BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33**(3) (2010), 469–478

Flat Surfaces in the Euclidean Space $\mathbb{E}^3$ with Pointwise 1-Type Gauss Map

UĞUR DURSUN
Department of Mathematics, Faculty of Science and Letters,
Istanbul Technical University, 34469 Maslak, Istanbul, Turkey
udursun@itu.edu.tr

Abstract. In this article we prove that a flat nonplanar surface in the Euclidean space $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind is either a right circular cone or a cylinder such that the curvature of the base curve satisfies a specific differential equation. We conclude that there is no tangent developable surface in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind.

2010 Mathematics Subject Classification: 53B25, 53C40

Key words and phrases: Right cone, cylinder, developable surface, mean curvature, pointwise 1-type, Gauss map.

1. Introduction

A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + \cdots + x_k$, where x_0 is a constant map, $x_1, \ldots, x_k$ are nonconstant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then M is said to be of k-type (cf. [5, 6]). In [7], this definition was similarly extended to differentiable maps, in particular, to Gauss maps of submanifolds. The notion of finite type Gauss map is an especially useful tool in the study of submanifolds (cf. [1, 2, 3, 4, 7, 17]). In [7], Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map, and for hypersurfaces they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if M is a hypersphere in $\mathbb{E}^{n+1}$.

If a submanifold M of a Euclidean space has 1-type Gauss map G, then $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. However, the Laplacian of the Gauss maps of several surfaces such as helicoid, catenoid and right cones in $\mathbb{E}^3$, and also some hypersurfaces has the form of the product

$$\Delta G = f(G + C) \tag{1.1}$$

Communicated by Lee See Keong.
Received: October 14, 2008; *Revised:* January 20, 2010.

for some smooth function f on M and some constant vector C. A submanifold of a Euclidean space is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1.1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of *the first kind* if the vector C in (1.1) is the zero vector. Otherwise, a submanifold with pointwise 1-type Gauss map is said to be of *the second kind*.

Remark 1.1. For a plane M in $\mathbb{E}^3$ the Gauss map G is a constant vector and $\Delta G = 0$. For $f = 0$ if we write $\Delta G = 0 \cdot G$, then M has pointwise 1-type Gauss map of the first kind. If we choose $C = -G$ for any nonzero smooth function f, then (1.1) holds. In this case M has pointwise 1-type Gauss map of the second kind. Therefore we say that a plane in $\mathbb{E}^3$ is a trivial surface with pointwise 1-type Gauss map of the first kind or the second kind.

Surfaces in Euclidean spaces and in pseudo-Euclidean spaces with pointwise 1-type Gauss map were recently studied in [8, 9, 12, 13, 14, 18]. Also, hypersurfaces of Euclidean space $\mathbb{E}^{n+1}$ with pointwise 1-type Gauss map were studied in [10]. In particular, the classification of surfaces of revolution with pointwise 1-type Gauss map was given in [8]. It was proved that the right circular cones are the only surfaces of revolution of polynomial kind with pointwise 1-type Gauss map of the second kind.

Here we prove that a right circular cone is the only cone in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind. Then we describe all cylinders in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind such that the curvature k of the base curve of the cylinders satisfies a specific differential equation which determines k implicitly. Also, we conclude that there is no tangent developable surface in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind.

2. Preliminaries

Let M be an oriented surface in the Euclidean space $\mathbb{E}^3$. The map $G : M \to S^2 \subset E^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of the surface M, where S^2 is the unit sphere in $\mathbb{E}^3$ centered at the origin. We denote by h, A_G, $\widetilde{\nabla}$ and ∇, the second fundamental form, the Weingarten map, the Levi-Civita connection of $\mathbb{E}^3$ and the induced Riemannian connection on M, respectively.

We choose an oriented orthonormal moving frame $\{e_1, e_2, e_3\}$ on M in $\mathbb{E}^3$ such that e_1, e_2 are tangent to M and $e_3 = G$ is normal to M.

