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Normality Criterion Concerning Sharing Functions II

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Abstract. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, and let $\varphi(z)(\not\equiv 0,\infty)$ be a meromorphic function in D such that f and $\varphi(z)$ have no common zeros for all $f\in\mathcal{F}$ and $\varphi(z)$ has no simple zeros in D, and all poles of $\varphi(z)$ have multiplicity at most k. If, for each $f\in\mathcal{F}$, all zeros of f have multiplicity at least $k+1$, $f^{(k)}(z)=0\Rightarrow f(z)=0, f^{(k)}(z)=\varphi(z)\Rightarrow f(z)=\varphi(z)$, then $\mathcal{F}$ is normal in D. This result improves and extends related results due to Schwick, Fang, Fang-Zalcman and Xu, *et al.*

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1. Introduction

Let f, g be two meromorphic functions in a domain D, and let a be a complex number. If $g(z)=a$ whenever $f(z)=a$, we denote it by $f=a\Rightarrow g=a$. $f=a\Leftrightarrow g=a$ means $f(z)=a$ if and only if $g(z)=a$, and we say that f and g share a.

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for any sequence $\{f_n\}\in\mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ converges spherically locally uniformly on D, to a meromorphic function or ∞ (see [5, 8, 14].

Schwick [9] discovered a connection between normality criteria and shared values. He proved:

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2, a_3 be distinct complex numbers. If, for each $f\in\mathcal{F}$, $f(z)=a_i\Leftrightarrow f'(z)=a_i (i=1,2,3)$, then $\mathcal{F}$ is normal in D.*

This result has undergone various extensions. The following result is due to Fang and Zalcman [4].

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Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, and let $a \neq 0$ be complex number. If, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, $f^{(k)}(z) = 0 \Rightarrow f(z) = 0, f^{(k)}(z) = a \Rightarrow f(z) = a$, then $\mathcal{F}$ is normal in D.*

In [3], Fang proved that Theorem 1.2 is still valid if a is replaced by a non-vanishing analytic function $\psi(z)$ for $k = 1$, as follows:

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let $\psi(z)$ be a non-vanishing analytic function in D, If, for each $f \in \mathcal{F}$, $f(z) = 0 \Leftrightarrow f'(z) = 0, f'(z) = \psi(z) \Rightarrow f(z) = \psi(z)$, then $\mathcal{F}$ is normal in D.*

Recently, Xu [12] proved the following result, which extends Theorems 1.2 and 1.3, and improves Theorem 1.1.

Theorem 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, and let $\varphi(z)(\not\equiv 0)$ be an analytic function in D such that f and $\varphi(z)$ have no common zeros for all $f \in \mathcal{F}$ and $\varphi(z)$ has no simple zeros in D. If, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, $f^{(k)}(z) = 0 \Rightarrow f(z) = 0, f^{(k)}(z) = \varphi(z) \Rightarrow f(z) = \varphi(z)$, then $\mathcal{F}$ is normal in D.*

Remark 1.1. There is an example in [12] that shows the hypothesis "$f \in \mathcal{F}$ and $\varphi(z)$ have no common zeros in D" is necessary in Theorem 1.4.

A natural problem arises: *What can we say if the analytic function $\varphi(z)(\not\equiv 0)$ in Theorem 1.4 is replaced by a meromorphic function $\varphi(z)(\not\equiv 0, \infty)$?*

In this paper, we first prove the following result.

Theorem 1.5. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, and let $\varphi(z)(\not\equiv \infty)$ be a non-vanishing meromorphic function in D such that all poles of $\varphi(z)$ have multiplicity at most k. If, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, $f^{(k)}(z) = 0 \Rightarrow f(z) = 0, f^{(k)}(z) = \varphi(z) \Rightarrow f(z) = \varphi(z)$, then $\mathcal{F}$ is normal in D.*

Since normality is a local property, combining the above theorem and Theorem 1.4, we can obtain the following theorem, which improves and generalizes Theorems 1.1–1.4.

