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Generalized Fuzzy Compactness in L-Topological Spaces

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Abstract. In this paper, we shall introduce generalized fuzzy compactness in L-spaces where L is a complete de Morgan algebra. This definition does not rely on the structure of basis lattice L and no distributivity is required. The intersection of a generalized fuzzy compact L-set and a generalized closed L-set is a generalized fuzzy compact L-set. The generalized irresolute image of a generalized fuzzy compact L-set is a generalized fuzzy compact L-set.

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1. Introduction and preliminaries

In 1976, Lowen first introduced the concepts of fuzzy compactness in $[0, 1]$-spaces in [6]. Subsequently its characterization was given by Wang in terms of α-net in [11]. In 1988, it is again extended to L-spaces [12], where L is a completely distributive de Morgan algebra (i.e., a F lattice). However the above mentioned definitions of fuzzy compactness seriously depend on the structure of the basis lattice L and complete distributivity was required.

Kubiák also extended fuzzy compactness to L-spaces by means of closed L-sets and the way below relation in [4], where complete distributivity was not required. But his definition still depend on the structure of the basis lattice L and can't be restated in terms of open L-sets by simply using quasi-complementation.

In [9, 10], a new definition of fuzzy compactness in presented in L-topological space by means of an inequality, which doesn't depend on the structure of L and no distributivity is require in L. When L is a completely distributive de Morgan algebra, it is equivalent to the notion of fuzzy compactness in [5, 7, 12].

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The notions of generalized open sets, generalized closed sets and generalized-irresolute mapping were introduced by Balasubramanian and Sundaram in [1].

In this paper, following the lines of [9, 10], we shall introduce a concept of generalized compactness in L-topological spaces in terms of generalized open L-sets and their inequality, where L is a complete de Morgan algebra. This definition doesn't rely on the structure of basis lattice L and no distributivity in L is required. It can also be characterized by generalized closed L-sets and their inequality. When L is a completely distributive de Morgan algebra, its many characterizations are presented.

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ is a complete de Morgan algebra. 0 and 1 denote the smallest element and the largest element in L, respectively.

A complete lattice L is a complete Heyting algebra if it satisfies the following infinite distributive law: For all $a \in L$ and all $B \subset L$, $a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\}$.

For a nonempty set X, L^X denotes the set of all L-topological fuzzy sets (or L-sets for short) on X. $\underline{0}$ and $\underline{1}$ denote the smallest element and the largest element in L^X, respectively. An L-space (L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily L^X which contains $\underline{0}, \underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its quasi-complementation is called a closed L-set. An element a in L is called a prime element if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. a in L is called co-prime element if a' is a prime element. The set of all nonzero co-prime elements in L is denoted by $M(L)$. It is easy to see that $M(L^X) = \{x_\alpha \mid x \in X, \alpha \in M(L)\}$ is exactly the set of all nonzero $\vee$-irreducible elements in L^X.

According to [12], we know that L is completely distributive if and only if each element a in L has the greatest minimal family (the greatest maximal family), denoted by $\beta(a)(\alpha(a))$. Obviously $\beta^*(a) = \beta(a) \bigcap M(L)$ is a minimal family of a and $\alpha^*(a) = \beta(a) \bigcap P(L)$ is a maximal family of a.

For a subfamily $\Phi \subset L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamily of Φ.

In [1], the notions of generalized open sets, generalized closed sets and generalized-irresolute mapping were introduced in [0,1]-fuzzy set theory by Balasubramanian and Sundaram. They can easily be extended to L-sets as follows:

Definition 1.1. *Let $(X, \mathcal{T})$ be an L-space and $A \in L^X$. Then A is called generalized closed L-set (or gl-closed for short) if $cl(A) \leq U$ whenever $A \leq U$ and U is open L-set. A is called generalized open (gl-open for short) if A' is gl-closed.*

GLO(X) *and* **GLC(X)** *will always denote the family of all generalized open L-sets and family of all generalized closed L-sets in X, respectively.*

Definition 1.2. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces, $f : X \to Y$ be a mapping and $f_L^{\rightarrow} : L^X \to L^Y$ be the extension of f. Then f called a generalized irresolute mapping if $f_L^{\leftarrow}(B)$ is generalized open in $(X, \mathcal{T}_1)$ for each generalized open L-set B in $(Y, \mathcal{T}_2)$.*

