theorem 29.5], $End_R(Y_i)$ is local; hence, it has stable range one. Therefore, E has stable range one, as asserted. ∎

Theorem 2.2. *Let A be a right R-module having the finite exchange property, let $E = End_R(A)$, and let $A = \bigoplus_{i\in I} A_i$. Suppose that each A_i is fully invariant, equal to a direct sum of isomorphic indecomposable submodules. Then:*

(1) *E is weakly stable if and only if A_i is directly finite for all but (possibly) a single $i \in I$.*
(2) *E has stable range one if and only if A is a directly finite.*

Proof.

(1) Suppose that E is weakly stable. If $i_1, i_2 \in I$ are two distinct indices such that A_{i_1} and A_{i_2} both fail to be directly finite. In view of [9, Lemma 5.1], $End_R(A_{i_1})$ and $End_R(A_{i_2})$ are both directly infinite. Thus, we can find some $s_1, t_1 \in End_R(A_{i_1})$ and $s_2, t_2 \in End_R(A_{i_2})$ such that $s_1t_1 = 1, t_1s_1 \neq 1, s_2t_2 \neq 1$ and $t_2s_2 = 1$. It is easy to check that $(s_1, s_2) = (s_1, s_2)(t_1, t_2)(s_1, s_2)$, i.e., $(s_1, s_2) \in End_R(A_{i_1}) \oplus End_R(A_{i_2})$ is regular. If (s_1, s_2) is one-sided unit-regular, there exists a right or left invertible (u_1, u_2) such that $(s_1, s_2) = (s_1, s_2)(u_1, u_2)(s_1, s_2)$; hence, $s_1 = s_1u_1s_1$ and $s_2 = s_2u_2s_2$. As a result, $s_1u_1 = 1$ and $u_2s_2 = 1$. If (u_1, u_2) is right invertible, $u_1 \in End_R(A_{i_1})$ is invertible. If (u_1, u_2) is left invertible, $u_2 \in End_R(A_{i_2})$ is invertible. Thus, either s_1 or s_2 is invertible. This gives a contradiction. By [10, Theorem 3.4], $End_R(A_{i_1}) \oplus End_R(A_{i_2})$ is not weakly stable. Since each A_i is a fully invariant submodule, we see that $Hom_R(A_i, A_j) = 0$ for $i \neq j$. Thus, $E \cong \prod_{i\in I} End_R(A_i)$(cf. [3]). According to [5, Proposition 3], $End_R(A_{i_1}) \oplus End_R(A_{i_2})$ is weakly stable, a contradiction. Therefore we conclude that A_i is directly finite for all but (possibly) a single $i \in I$.

Conversely, assume that all but possibly a single one of the A_i is directly infinite. If all of the A_i are directly finite, then $End_R(A_i)$ has stable range one by Lemma 2.2. As $E \cong \prod_{i\in I} End_R(A_i)$, we see that E has stable range one. If there exists a $j \in I$ such that A_j is directly infinite while for all $i \neq j(i \in I), A_i$ is directly finite. It follows by Lemma 2.2 that $End_R(A_j)$ is weakly stable, while for all $i \neq j(i \in I)$, $End_R(A_i)$ has stable range one. From this, E is weakly stable, as required.

(2) If E has stable range one, it easily follows that E is directly finite. Conversely, suppose that E is directly finite. Then $End_R(A_i)$ is directly finite. That is, A_i is not isomorphic to a proper submodule of itself. In view of Lemma 2.2, $End_R(A_i)$ has stable range one. As $E \cong \prod_{i\in I} End_R(A_i)$, we see that E has stable range one. ∎

Corollary 2.3. *Let G be an abelian group such that $End(G)$ is regular. If G is a reduced torsion group, then $End(G)$ has stable range one if and only if it is directly finite.*

Proof. As is known, a reduced abelian torsion group has a regular endomorphism ring if and only if it is a direct sum of cyclic groups of prime order. Thus, the result follows by Theorem 2.2. ∎

3. Related comparability

An element $w \in R$ is called a related unit if there exists an $e \in B(R)$ such that $ew \in eRe$ is right invertible and $(1-e)w \in (1-e)R(1-e)$ is left invertible. We use $U_r(R)$ to denote the set of all related units in R. Now we investigate some elementary properties of related comparability which generalize the corresponding results for exchange rings.