Denote by $\{\omega^1, \omega^2, \omega^3\}$ the dual 1-forms to $\{e_1, e_2, e_3\}$ and by $\{\omega_B^A\}, A, B = 1, 2, 3$, the connection 1-forms associated with $\{\omega^1, \omega^2, \omega^3\}$ satisfying $\omega_B^A + \omega_A^B = 0$. Then we have $\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^2 \omega_i^j(e_k)e_j + h_{ik}e_3$, $\widetilde{\nabla}_{e_k} e_3 = \sum_{j=1}^2 \omega_3^j(e_k)e_j$, where h_{ik} are the coefficients of the second fundamental form h. By Cartan's Lemma, we also have $\omega_j^3 = \sum_{k=1}^2 h_{jk}\omega^k$, $h_{jk} = h_{kj}$.

The mean curvature H and the Gauss curvature K are, respectively, defined by $H = (h_{11} + h_{22})/2$ and $K = h_{11}h_{22} - h_{12}h_{21}$.

Let I be an open interval containing zero in the real line $\mathbb{R}$. A ruled surface M is parametrized by

$$x(s,t) = \alpha(s) + t\beta(s), \; s \in I, \; t \in \mathbb{R},$$

where α and β are smooth mappings from I into $\mathbb{E}^3$ and β is nowhere zero. The map $\alpha = \alpha(s)$ is called a base curve and $\beta = \beta(s)$ is called a director curve. We say that a ruled surface M is a cylinder if β is a constant vector, M is a cone if α is a constant vector, and M is a tangent surface if β is tangent to α.

A surface in the Euclidean space $\mathbb{E}^3$ whose Gaussian curvature vanishes on the regular part is called a developable surface. Then we have the following well-known classification theorem of developable surfaces [16].

Theorem 2.1. [16] *A developable surface is one of the following:*

(1) *A part of cylindrical surface.*
(2) *A part of a conical surface.*
(3) *A part of a tangent developable surface.*
(4) *The result of gluing two or more surfaces of the above three types.*

3. Surfaces with pointwise 1-type Gauss map of second kind

We study flat surfaces (developable ruled surface) of $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind.

Lemma 3.1. [10] *Let M be an oriented hypersurface of a Euclidean space $\mathbb{E}^{n+1}$. Then the Laplacian of the Gauss map G is given by*

$$\Delta G = \|A_G\|^2 G + n\nabla H, \tag{3.1}$$

where ∇H is the gradient of the mean curvature H and $\|A_G\|^2 = tr(A_G A_G)$.

We prove the following lemma for later use.

Lemma 3.2. *Let M be an oriented surface in the Euclidean space $\mathbb{E}^3$. Let e_1, e_2 be the unit principal directions of the shape operator A_G of M. If C is a constant vector in $\mathbb{E}^3$, then the components of $C = C_1 e_1 + C_2 e_2 + C_3 G$ in the basis $\{e_1, e_2, G\}$ of $\mathbb{E}^3$ satisfy the following equations:*

$$e_1(C_1) + \omega_2^1(e_1)\, C_2 - h_{11}\, C_3 = 0, \tag{3.2}$$

$$e_1(C_2) - \omega_2^1(e_1)\, C_1 = 0, \tag{3.3}$$

$$e_1(C_3) + h_{11}\, C_1 = 0, \tag{3.4}$$

$$e_2(C_1) + \omega_2^1(e_2)\, C_2 = 0, \tag{3.5}$$

$$e_2(C_2) - \omega_2^1(e_2)\, C_1 - h_{22}\, C_3 = 0, \tag{3.6}$$

$$e_2(C_3) + h_{22}\, C_2 = 0, \tag{3.7}$$

where $C_i = \langle C, e_i \rangle$, $i = 1, 2$ and $C_3 = \langle C, G \rangle$.