Definition 1.3. [9, 10] *Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. Then G is called fuzzy compact if for every family $\mathcal{U} \subset \mathcal{T}$, it follows that*

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Lemma 1.1. [10] *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces, where L is a complete Heyting algebra, $f : X \to Y$ be a mapping, $f_L^{\rightarrow} : L^X \to L^Y$ is the extension of f. Then for any $\mathcal{P} \subset L^Y$, we have that*

$$\bigvee_{y \in Y} \left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in P} B(y) \right) = \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x) \right).$$

2. Generalized fuzzy compactness of L-subsets

Definition 2.1. *Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. Then G is called generalized fuzzy compact if for every family $\mathcal{U} \subset$**GLO**(X), it follows that*

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Now we consider characterizations of generalized fuzzy compactness. First we introduce the following concept.

Definition 2.2. *Let $(X, \mathcal{T})$ be an L-space, $a \in L \setminus \{1\}$ and $G \in L^X$. A family $\mathcal{U} \subset$**GLO**(X) is said to be a generalized open a-shading of G if for any $x \in X$ with $G(x) \geq a'$, there exists an $A \in \mathcal{U}$ such that $A(x) \not\leq a$. $\mathcal{U}$ is said to be a generalized open strong a-shading of G if*

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a$$

for any $x \in X$.

Obviously, a generalized open strong a-shading of G is a generalized open a-shading of G and $\mathcal{U}$ is a generalized open a-shading of G if and only if

$$G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \not\leq a.$$

By Definition 2.1 and Definition 2.2 we obtain the following result.

Theorem 2.1. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is generalized fuzzy compact if and only if for any $a \in L \setminus \{1\}$, each generalized open strong a-shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is still a generalized open strong a-shading of G.*

Proof. Suppose that G is generalized fuzzy compact and for any $a \in L \setminus \{1\}$, $\mathcal{U}$ is any generalized open strong a-shading of G. Then

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right)$$

and

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a.$$

So that

$$\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x) \right) \not\leq a,$$

hence there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that

$$\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x) \right) \not\leq a.$$

Thus $\mathcal{V}$ is finite subfamily of $\mathcal{U}$ and $\mathcal{V}$ is a generalized open strong a-shading of G.

Conversely, suppose that for any $a \in L \setminus \{1\}$, each generalized open strong a-shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is still a generalized open strong a-shading of G. Hence we have that

$$\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \right) \not\leq a \text{ implies that } \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x) \right) \not\leq a,$$

therefore

$$\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \right) \leq \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x) \right).$$

Thus we obtain that

$$\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x) \right).$$

Hence G is generalized fuzzy compact from Definition 2.1. ∎

Moreover from Definition 2.1 we easily obtain the following theorem by simply using quasi-complementation.

Theorem 2.2. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is generalized fuzzy compact if and only if for every subfamily $\mathcal{P} \subset$* **GLC**(X), *it follows that*

$$\bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x) \right) \geq \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x) \right).$$

Definition 2.3. *Let $(X, \mathcal{T})$ be an L-space, $a \in L \setminus \{1\}$ and $G \in L^X$. A family $\mathcal{P} \subset$* **GLC**(X) *is said to be a generalized closed a-remote family of G if for any $x \in X$ with $G(x) \geq a$, there exists a $B \in \mathcal{P}$ such that $B(x) \not\geq a$. $\mathcal{P}$ is said to be a generalized closed strong a-remote family of G if*

$$\bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x) \right) \not\geq a.$$

It is obvious that a generalized closed strong a-remote family of G is a generalized closed a-remote family of G, $\mathcal{P}$ is a generalized closed a-remote family of G if and only if

$$G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x) \not\geq a$$

and $\mathcal{P}$ is a generalized closed strong a-remote family of G if and only if $\mathcal{P}'$ is a generalized open strong a-shading of G.

From Theorem 2.2 we obtain the following result.