Theorem 1.6. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, and let $\varphi(z)(\not\equiv 0, \infty)$ be a meromorphic function in D such that f and $\varphi(z)$ have no common zeros for all $f \in \mathcal{F}$ and $\varphi(z)$ has no simple zeros in D, and all poles of $\varphi(z)$ have multiplicity at most k. If, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, $f^{(k)}(z) = 0 \Rightarrow f(z) = 0, f^{(k)}(z) = \varphi(z) \Rightarrow f(z) = \varphi(z)$, then $\mathcal{F}$ is normal in D.*

Remark 1.2. The restriction on the poles of $\varphi(z)$ in Theorems 1.5 and 1.6 can not be omitted, which is shown by the following example.

Example 1.1. [2] Let $k \in \mathbb{N}$, $D = \{z : |z| < 1\}$, $\varphi(z) = \frac{1}{z^{k+1}}$, and

$$\mathcal{F} = \left\{ f_n(z) = \frac{1}{nz}, z \in D, n = 1, 2, \cdots \right\}.$$

Since $f_n(z)$ and $f_n^{(k)}(z)$ have no zeros, $f_n^{(k)}(z) = 0 \Rightarrow f_n(z) = 0$. Obviously, there exists $n_0 \in \mathbb{N}$ such that $f_n^{(k)}(z) - \varphi(z) \neq 0$ for $n \geq n_0$, hence $f_n^{(k)}(z) = \varphi(z) \Rightarrow f_n(z) = \varphi(z)$. But $\mathcal{F}$ is not normal in D.

Remark 1.3. We conjecture that Theorem 1.5 and 1.6 still hold if we replace "all poles of $\varphi(z)$ have multiplicity at most k" by "for all $f \in \mathcal{F}$, f and φ have no common poles in D".

2. Some lemmas

To prove our results, we need the following lemmas.

Lemma 2.1. [3] *Let f be a meromorphic function of finite order in the plane $\mathbb{C}$. If $f(z) = 0 \Leftrightarrow f'(z) = 0, f'(z) \neq 1$, then f is a constant.*

Lemma 2.2. [4] *Let f be a meromorphic function of finite order in the plane $\mathbb{C}$ and $k \geq 2$ be a positive integer. If all zeros of f have multiplicity at least $k+1$, $f^{(k)}(z) = 0 \Rightarrow f(z) = 0, f^{(k)}(z) \neq 1$, then f is a constant.*

Lemma 2.3. [11] *Let f be a transcendental meromorphic function, let $R(z)(\not\equiv 0)$ be a rational function, and k be a positive integer. If all zeros of f have multiplicity at least $k+1$, except for finitely many, and $f^{(k)}(z) = 0 \Rightarrow f(z) = 0$, then $f^{(k)}(z) - R(z)$ has infinitely many zeros.*

Lemma 2.4. [13] *Let k, l be positive integers, and let $Q(z)$ be a rational function all of whose zeros are of order at least k. If $Q^{(k)}(z) \neq z^{-l}$, then $Q(z)$ is constant.*

The well-known Zalcman's lemma is a very important tool in the study of normal families. It has also undergone various extensions and improvements. The following is one up-to-date local version, which is due to Pang and Zalcman [7] (cf. [1, 2, 10, 15]).

Lemma 2.5. *Let k be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, such that each function $f \in \mathcal{F}$ has only zeros of order at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathcal{F}$ is not normal at $z_0 \in D$, thus, for each $0 \leq \alpha \leq k$, there exist*

(a) *a sequence of points $z_n \in D$, $z_n \to z_0$;*
(b) *a sequence of positive numbers $\rho_n \to 0$;*
(c) *a sequence of functions $f_n \in \mathcal{F}$,*

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity $\geq k$, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. Moreover, g has order at most 2.