Lemma 3.1. *Let A be a right R-module, let $E = End_R(A)$, and let $e, f \in E$ be idempotents. Then the following hold:*

(1) *$eA \lesssim^{\oplus} fA$ if and only if there exist some $a \in eEf$ and $b \in fEe$ such that $e = ab$.*
(2) *$eA \cong fA$ if and only if there exist some $a, b \in E$ such that $e = ab$ and $f = ba$.*

Proof.

(1) Suppose that $eA \lesssim^{\oplus} fA$. Then there exist R-morphisms $\alpha : eA \to fA$ and $\beta : fA \to eA$ such that $\beta\alpha = 1_{eA}$. Let

$$a : A = fA \oplus (1-f)A \overset{f}{\twoheadrightarrow} fA \overset{\beta}{\to} eA \hookrightarrow A$$

and

$$b : A = eA \oplus (1-e)A \overset{e}{\twoheadrightarrow} eA \overset{\alpha}{\to} fA \hookrightarrow A.$$

Then $e = ab$ with $a = eaf \in eEf$ and $b = fbe \in fEe$.

Suppose that there exist some $a \in eEf$ and $b \in fEe$ such that $e = ab$. Construct two R-morphisms $\varphi : eA \to fA$ given by $\varphi(er) = ber$ for any $r \in A$ and $\phi : fA \to eA$ given by $\phi(fr) = afr$ for any $r \in A$. It is easy to verify that $\phi\varphi = 1_{eA}$, i.e., φ is a split R-monomorphism. Thus, we have a right R-module D such that $eA \oplus D \cong fA$. Therefore $eA \lesssim^{\oplus} fA$.

(2) Suppose that $eA \cong fA$. Then there exist R-morphisms $\alpha : eA \to fA$ and $\beta : fA \to eA$ such that $\beta\alpha = 1_{eA}$ and $\alpha\beta = 1_{fA}$. Let

$$a : A = fA \oplus (1-f)A \overset{f}{\twoheadrightarrow} fA \overset{\beta}{\to} eA \hookrightarrow A$$

and

$$b : A = eA \oplus (1-e)A \overset{e}{\twoheadrightarrow} eA \overset{\alpha}{\to} fA \hookrightarrow A.$$

Then $e = ab$ and $f = ba$ with $a = eaf \in eEf$ and $b = fbe \in fEe$.

Suppose that there exist some $a, b \in E$ such that $e = ab$ and $f = ba$. Let $c = eaf$ and $d = fbe$. Then $e = cd$ and $f = dc$ with $c \in eEf$ and $d \in fEe$. Construct two R-morphisms $\varphi : eA \to fA$ given by $\varphi(er) = der$ for any $r \in A$ and $\phi : fA \to eA$ given by $\phi(fr) = cfr$ for any $r \in A$. It is easy to verify that $\phi\varphi = 1_{eA}$ and $\varphi\phi = 1_{fA}$, i.e., φ is an isomorphism. Therefore $eA \cong fA$, as asserted. ∎

As is well known, an exchange ring satisfies related comparability if and only if $R = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$ implies there exists some $e \in B(R)$ such that $Be \lesssim^{\oplus} Ce$ and $C(1-e) \lesssim^{\oplus} B(1-e)$. We extend this result to a general case.

Theorem 3.1. *Let A be a right R-module, and let $E = End_R(A)$. Then the following are equivalent:*

(1) *E satisfies related comparability.*
(2) *Every regular element in E is related unit-regular.*
(3) *$A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$ implies that there exists a $u \in B(E)$ such that $uB \lesssim^{\oplus} uC$ and $(1-u)C \lesssim^{\oplus} (1-u)B$.*