Theorem 2.3. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is generalized fuzzy compact if and only if for any $a \in L \setminus \{0\}$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ which is still a generalized closed strong a-remote family of G.*

Proof. Analogous to the proof of Theorem 2.1. ∎

Theorem 2.4. *Let L be a complete Heyting algebra. If both G and H are generalized fuzzy compact, then $G \vee H$ is generalized fuzzy compact.*

Proof. For any family $\mathcal{P} \subset \mathbf{GLC}(X)$, by Theorem 2.2 we have that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\vee H)(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right)\\
&=\left\{\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(H(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right)\right\}\\
&\geq\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right)\right\}\vee\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(H(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right)\right\}\\
&=\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left((G\vee H)(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right).
\end{aligned}$$

This shows that $G \vee H$ is generalized fuzzy compact. ∎

Theorem 2.5. *If G is a generalized fuzzy compact L-set and H is a generalized closed L-set, then $G \wedge H$ is a generalized fuzzy compact L-set.*

Proof. Since G is a generalized fuzzy compact L-set, for any family $\mathcal{P} \subset \mathbf{GLC}(X)$, by Theorem 2.2 we have that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right)\\
&=\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{P}\cup\{H\}} B(x)\right)\geq \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P}\cup\{H\})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right)\\
&=\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right)\right\}\\
&\quad\wedge\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge\left(H(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right)\right)\right\}\\
&=\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right).
\end{aligned}$$

This shows that $G \wedge H$ is a generalized fuzzy compact L-set. ∎

Theorem 2.6. *Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two L-spaces, where L is a complete Heyting algebra, $f : X \to Y$ be a generalized irresolute mapping. If G is generalized fuzzy compact in $(X,\mathcal{T}_1)$, then so is $f_L^{\to}(G)$ is in $(Y,\mathcal{T}_2)$.*

Proof. For any $\mathcal{P} \subset$**GLC**(X), by Lemma 1.1 and Theorem 2.2, we have that

$$\begin{aligned}\bigvee_{y\in Y}\left(f_L^{\to}(G)(y)\wedge\bigwedge_{B\in\mathcal{P}}B(y)\right) &= \bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}f_L^{\leftarrow}(B)(x)\right)\\ &\geq \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}f_L^{\leftarrow}(B)(x)\right)\\ &= \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{y\in Y}\left(f_L^{\to}(G)(y)\wedge\bigwedge_{B\in\mathcal{F}}B(y)\right).\end{aligned}$$

Therefore $f_L^{\to}(G)$ is generalized fuzzy compact. ∎

3. Some characterizations of generalized fuzzy compact

In this section, we assume that L is a completely distributive de Morgan algebra. We give many characterizations of generalized fuzzy compact.

Theorem 3.1. *Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is generalized fuzzy compact;*
(2) *For any $a \in L\setminus\{0\}$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ which is a generalized closed strong a-remote family of G;*
(3) *For any $a \in L\setminus\{0\}$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ which is a generalized closed a-remote family of G;*
(4) *For any $a \in L\setminus\{0\}$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ and $b \in \beta(a)$ such that $\mathcal{F}$ is a generalized closed strong b-remote family of G;*
(5) *For any $a \in L\setminus\{0\}$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ such that $\mathcal{F}$ is a generalized closed b-remote family of G;*
(6) *For any $a \in M(L)$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ which is a generalized closed strong a-remote family of G;*
(7) *For any $a \in M(L)$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ which is a generalized closed a-remote family of G;*
(8) *For any $a \in M(L)$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{F}$ is a generalized closed strong b-remote family of G;*
(9) *For any $a \in M(L)$, each generalized closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{F}$ is a generalized closed b-remote family of G.*

Proof. By Theorem 2.3 we can obtain (1)$\Longleftrightarrow$(2). (2)$\Longrightarrow$(3) is obvious. Now to prove (3)$\Longrightarrow$(4), suppose that $a \in L \setminus \{0\}$ and $\mathcal{P}$ is a generalized closed strong a-remote family of G, then we obtain that

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\not\geq a,$$

take $c \in \beta(a)$ such that

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\not\geq c,$$

obviously $\mathcal{P}$ is a strong generalized closed c-remote family of G, by (3) we know that $\mathcal{P}$ has a finite subfamily $\mathcal{F}$ which is a generalized closed c-remote family of G. Take $b \in \beta(a)$ such that $c \in \beta(b)$, then $\mathcal{F}$ is a generalized closed strong b-remote family of G. (4) is shown. (4)$\Longrightarrow$(5) is obvious, we prove (5)$\Longrightarrow$(2). For any $a \in L \setminus \{0\}$, suppose that $\mathcal{P}$ is any generalized closed strong a-remote family of G, by (5), $\mathcal{P}$ has a finite subfamily $\mathcal{F}$ and $b \in \beta(a)$ such that $\mathcal{F}$ is a generalized closed b-remote family of G. So that for any

$$x \in X, G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\not\geq b,$$

we obtain

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\not\geq a,$$

in fact, if

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\geq a,$$

then by $b \in \beta(a)$, there exists $x_0 \in X$ such that

$$G(x_0)\wedge\bigwedge_{B\in\mathcal{F}}B(x_0)\geq b,$$

a contradiction. So that

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\not\geq a.$$

This implies that $\mathcal{F}$ is a generalized closed strong a-remote family of G. Similarly we can prove that (2)$\Longrightarrow$(6)$\Longrightarrow$(7)$\Longrightarrow$(8)$\Longrightarrow$(9)$\Longrightarrow$(1). ∎