Proof. Let e_1, e_2 be the unit principal directions of the shape operator A_G. Then we have $A_G(e_i) = h_{ii} e_i$, $i = 1, 2$, and $h_{12} = h_{21} = 0$. When we take derivative of the vector C in direction e_k and use the formulas of Gauss and Weingarten, we obtain

$$\begin{aligned}\widetilde{\nabla}_{e_k} C = {} & [e_k(C_1) + \omega_2^1(e_k)\, C_2 - h_{k1}\, C_3] e_1 \\ & + [e_k(C_2) - \omega_2^1(e_k)\, C_1 - h_{k2}\, C_3] e_2\end{aligned}$$

$$+ [e_k(C_3) + h_{1k}\, C_1 + h_{2k}\, C_2]G = 0$$

which produces equations (3.2)–(3.7) for $k = 1, 2$. ∎

In [8], it was shown that a right circular cone is the only surface of revolution of polynomial kind with pointwise 1-type Gauss map of the second kind in $\mathbb{E}^3$. We prove:

Theorem 3.1. *Let M be an oriented flat regular surface in the Euclidean space $\mathbb{E}^3$. Then M has pointwise 1-type Gauss map of the second kind if and only if M is an open part of the following surfaces:*

(1) *A right circular cone in $\mathbb{E}^3$,*
(2) *a plane in $\mathbb{E}^3$,*
(3) *a cylinder given, up to a rigid motion, by*

$$x(s,t) = \left(\pm\frac{q_0^2}{d_0}\mu(s) - \frac{s}{d_0} + d_1, -\frac{q_0}{2d_0k^2(s)} + d_2, t\right), \tag{3.8}$$

where $d_0 \neq 0$, $q_0 \neq 0$, d_1, and d_2 are arbitrary constants, while the function $\mu(s)$ and the curvature function $k(s)$ of the base curve are related by

$$\mu(s) = \int \frac{dk}{k^3\sqrt{(d_0^2-1)k^2 + 2q_0k - q_0^2}},$$

and $k(s)$ satisfies the differential equation $q_0^2{k'}^2 = k^4[(d_0^2-1)k^2 + 2q_0k - q_0^2]$.

Proof. Suppose that M is a flat nonplanar surface of $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind. Then the gradient vector ∇H of the mean curvature H is nonzero on M because of (3.1). If ∇H were zero, then the Gauss map would be of pointwise 1-type of the first kind. So the mean curvature H is a nonconstant function on M.

Let e_1, e_2 be the unit principal directions of A_G, i.e., $A_G(e_i) = h_{ii}e_i$, $i = 1, 2$, and $h_{12} = h_{21} = 0$. By (1.1) and (3.1) we have

$$\|A_G\|^2 G + 2\nabla H = f(G + C) \tag{3.9}$$

for some nonzero smooth function f on M and some nonzero constant vector $C \in \mathbb{E}^3$. In the basis $\{e_1, e_2, G\}$ we can write

$$C = C_1\, e_1 + C_2 e_2 + C_3 G,$$

where $C_i = \langle C, e_i\rangle$, $i = 1, 2$ and $C_3 = \langle C, G\rangle$ which satisfy equations (3.2)–(3.7) in Lemma 3.2. Hence equation (3.9) implies

$$||A_G||^2 = f(1 + C_3), \tag{3.10}$$

$$2\, e_1(H) = f\, C_1, \tag{3.11}$$

$$2\, e_2(H) = f\, C_2. \tag{3.12}$$

As the Gauss curvature is zero, that is, M is a developable ruled surface, then from Theorem 2.1 M is a part of a cone, a cylinder or a tangent developable surface. So, to prove the theorem we consider three cases.