Lemma 2.6. *Let k be a positive integer and $\mathcal{F} = \{f_n\}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+1$, and let $\varphi_n(z)$ be a sequence of holomorphic functions in D such that $\varphi_n(z) \to \varphi(z)(\neq 0)$ locally uniformly in D. If there exist a sequence of points $a_n \to 0$, such that $f_n^{(k)}(z) = 0 \Rightarrow f_n(z) = 0, f_n^{(k)}(z) = \varphi_n(z) \Rightarrow f_n(z) = a_n^{-k}\varphi_n(z)$, then $\mathcal{F}$ is normal in D.*

Proof. Suppose $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 2.5, there exist a sequence of complex numbers $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a subsequence of $\mathcal{F}$, which we continue to denote by $\{f_n\}$, such that

$$F_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \to F(\zeta)$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric, where F is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity $\geq k+1$. We claim

(1) $F^{(k)}(\zeta) = 0 \Rightarrow F(\zeta) = 0$, and

(2) $F^{(k)}(\zeta) \neq \varphi(z_0)$.

Suppose that $F^{(k)}(\zeta_0) = 0$, since all zeros of $F(\zeta)$ have multiplicity at least $k+1$, we have $F^{(k)}(\zeta) \not\equiv 0$. Then there exist $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$F_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n\zeta_n) = 0.$$

Since $f_n^{(k)}(z) = 0 \Rightarrow f_n(z) = 0$, thus $f_n(z_n + \rho_n\zeta_n) = 0$, and then

$$F(\zeta_0) = \lim_{n\to\infty} F_n(\zeta_n) = \lim_{n\to\infty} \frac{f_n(z_n + \rho_n\zeta_n)}{\rho_n^k} = 0.$$

This proves (1).

Next we prove (2). Suppose $F^{(k)}(\zeta_0) = \varphi(z_0)(\neq 0, \infty)$, clearly, $F^{(k)}(\zeta) \not\equiv \varphi(z_0)$ since all zeros of $F(\zeta)$ have multiplicity at least $k+1$. Noting that

$$F_n^{(k)}(\zeta) - \varphi_n(z_n + \rho_n\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) - \varphi_n(z_n + \rho_n\zeta) \to F^{(k)}(\zeta_0) - \varphi(z_0).$$

By Hurwitz's theorem, there exist $\zeta_n \to \zeta_0$ such that (for n sufficiently large) $f_n^{(k)}(z_n + \rho_n\zeta_n) = \varphi_n(z_n + \rho_n\zeta_n)$, and thus $f_n(z_n + \rho_n\zeta_n) = a_n^{-k}\varphi_n(z_n + \rho_n\zeta_n)$. Hence

$$F(\zeta_0) = \lim_{n\to\infty} F_n(\zeta_n) = \lim_{n\to\infty} \frac{\varphi_n(z_n + \rho_n\zeta_n)}{\rho_n^k a_n^k} = \infty.$$

This contradicts that $F^{(k)}(\zeta_0) = \varphi(z_0) \neq \infty$. This proves (2).

Hence, by Lemma 2.1 and 2.2, $F(\zeta)$ must be a constant, a contradiction. Lemma 2.6 is proved. ∎

Using the argument as the proof of Lemma 2.6, we can obtain the following lemma.

Lemma 2.7. *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+1$, and let $\varphi(z)(\neq 0)$ be a holomorphic function in D. If, for each function $f \in \mathcal{F}$, $f^{(k)}(z) = 0 \Rightarrow f(z) = 0$ and $f^{(k)}(z) = \varphi(z) \Rightarrow f(z) = \varphi(z)$, then $\mathcal{F}$ is normal in D.*

3. Proof of Theorem 1.5

Since normality is a local property, by Lemma 2.7, we only need to prove that $\mathcal{F}$ is normal at every pole of $\varphi(z)$. Without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$, and

$$\varphi(z) = \frac{1}{z^l} + \frac{a_{-l+1}}{z^{l-1}} + \cdots = \frac{\phi(z)}{z^l} \quad (z \in \Delta),$$

where $l \leq k$ is a positive integer , $\phi(0) = 1, \phi(z) \neq 0, \infty$ for $0 < |z| < 1$. So it is enough to show that $\mathcal{F}$ is normal at $z = 0$.