Proof.
(1) $\Longrightarrow$ (3) Given $A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$, then we have idempotents $e, f \in E$ such that $A_1 = (1-e)A, B = eA, A_2 = (1-f)A$ and $C = fA$. As $(1-e)A \cong (1-f)A$, it follows by Lemma 3.1 that there exist a, b such that $e = 1+ab$ and $f = 1 + ba$. By hypothesis, there exists a $u \in B(E)$ such that $ueE \lesssim^{\oplus} ufE$ and $(1-u)fE \lesssim^{\oplus} (1-u)eE$. By Lemma 3.1 again, we have some $s \in ueEuf$ and $t \in ufEue$ such that $ue = ab$. According to Lemma 3.1, we get $ueA \lesssim^{\oplus} ufA$. That is, $uB \lesssim^{\oplus} uC$. Likewise, $(1-u)C \lesssim^{\oplus} (1-u)B$, as required.
(3) $\Longrightarrow$ (2) For any regular $x \in E$, there exists a $y \in E$ such that $x = xyx$. Since xy and yx are both idempotents, $A = yxA \oplus (1-yx)A = xyA \oplus (1-xy)A = xA \oplus (1-xy)A$. Obviously, $\varphi : xA \to yxA$, given by $xr \mapsto yxr$, is an isomorphism. So, there exists $f \in B(E)$ such that $f(1-xy)A \lesssim^{\oplus} f(1-yx)A$ and $(1-f)(1-yx)A \lesssim^{\oplus} (1-f)(1-xy)A$. Thus, there exists a split R-monomorphism $\phi : f(1-xy)A \to f(1-yx)A$. Let $\alpha : fA \to fA$ with $\alpha(b+c) = \varphi(b) + \phi(c)$ for any $b \in fxA$, $c \in f(1-xy)A$. It is easy to verify that $\alpha \in End_R(fA)$ is left invertible. Furthermore, we see that $fx = x\alpha fx$. Furthermore, there exists a split R-epimorphism $\psi : (1-f)(1-xy)A \to (1-f)(1-yx)A$. Let $\beta : (1-f)A \to (1-f)A$ with $\beta(b+c) = \varphi(b) + \psi(c)$ for any $b \in (1-f)xA$, $c \in (1-f)(1-xy)A$. One easily checks that $\beta \in End_R\big((1-f)A\big)$ is right invertible. In addition, we get $(1-f)x = x\beta(1-f)x$. Define $w : A = fA \oplus (1-f)A \to fA \oplus (1-f)A$ given by $w(s+t) = \alpha(s) + \beta(t)$ for any $s \in eA, t \in (1-e)A$. Then $w \in U_r(E)$. Furthermore, $x = fx + (1-f)x = xwx$, as desired.
(2) $\Longrightarrow$ (1) For any idempotents $e, f \in E$ with $e = 1 + ab$ and $f = 1 + ba$ for some $a, b \in E$, we see that $1 - e = (-a)(1-f)b$ and $1 - f = b(1-e)(-a)$. Let $c = (1-e)(-a)(1-f)$ and $d = (1-f)b(1-e)$. Then $1 - e = cd$ and $1 - f = dc$. In addition, $dcd = (1-f)d = d$. By hypothesis, there exists a $w \in U_r(E)$ such that $d = dwd$. Set $u = (e - wd)w(f - dw)$. Then $(e-wd)^2 = 1 = (f-dw)^2$, whence, $u \in U_r(E)$. Furthermore, we see that $eu = w - wdw = uf$. As $u \in U_r(E)$, there is a $g \in B(E)$ such that $gus = g$ and $t(1-g)u = 1-g$. Thus, $eg = ufsg = eufg \cdot fseg$. In view of Lemma 3.1, we get $geE \lesssim^{\oplus} gfE$. Analogously, we deduce that $(1-g)fE \lesssim^{\oplus} (1-g)eE$. Therefore E satisfies related comparability. ∎

Theorem 3.1 shows that related comparability is right and left symmetric. That is, a ring R satisfies related comparability if and only if so does its opposite ring R^{op}. Also we note that every commutative ring satisfies related comparability from Theorem 3.1.