Case 1. M is an open part of a cone. Then, by an appropriate rigid motion, M can be parametrized locally by

$$x(s,t) = \alpha_0 + t\beta(s), \quad t \neq 0,$$

where $\langle \beta(s), \beta(s)\rangle = 1$ and $\langle \beta'(s), \beta'(s)\rangle = 1$, while α_0 is a constant vector. The coordinate vector fields $x_s = t\beta'(s)$ and $x_t = \beta(s)$ are orthogonal as $\langle \beta(s), \beta(s)\rangle = 1$. So we take the orthonormal tangent frame $\{e_1, e_2\}$ on M such that $e_1 = \frac{1}{t}\frac{\partial}{\partial s}$ and $e_2 = \frac{\partial}{\partial t}$. The Gauss map of M is given by $G = e_1 \times e_2 = \beta'(s) \times \beta(s)$.

By calculation we obtain

$$\widetilde{\nabla}_{e_1} e_1 = -\frac{1}{t}e_2 - \frac{k_g(s)}{t}G, \quad \widetilde{\nabla}_{e_1} e_2 = \frac{1}{t}e_1, \quad \widetilde{\nabla}_{e_2} e_1 = \widetilde{\nabla}_{e_2} e_2 = 0,$$

where $k_g(s) = \langle \beta(s), \beta'(s) \times \beta''(s)\rangle \neq 0$ which is the geodesic curvature of β in the unit sphere $\mathbb{S}^2(1)$. All these relations imply that

$$\omega_1^2(e_1) = -\frac{1}{t},\ \omega_2^1(e_2) = 0,\ h_{11} = -\frac{k_g(s)}{t},\ h_{12} = h_{21} = h_{22} = 0.$$

Thus, e_1, e_2 are principal vectors of the surface, $H = -\frac{k_g(s)}{2t}$, and $\|A_G\|^2 = \frac{k_g^2(s)}{t^2}$.

Now (3.5)–(3.7) imply that C_1, C_2, and C_3 are functions of s, and equations (3.2)–(3.4) become

$$C_1'(s) + C_2(s) + k_g(s)C_3(s) = 0, \tag{3.13}$$

$$C_2'(s) - C_1(s) = 0, \tag{3.14}$$

$$C_3'(s) - k_g(s)C_1(s) = 0. \tag{3.15}$$

On the other hand, we have from (3.10), (3.11), and (3.12),

$$\frac{k_g^2(s)}{t^2} = f(1 + C_3), \tag{3.16}$$

$$-\frac{1}{t^2}\frac{dk_g(s)}{ds} = f\,C_1, \tag{3.17}$$

$$\frac{k_g(s)}{t^2} = f\,C_2. \tag{3.18}$$

It follows from (3.18) that $C_2 \neq 0$. Also, (3.16) and (3.18) give

$$k_g(s)C_2(s) - C_3(s) = 1 \tag{3.19}$$

from which by taking derivative with respect to s we get

$$k_g'(s)C_2(s) + k_g(s)C_2'(s) = C_3'(s) \tag{3.20}$$

which equals $k_g'(s)C_2(s) = 0$ in view of (3.14) and (3.15). Hence we obtain $k_g'(s) = 0$ as $C_2 \neq 0$, that is, $k_g(s)$ is a nonzero constant. It is well-known that a spherical curve in $\mathbb{S}^2$ is completely determined by its geodesic curvature, in particular, if k_g is a nonzero constant, then it is a small circle of $\mathbb{S}^2$. Therefore, β is a part of a small circle in the unit sphere. As a result, M is an open part of a right circular cone.

Moreover we have $C_1 = 0$ from (3.17), and $C_2' = 0$ and $C_3' = 0$ from (3.14) and (3.15), respectively. That is, C_2 and C_3 are constants, and thus, from (3.13) and (3.19) we get

$$C_2 = \frac{k_g}{1+k_g^2}, \quad C_3 = -\frac{1}{1+k_g^2}.$$

Also, we have from (3.18) $f = \frac{1+k_g^2}{t^2}$. Therefore M has pointwise 1-type Gauss map of the second kind, that is, equation (1.1) holds for $f = \frac{1+k_g^2}{t^2}$ and for the constant vector $C = \frac{k_g}{1+k_g^2} e_2 - \frac{1}{1+k_g^2} G$.