Now we present some characterizations of generalized fuzzy compactness by means of generalized open L-sets.

Theorem 3.2. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is generalized fuzzy compact;*
(2) *For any $a \in L \setminus \{1\}$, each generalized open strong a-shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a generalized open strong a-shading of G;*
(3) *For any $a \in L \setminus \{1\}$, each generalized open strong a-shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a generalized open a-shading of G;*

(4) *For any* $a \in L\setminus\{1\}$, *each generalized open strong* a*-shading* $\mathcal{U}$ *of* G, *there exists a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *and* $b \in \alpha(a)$ *such that* $\mathcal{V}$ *is a strong generalized open* b*-shading of* G;

(5) *For any* $a \in L \setminus \{1\}$, *each generalized open strong* a*-shading* $\mathcal{U}$ *of* G, *there exists a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *and* $b \in \alpha(a)$ *such that* $\mathcal{V}$ *is a generalized open* b*-shading of* G;

(6) *For any* $a \in P(L)$, *each generalized open strong* a*-shading* $\mathcal{U}$ *of* G *has a finite subfamily* $\mathcal{V}$ *which is a generalized open strong* a*-shading of* G;

(7) *For any* $a \in P(L)$, *each generalized open strong* a*-shading* $\mathcal{U}$ *of* G *has a finite subfamily* $\mathcal{V}$ *which is a generalized open* a*-shading of* G;

(8) *For any* $a \in P(L)$, *each generalized open strong* a*-shading* $\mathcal{U}$ *of* G *has a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *and* $b \in \alpha^*(a)$ *such that* $\mathcal{V}$ *is a strong generalized open* b*-shading of* G;

(9) *For any* $a \in P(L)$, *each generalized open strong* a*-shading* $\mathcal{U}$ *of* G *has a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *and* $b \in \alpha^*(a)$ *such that* $\mathcal{V}$ *is a generalized open* b*-shading of* G.

Proof. By Theorem 2.1 we can obtain (1)$\Longleftrightarrow$(2).
(2)$\Longrightarrow$(3) is obvious.
(3)$\Longrightarrow$(4). Suppose that $a \in L \setminus \{1\}$ and $\mathcal{U}$ is a generalized open strong a-shading of G, then

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{B\in\mathcal{U}} B(x)\right) \not\leq a.$$

Take $c \in \alpha(a)$ such that

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{B\in\mathcal{U}} B(x)\right) \not\leq c,$$

obviously $\mathcal{U}$ is a generalized open strong c-shading of G and by (3) we know that $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ which is a generalized open c-shading of G. Take $b \in \alpha(a)$ such that $c \in \alpha(b)$, then $\mathcal{V}$ is a generalized open strong b-shading of G, (4) is shown.
(4)$\Longrightarrow$(5) is obvious.
(5)$\Longrightarrow$(2). For any $a \in L \setminus \{1\}$, suppose that $\mathcal{U}$ is any generalized open strong a-shading of G, by (5), $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ and $b \in \alpha(a)$ such that $\mathcal{V}$ is a generalized open b-shading of G. So that for any $x \in X$,

$$G'(x) \vee \bigvee_{B\in\mathcal{V}} B(x) \not\leq b,$$

we obtain

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{B\in\mathcal{V}} B(x)\right) \not\leq a,$$

in fact, if

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{B\in\mathcal{V}} B(x)\right) \leq a,$$

then by $b \in \alpha(a)$, there exists $x_0 \in X$ such that

$$G(x_0) \vee \bigvee_{B \in \mathcal{V}} B(x_0) \leq b,$$

a contradiction. So that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{B \in \mathcal{V}} B(x) \right) \not\leq a.$$

This implies that $\mathcal{V}$ is a generalized open strong a-shading of G.