Corollary 3.1. *Let R be a ring. Then the following are equivalent:*

(1) *R satisfies related comparability.*
(2) *For any regular $a \in R$, $aR + bR = R$ implies that there exists a $y \in R$ such that $a + by \in U_r(R)$.*
(3) *Whenever $ax + b = 1$ with $ba = 0$, then there exists a $y \in R$ such that $a + by \in U_r(R)$.*

Proof.
(1) $\Longrightarrow$ (2) For any regular $a \in R$, $aR + bR = R$ implies that there exist $x, y \in R$ such that $ax + by = 1$. In view of Theorem 3.1, $a = awa$ for some $w \in U_r(R)$. Thus, we have an $e \in B(R)$ such that $ews = e$ and $t(1-e)w = 1-e$ for some $s, t \in R$. Let $f = wa$. Then, $fx + wc = w$, where $c = by$. So $f(x+wc) + (1-f)wc = w$. Clearly, $(1-f)wc = (1-f)w$. It is easy to verify that $(1-f)w = (1-f)w\big(es+(1-e)t\big)(1-f)w$. Let $g = (1-f)w\big(es+(1-e)t\big)(1-f)$. Then $g = g^2, fg = gf = 0$. This implies that $f(x+wc) = fw$ and $(1-f)wc = gw$. As a result, we deduce that $w\big(a + c(es + (1-e)t)(1-f)(1+fwc(es+(1-e)t)(1-f))\big)\big(1 - fwc(es+(1-e)t)(1-f)\big)w = \big(f + wc(es+(1-e)t)(1-f)(1+fwc(es+(1-e)t)(1-f))\big)\big(1 - fwc(es+(1-e)t)(1-f)\big)w = \big(f(1-fwc(es+(1-e)t)(1-f)) + wc(es+(1-e)t)(1-f)\big)w = \big(f + (1-f)w(es+(1-e)t)(1-f)\big)w = (f+g)w = w$. As $w \in U_r(R)$, we deduce that $a + byz \in U_r(R)$, where $z = \big(es+(1-e)t\big)(1-f)\big(1+fwc(es+(1-e)t)(1-f)\big)$.
(2) $\Longrightarrow$ (3) Whenever $ax + b = 1$ with $ba = 0$, then $axa = a$, i.e., $a \in R$ is regular. Thus, there exists a $y \in R$ such that $a + by \in U_r(R)$.
(3) $\Longrightarrow$ (1) Given any regular $a \in R$, there exists some $x \in R$ such that $a = axa$ and $x = xax$. Hence, $xa + (1-xa) = 1$ with $(1-xa)x = 0$. By hypothesis, we have a $y \in R$ such that $u := x + (1-xa)y \in U_r(R)$. Thus, $a = axa = aua$. According to Theorem 3.1, R satisfies related comparability. ∎

Corollary 3.2. *Let $e \in R$ be an idempotent. If R satisfies related comparability, then so does eRe.*

Proof. Assume that $ax + b = e$ and $ba = 0$, where $a, x, b \in eRe$. Then $(a+1-e)(x+1-e) + b = 1$ with $b(a+1-e) = 0$. Since R satisfies related comparability, by virtue of Corollary 3.1, there exists a $y \in R$ such that $a + 1 - e + by \in U_r(R)$. Thus, we have an $f \in B(R)$ such that $f(a+1-e+by)$ is right invertible in fRf and $(1-f)(a+1-e+by)$ is left invertible in $(1-f)R(1-f)$. Assume that $f(a+1-e+by)s = f$ for some $s \in R$. Then $f(1-e)se = 0$; hence, $fse = fese$. Thus, $f\big(a+b(eye)\big)(ese) = fe$. Assume that $(1-f)t(a+1-e+by) = 1-f$ for some $t \in R$. Then $(1-f)(ete)\big(a+b(eye)\big) = (1-f)e$. This implies that $a + b(eye) \in U_r(eRe)$. By Corollary 3.1 again, eRe satisfies related comparability. ∎

If $M_n(R)(n \in \mathbb{N})$ satisfies related comparability, then so does R from Corollary 3.2.