Case 2. M is an open part of a cylinder. Locally it can be parametrized by

$$x(s,t) = \alpha(s) + t\beta, \tag{3.21}$$

where $\alpha(s)$ is a base curve of the cylinder parametrized by arc length that lies in a plane with unit normal vector β which is the director of the cylinder. By an appropriate rigid motion, we may assume that $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ and $\beta = (0,0,1)$ without lose of generality.

If the curvature $k(s)$ of $\alpha(s)$ is zero, then α is a line, and the cylinder M is a plane which has pointwise 1-type Gauss map of the second kind by choosing $C = -G$ for any nonzero smooth function f. This proves the part 2 of Theorem 3.1. We then assume that k is a nonconstant function because if $k(s)$ were a nonzero constant, then M would be a circular cylinder which has pointwise 1-type Gauss map of the first kind.

Now we take an orthonormal tangent frame $\{e_1, e_2\}$ on M such that $e_1 = \frac{\partial}{\partial t}$ and $e_2 = \frac{\partial}{\partial s}$ since $\langle \alpha'(s), \alpha'(s)\rangle = 1$, $\langle \beta, \beta\rangle = 1$ and $\langle \alpha'(s), \beta\rangle = 0$. Thus the Gauss map is $G = e_1 \times e_2$.

By a direct calculation we obtain $\widetilde{\nabla}_{e_1} e_1 = \widetilde{\nabla}_{e_1} e_2 = \widetilde{\nabla}_{e_2} e_1 = 0$ and $\widetilde{\nabla}_{e_2} e_2 = k(s)G$, where $k(s)$ is the curvature of $\alpha(s)$. All these imply that $\omega_2^1(e_1) = \omega_2^1(e_2) = 0$, $h_{11} = h_{12} = h_{21} = 0$, and $h_{22} = k(s)$. Therefore e_1 and e_2 are the principal vectors of the surface, $H = k(s)/2$ which is the function of s, and $\|A_G\|^2 = k^2(s)$. Hence (3.11) and (3.12) give, respectively, $C_1 = 0$ and $C_2 \neq 0$.

On the other hand it follows from (3.2)–(3.4) that C_1, C_2, and C_3 are functions of s, and equations (3.6) and (3.7) give, respectively

$$C_2'(s) - k(s)\,C_3(s) = 0 \tag{3.22}$$

and

$$C_3'(s) + k(s)\,C_2(s) = 0 \tag{3.23}$$

which yield $C_2^2(s) + C_3^2(s) = d_0^2$, where d_0 is a nonzero constant. We may put

$$C_2(s) = d_0 \sin\lambda(s), \quad C_3(s) = d_0 \cos\lambda(s) \tag{3.24}$$

which implies equations (3.22) and (3.23) if $\lambda(s) = k_0 + \int k(s)ds$, where k_0 is an integration constant.

If we make use of (3.10) and (3.12) together with (3.24) we obtain

$$\frac{k'(s)}{k^2(s)} = \frac{d_0 \sin\lambda(s)}{1 + d_0 \cos\lambda(s)}$$

from which by the integration we obtain

$$d_0 \cos \lambda(s) = \frac{q_0}{k(s)} - 1, \tag{3.25}$$

where q_0 is a nonzero constant. By the last two equations we have

$$d_0 \sin \lambda(s) = \frac{q_0 k'(s)}{k^3(s)}. \tag{3.26}$$

Now, by (3.25) and (3.26) the equation $C_2^2(s) + C_3^2(s) = d_0^2$ yields the differential equation

$$q_0^2 {k'}^2 = k^4[(d_0^2 - 1)k^2 + 2q_0 k - q_0^2]. \tag{3.27}$$

One can obtain the solution of the differential equation which defines k implicitly as a function s.