Suppose that $\mathcal{F}$ is not normal at $z = 0$. By Lemma 2.5 ($\alpha = k - l$), there exist a sequence of complex numbers $z_n \to 0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$, such that

$$F_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^{k-l}} \to F(\zeta)$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric, where F is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity $\geq k+1$. Using almost the same argument as in the proof of Lemma 2.6, we deduce that $F^{(k)}(\zeta) = 0 \Rightarrow F(\zeta) = 0$.

We distinguish two cases.

Case 1. $z_n/\rho_n \to \infty$.

Consider

$$\psi_n(\zeta) = z_n^{l-k} f_n(z_n + z_n\zeta) = z_n^{l-k} f_n(z_n(1+\zeta)).$$

By the assumptions of f_n, we see that all zeros of $\psi_n(\zeta)$ have multiplicity at least $k+1$, and $\psi_n^{(k)}(\zeta) = 0 \Rightarrow \psi_n(\zeta) = 0$.

Next we prove

$$\psi_n^{(k)}(\zeta) = \frac{\phi(z_n(1+\zeta))}{(1+\zeta)^l} \Rightarrow \psi_n(\zeta) = z_n^{-k}\frac{\phi(z_n(1+\zeta))}{(1+\zeta)^l}.$$

Indeed, if $\psi_n^{(k)}(\zeta) = z_n^l f_n^{(k)}(z_n(1+\zeta)) = \phi(z_n(1+\zeta))/(1+\zeta)^l$, then

$$f_n^{(k)}(z_n(1+\zeta)) = \frac{\phi(z_n(1+\zeta))}{z_n^l(1+\zeta)^l} = \varphi(z_n(1+\zeta)).$$

Since $f_n^{(k)}(z) = \varphi(z) \Rightarrow f_n(z) = \varphi(z)$, we have

$$f_n(z_n(1+\zeta)) = \varphi(z_n(1+\zeta)) = \frac{\phi(z_n(1+\zeta))}{z_n^l(1+\zeta)^l}.$$

Thus

$$\psi_n(\zeta) = z_n^{l-k}\frac{\phi(z_n(1+\zeta))}{z_n^l(1+\zeta)^l} = z_n^{-k}\frac{\phi(z_n(1+\zeta))}{(1+\zeta)^l}.$$

Obviously, for each n, $\phi(z_n(1+\zeta)/(1+\zeta)^l$ is holomorphic on Δ. Noting that $z_n \to 0$ and $\phi(z_n(1+\zeta)/(1+\zeta)^l \to 1/(1+\zeta)^l (\neq 0)$ on Δ. Then, by Lemma 2.6, the family $\{\psi_n(\zeta)\}$ is normal on Δ.

Now we can find a subsequence $\{\psi_{n_j}(\zeta)\}$ and a function $\psi(z)$ such that

$$\psi_{n_j}(\zeta) = z_{n_j}^{l-k} f_{n_j}(z_{n_j}(1+\zeta)) \to \psi(\zeta).$$

If $\psi(0) \neq \infty$, then

$$\begin{aligned}F^{(k-l)}(\zeta) &= \lim_{j\to\infty} f_{n_j}^{(k-l)}(z_{n_j} + \rho_{n_j}\zeta) = \lim_{j\to\infty} f_{n_j}^{(k-l)}(z_{n_j} + z_{n_j}(\frac{\rho_{n_j}}{z_{n_j}}\zeta)) \\ &= \lim_{j\to\infty} \psi_{n_j}^{(k-l)}(\frac{\rho_{n_j}}{z_{n_j}}\zeta) = \psi^{(k-l)}(0).\end{aligned}$$

This implies that $F^{(k-l)}(\zeta)$ is a constant, and then $F^{(k)}(\zeta) \equiv 0$. It follows that $F(\zeta) = a_{k-1}\zeta^{k-1} + \cdots + a_1\zeta + a_0$. We arrive at a contradiction since $F(\zeta)$ is nonconstant and all zeros of $F(\zeta)$ have multiplicity $\geq k+1$.