Theorem 3.2. *Let A be a right R-module having the finite exchange property, let $E = End_R(A)$, and let $A = \bigoplus_{i\in I} A_i$. Suppose that each A_i is fully invariant, equal to a direct sum of isomorphic indecomposable submodules. Then E satisfies related comparability.*

Proof. As each A_i is fully invariant, $E \cong \prod_{i\in I} End_R(A_i)$ (see [3]). According to Lemma 2.2, each $End_R(A_i)$ is weakly stable. Given $(a_i)(x_i) + (b_i) = (1_{R_1}, 1_{R_2}, \cdots, 1_{R_i}, \cdots)$ in $\prod_{i=1}^{\infty} R_i$, where $R_i = End_R(A_i)$, then $a_i x_i + b_i = 1_{R_i}$ for all $i \in \mathbb{N}$. For each i, since R_i is weakly stable, there exists some $y_i \in R$ such that $u_i := a_i + b_i y_i \in R_i$ is right or left invertible. If $u_i \in R_i$ is right invertible, choose $e_i = 1_{R_i}$. If $u_i \in R_i$ is left invertible, choose $e_i = 0$. Then $e_i u_i \in e_i R_i e_i$ is right invertible and $(1_{R_i} - e_i)u_i \in (1_{R_i} - e_i)R_i(1_{R_i} - e_i)$ is left invertible. Let $e = (e_i)$ and $y = (y_i)$. Then $e \in B\big(\prod_{i=1}^{\infty} R_i\big)$, and that $e\big((a_i) + (b_i)y\big) \in e\big(\prod_{i=1}^{\infty} R_i\big)e$ is right invertible and $\big((1_{R_1}, 1_{R_2}, \cdots, 1_{R_i}, \cdots) - e\big)\big((a_i) + (b_i)y\big) \in \big((1_{R_1}, 1_{R_2}, \cdots, 1_{R_i}, \cdots) - e\big)\big(\prod_{i=1}^{\infty} R_i\big)\big((1_{R_1}, 1_{R_2}, \cdots, 1_{R_i}, \cdots) - e\big)$ is left invertible. Hence, we get $(a_i) + (b_i)y \in U_r\big(\prod_{i=1}^{\infty} R_i\big)$, and so $\prod_{i=1}^{\infty} R_i$ satisfies related comparability by Corollary 3.1. Therefore E satisfies related comparability. ∎

Corollary 3.3. *Let G be an abelian group such that $End(G)$ is regular. If G is a reduced torsion group, then $End(G)$ satisfies related comparability.*

Proof. As is well known, a reduced abelian torsion group has a regular endomorphism ring if and only if it is a direct sum of cyclic groups of prime order. So we may assume that $G \cong \bigoplus_p \bigoplus_i G_{(p,i)}$, where each $G_{(p,i)}$ is a cyclic group of prime order p. Clearly, each $\bigoplus_i G_{(p,i)}$ is fully invariant. In addition, $G_{(p,i)} \cong G_{(p,j)}$ for any i, j. Thus, the result follows by Theorem 3.2. ∎

Take $G = H \oplus K$, with H the direct sum of infinitely many isomorphic copies of $\mathbb{Z}_p$, K the direct sum of infinitely many isomorphic copies of $\mathbb{Z}_q (p, q$ distinct primes). In view of Corollary 3.3, $End(G)$ is a regular ring satisfying related comparability. But $End(G)$ is not one-sided unit-regular. Let R_i be purely infinite, simple ring. As in the proof of Theorem 3.2, we see that $\prod_{i=1}^{\infty} R_i$ is an exchange ring satisfies related comparability. As in the proof of Theorem 2.2, we claim that $\prod_{i=1}^{\infty} R_i$ is not weakly stable.

Following Ara *et al.*, a ring R is separative provided that for all finitely generated projective right R-modules A, B, $2A \cong A \oplus B \cong 2B \Longrightarrow A \cong B$. Separativity plays a key role in the direct sum decomposition theory of exchange rings (cf. [1]).