Similarly we can prove that (2)$\Longrightarrow$(6)$\Longrightarrow$(7)$\Longrightarrow$(9) $\Longrightarrow$(9)$\Longrightarrow$(1). ∎

Definition 3.1. *Let $(X, \mathcal{T})$ be an L-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{U} \subset$* **GLO**(X) *is said to be a generalized open β_a-cover of G if for any $x \in X$ with $a \notin \beta(G'(x))$, there exists $A \in \mathcal{U}$ such that $a \in \beta(A(x))$. $\mathcal{U}$ is said to be a generalized open strong β_a-cover of G if*

$$a \in \beta\left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \right).$$

It is obvious that a generalized open strong β_a-cover of G is generalized open β_a-cover G and $\mathcal{U}$ is a generalized open β_a-cover of G if and only if for any $x \in X$,

$$a \in \beta\left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right).$$

Theorem 3.3. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is generalized fuzzy compact;*

(2) *For any $a \in L \setminus \{0\}$, each generalized open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a generalized open strong β_a-cover of G;*

(3) *For any $a \in L \setminus \{0\}$, each generalized open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a generalized open β_a-cover of G;*

(4) *For any $a \in L \setminus \{0\}$, any generalized open strong β_a-cover $\mathcal{U}$ of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a generalized open strong β_a-cover of G;*

(5) *For any $a \in L \setminus \{0\}$, any generalized open strong β_a-cover $\mathcal{U}$ of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a generalized open β_a-cover of G;*

(6) *For any $a \in M(L)$, each generalized open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a generalized open strong β_a-cover of G;*

(7) *For any $a \in M(L)$, each generalized open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a generalized open β_a-cover of G;*

(8) *For any $a \in M(L)$ and any generalized open strong β_a-cover $\mathcal{U}$ of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a generalized open strong β_a-cover of G;*

(9) *For any $a \in M(L)$ and any generalized open strong β_a-cover $\mathcal{U}$ of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a generalized open β_a-cover of G.*

Proof. We only prove (1)$\Longleftrightarrow$(2).
(1)$\Longrightarrow$(2). Suppose that G is generalized fuzzy compact and for any $a \in L \setminus \{0\}$, $\mathcal{U}$ is any generalized open strong β_a-cover of G. Then

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right).$$

So

$$\beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)\right) \leq \beta\left(\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right)\right).$$

By

$$a \in \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)\right),$$

we obtain

$$a \in \beta\left(\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right)\right),$$

therefore

$$a \in \bigcup_{\mathcal{V}\in 2^{(\mathcal{U})}} \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right)\right),$$

hence there exists a $\mathcal{V} \in 2^{(\mathcal{U})}$ such that

$$a \in \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right)\right).$$

Thus $\mathcal{V}$ is a generalized open strong β_a-cover of G.
(2)$\Longrightarrow$(1). Suppose that for any $a \in L \setminus \{0\}$, each generalized open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a generalized open strong β_a-cover of G, then we know that

$$a \in \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)\right) \text{ implies that } a \in \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right)\right)$$

where $\mathcal{V} \in 2^{(\mathcal{U})}$. Hence

$$\beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)\right) \leq \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right)\right).$$

Thus

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right).$$

This prove that G is generalized fuzzy compact. ∎

Definition 3.2. *Let $(X, \mathcal{T})$ be an L-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{U} \subset$***GLO**(X) *is said to be a generalized open Q_a-cover of G if for any $x \in X$ it follows that*

$$G' \vee \bigvee_{A \in \mathcal{U}} A(x) \geq a.$$

It is obvious that a generalized open β_a-cover of G is a generalized open Q_a-cover of G. Moreover form Definition 2.1 we also can obtain the following result.

Theorem 3.4. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is generalized fuzzy compact;*
(2) *For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each generalized open Q_a-cover of G, has a finite subfamily which is a generalized open Q_b-cover of G;*
(3) *For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each generalized open Q_a-cover of G, has a finite subfamily which is a generalized open β_a-cover of G;*
(4) *For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each generalized open Q_a-cover of G, has a finite subfamily which is a generalized open strong β_a-cover of G;*
(5) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each generalized open Q_a-cover of G, has a finite subfamily which is a generalized open Q_b-cover of G;*
(6) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each generalized open Q_a-cover of G, has a finite subfamily which is a generalized open β_b-cover of G;*
(7) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each generalized open Q_a-cover of G, has a finite subfamily which is a generalized open strong β_b-cover of G.*

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