Also, from (3.10) and (3.25) we get $f = \frac{k^3}{q_0}$. Therefore, M has pointwise 1-type Gauss map of the second kind, that is, equation (1.1) holds for $f = \frac{k^3}{q_0}$ and for the constant vector $C = \frac{q_0 k'}{k^3} e_2 + (\frac{q_0}{k} - 1)G = (0, d_0, 0)$ if the curvature $k(s)$ satisfies (3.27).

It is well-known that given a differentiable function $k(s)$, a parametrized plane curve having $k(s)$ as curvature is determined uniquely, up to a rigid motion, by

$$\left(\int \cos \lambda(s) ds + d_1, \int \sin \lambda(s) ds + d_2 \right),$$

where $\lambda(s) = \int k(s)ds + k_0$ and s is the arc length parameter of the curve. Therefore, we can write the base curve $\alpha(s)$ of the cylinder by considering (3.25) and (3.26) as follows

$$\alpha(s) = \left(\frac{q_0}{d_0} \int \frac{ds}{k(s)} - \frac{s}{d_0} + d_1, \ \frac{q_0}{d_0} \int \frac{k'(s)}{k^3(s)} ds + d_2, 0 \right). \tag{3.28}$$

When we evaluate the integral in the second component of $\alpha(s)$ we have

$$\int \frac{k'(s)ds}{k^3(s)} = -\frac{1}{2k^2(s)}. \tag{3.29}$$

Also, using (3.27) we write the integral in the first component of $\alpha(s)$ as

$$\int \frac{ds}{k(s)} = \pm q_0 \int \frac{dk}{k^3 \sqrt{(d_0^2 - 1)k^2 + 2q_0 k - q_0^2}} \tag{3.30}$$

which can be evaluated in terms of elementary functions. Therefore, we have (3.8) from (3.21), (3.28), (3.29), and (3.30).

Case 3. M is an open part of a tangent developable surface. We will show that there is no tangent developable surface in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind. The surface M is locally parametrized by

$$x(s,t) = \alpha(s) + t\alpha'(s), \quad t \neq 0,$$

where $\alpha(s)$ is a unit speed curve with nonzero curvature $k(s)$ in $\mathbb{E}^3$. We assume that the torsion $\tau(s)$ of $\alpha(s)$ is nonzero. If $\tau = 0$, then the tangent surface is a part of a plane which has no pointwise 1-type Gauss map of the second kind.

Let T, N and B denote the unit tangent vector, principal normal vector and binormal vector of the curve α, respectively. The coordinate vector fields are $x_s = \alpha'(s) + t\alpha''(s) = T + tk(s)N$ and $x_t = \alpha'(s) = T$ which are not orthogonal. The parametrization x is regular if $tk \neq 0$. We take the orthonormal tangent frame $\{e_1, e_2\}$ on M such that $e_1 = \frac{\partial}{\partial t}$ and $e_2 = \frac{1}{tk(s)}\big(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\big)$. It is seen that $e_1 = T$ and $e_2 = N$. Then the Gauss map of M is given by $G = e_1 \times e_2 = T \times N = B$.

By a direct calculation we obtain

$$\widetilde{\nabla}_{e_1} e_1 = \widetilde{\nabla}_{e_1} e_2 = 0, \ \ \widetilde{\nabla}_{e_2} e_1 = \frac{1}{t} e_2, \ \ \widetilde{\nabla}_{e_2} e_2 = -\frac{1}{t} e_1 + \frac{\tau}{tk} G.$$

These relations imply that $\omega_2^1(e_1) = 0$, $\omega_1^2(e_2) = \frac{1}{t}$, $h_{11} = h_{12} = h_{21} = 0$ and $h_{22} = \frac{\tau}{tk}$. Therefore e_1, e_2 are principal vectors of the surface, $H = \frac{\tau}{2tk}$, and $\|A_G\|^2 = (\frac{\tau}{tk})^2$.