Lemma 3.2. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is separative.*
(2) *For any $A, B, C \in FP(R)$, $A \oplus C \cong B \oplus C$ with $C \lesssim^{\oplus} A, B \Longrightarrow Ae \lesssim^{\oplus} Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$ for some $e \in B(R)$.*
(3) *For any $A, B \in FP(R)$, $A \oplus A \cong A \oplus B \cong B \oplus B \Longrightarrow Ae \lesssim^{\oplus} Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$ for some $e \in B(R)$.*
(4) *For any $A, B \in FP(R)$, $2A \cong 2B$ and $3A \cong 3B \Longrightarrow Ae \lesssim^{\oplus} Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$ for some $e \in B(R)$.*

Proof. Clearly, R can be seen as an ideal of itself. Therefore the proof is an immediate consequence of [7, Lemma 4.1]. ∎

Lemma 3.3. *Every exchange ring satisfying related comparability is separative.*

Proof. Let R be an exchange ring satisfying related comparability. Suppose that $A, B \in FP(R)$ such that $A \oplus A \cong A \oplus B \cong B \oplus B$. Since R satisfies related comparability, by [6, Theorem 6], we have $Ae \lesssim^{\oplus} Be$ and $B(1-e) \lesssim^{\oplus} A(1-e)$ for some $e \in B(R)$. Therefore we get the result by Lemma 3.2. ∎

Recall that a ring R has stable rank m if there exists the least number m such that $a_1R + \cdots + a_{m+1}R = R$ with $a_1, \cdots, a_{m+1} \in R$ implies that there exist $b_1, \cdots, b_m \in R$ such that $(a_1 + a_{m+1}b_1)R + \cdots + (a_m + a_{m+1}b_m)R = R$. If this m does not exist, we say that R has stable rank ∞.

Theorem 3.3. *Let G be an abelian group such that $End(G)$ is regular. If G is a reduced torsion group, then $End(G)$ is separative, so has stable rank $1, 2$ or ∞.*

Proof. In view of Corollary 3.3, $End(G)$ satisfies related comparability. Therefore $End(G)$ is separative from Lemma 3.3. Therefore we complete the proof by [1, Theorem 3.3]. ∎

Proposition 3.1. *Let R be a simple exchange ring. Then the following are equivalent:*

(1) *R satisfies related comparability.*
(2) *R is weakly stable.*
(3) *R is separative.*

Proof.
(1) $\Longrightarrow$ (3) is obvious by Lemma 3.3.
(3) $\Longrightarrow$ (2) If R is directly finite, R has stable range one from [1, Theorem 3.4]. If R is directly infinite, then $R \oplus D \cong R$ for some nonzero right R-module D. Let $e, f \in R$ be idempotents. If $e = 0$ or $f = 0$, then $eR \lesssim^{\oplus} fR$ or $fR \lesssim^{\oplus} eR$. Now we assume that $e \neq 0, f \neq 0$. Clearly, there exists a nonzero idempotent $g \in R$ such that $D \cong gR$. Since R is simple, we see that $RgR = R$. Thus, there are some $s_i, t_i \in R(1 \leq i \leq n)$ such that $\sum_{i=1}^{n} s_i g t_i = 1$. Construct a R-morphism $\varphi : n(gR) \to R$ given by $\varphi(gr_1, \cdots, gr_n) = \sum_{i=1}^{n} s_i g r_i$. It is easy to verify that φ is a R-epimorphism. As R is projective, there exists a right R-module N such that $R \oplus N \cong n(gR)$. Hence, $R \lesssim^{\oplus} nD$; whence, $eR \lesssim^{\oplus} nD$. Thus $eR \oplus R \lesssim^{\oplus} nD \oplus R \cong R$, and so $eR \oplus R \lesssim^{\oplus} R \lesssim^{\oplus} fR \oplus R$. Hence, $R \oplus \big(eR \oplus E\big) \cong R \oplus fR$ for a right R-module E. As eR and fR are both nonzero, we also have $R \lesssim^{\oplus} s(eR) \lesssim^{\oplus} s\big(eR \oplus E\big)$ and $R \lesssim^{\oplus} t(fR)$ for some $s, t \in \mathbb{N}$. Applying [1, Lemma 2.1], $eR \lesssim^{\oplus} eR \oplus E \cong fR$. In view of Corollary 2.2, R is weakly stable.
(2) $\Longrightarrow$ (1) is trivial. ∎

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