Now, it follows from (3.2)–(3.4) that C_1, C_2 and C_3 are functions of s, and thus equations (3.5)–(3.7) become

$$C_1'(s) - k(s)C_2(s) = 0, \tag{3.31}$$

$$C_2'(s) + k(s)C_1(s) - \tau(s)C_3(s) = 0, \tag{3.32}$$

$$C_3'(s) + \tau(s)C_2(s) = 0. \tag{3.33}$$

On the other hand, we have from (3.10), (3.11) and (3.12)

$$\frac{\tau^2}{t^2k^2} = f(1 + C_3), \tag{3.34}$$

$$-\frac{\tau}{t^2k} = f\, C_1, \tag{3.35}$$

$$\frac{1}{t^2k}\left(\frac{d}{ds}\left(\frac{\tau}{k}\right) + \frac{\tau}{tk}\right) = f\, C_2. \tag{3.36}$$

Equation (3.35) implies that $C_1 \neq 0$ as $\tau \neq 0$. So, by the equations (3.34) and (3.35) we obtain

$$\tau(s)C_1(s) + k(s)C_3(s) = -k(s) \tag{3.37}$$

from which by taking derivative with respect to s we get

$$\tau'(s)C_1(s) + \tau(s)C_1'(s) + k'(s)C_3(s) + k(s)C_3'(s) = -k'(s). \tag{3.38}$$

In view of (3.31) and (3.33), the equation (3.38) turns into

$$\tau'(s)C_1(s) + \ k'(s)C_3(s) = -k'(s). \tag{3.39}$$

If $\tau'(s)k(s) - k'(s)\tau(s) \neq 0$, then equations (3.37) and (3.39) give $C_1 = 0$ and $C_3 = -1$. Hence, we have $\tau = 0$ from (3.34) or (3.35) which is a contradiction.

Now suppose that $\tau'(s)k(s) - k'(s)\tau(s) = 0$ which means that $\frac{\tau}{k} = r_0$ is a constant. In this case, by (3.35) and (3.36) we get $tk(s)C_2(s) + C_1(s) = 0$ which implies that $C_1 = C_2 = 0$, that is, $\tau = 0$ by (3.35). This is a contradiction. Therefore the torsion τ is zero, and there is no tangent developable surface with pointwise 1-type Gauss map of the second kind.

The converse of the proof follows from a straightforward calculation. ∎

We then have the following:

Corollary 3.1. *A right circular cone in the Euclidean space* $\mathbb{E}^3$ *is the only cone with pointwise* 1*-type Gauss map of the second kind.*

Corollary 3.2. *There is no tangent developable surface in the Euclidean space* $\mathbb{E}^3$ *with pointwise* 1*-type Gauss map of the second kind.*

Example 3.1. Let $d_0 = q_0 = 1$ and $d_1 = d_2 = 0$ in (3.8). Solving the differential equation (3.27) we obtain

$$s = \int \frac{dk}{k^2\sqrt{2k-1}} = \frac{\sqrt{2k-1}}{k} + 2\arctan\sqrt{2k-1} + k_1, \tag{3.40}$$

where k_1 is an integration constant. Now, if we evaluate the integral defining the function $\mu(s)$ in the second part of Theorem 3.1, then we obtain

$$\mu(s) = \frac{(1+3k)\sqrt{2k-1}}{2k^2} + 3\arctan\sqrt{2k-1} + k_2. \tag{3.41}$$

By taking the integration constants k_1 and k_2 zero and using (3.40) we get

$$\mu(s) = \frac{3s}{2} + \frac{\sqrt{2k-1}}{2k^2}.$$

Therefore the cylinder with pointwise 1-type Gauss map of the second kind with (3.8) is in this case parametrized by

$$x(s,t) = \left(\frac{s}{2} + \frac{\sqrt{2k(s)-1}}{2k^2(s)}, -\frac{1}{2k^2(s)}, t\right), \quad k > 1/2, \tag{3.42}$$

where k satisfies $\sqrt{2k-1} + 2k\arctan\sqrt{2k-1} - sk = 0$. Using this equation we can parametrize (3.42) in terms of k and t.

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