If $\psi(0) = \infty$, then

$$\psi_{n_j}\left(\frac{\rho_{n_j}}{z_{n_j}}\zeta\right) = z_{n_j}^{l-k} f_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \to \psi(0) = \infty,$$

and hence

$$\begin{aligned} F(\zeta) &= \lim_{j\to\infty} \frac{f_{n_j}(z_{n_j} + \rho_{n_j}\zeta)}{\rho_{n_j}^{k-l}} \\ &= \lim_{j\to\infty} \left(\frac{z_{n_j}}{\rho_{n_j}}\right)^{k-l} z_{n_j}^{l-k} f_{n_j}(z_{n_j} + \rho_{n_j}\zeta) = \infty. \end{aligned}$$

We arrive at a contradiction since F is a nonconstant meromorphic function.

Case 2. $z_n/\rho_n \not\to \infty$. Taking a subsequence and renumbering, we may assume that $z_n/\rho_n \to \alpha$, a finite complex number. Then

$$F_n^{(k)}(\zeta) - \frac{\rho_n^l \phi(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^l} \to F^{(k)}(\zeta) - \frac{1}{(\alpha+\zeta)^l}$$

on $\mathbb{C}\backslash\{-\alpha\}$.

We first prove that $F^{(k)}(\zeta) - 1/(\alpha+\zeta)^l \neq 0$ on $C\backslash\{-\alpha\}$. Suppose that there exists $\zeta_0 \in C\backslash\{-\alpha\}$ such that $F^{(k)}(\zeta_0) - 1/(\alpha+\zeta_0)^l = 0$. Since all poles of $F^{(k)}(\zeta)$ have multiplicity $\geq k+1 > l$, $F^{(k)}(\zeta) - 1/(\alpha+\zeta)^l \not\equiv 0$. Then, by Hurwitz's theorem, there exist $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$F_n^{(k)}(\zeta_n) - \frac{\rho_n^l \phi(z_n + \rho_n\zeta_n)}{(z_n + \rho_n\zeta_n)^l} = 0.$$

Since

$$F_n^{(k)}(\zeta) - \frac{\rho_n^l \phi(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^l} = \rho_n^l \left(f_n^{(k)}(z_n + \rho_n\zeta) - \frac{\phi(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^l}\right),$$

we have

$$f_n^{(k)}(z_n + \rho_n\zeta_n) - \frac{\phi(z_n + \rho_n\zeta_n)}{(z_n + \rho_n\zeta_n)^l} = 0,$$

and thus

$$f_n(z_n + \rho_n\zeta_n) - \frac{\phi(z_n + \rho_n\zeta_n)}{(z_n + \rho_n\zeta_n)^l} = 0.$$

Hence

$$F(\zeta_0) = \lim_{n\to\infty} F_n(\zeta_n) = \lim_{n\to\infty} \frac{\phi(z_n + \rho_n\zeta_n)}{\rho_n^k(\frac{z_n}{\rho_n} + \zeta_n)^l} = \infty.$$

But this contradicts the fact that $F^{(k)}(\zeta_0) = 1/(\alpha+\zeta_0)^l$.

Now we prove that $F^{(k)}(\zeta) \neq 1/(\alpha+\zeta)^l$. To do this, we need to prove that $F(-\alpha) \neq \infty$. For simplicity, we assume that $\alpha = 0$. Suppose that $\zeta = 0$ is a pole of $F(\zeta)$, thus $F^{(k)}(\zeta)$ has a pole of order at least $k+1 \geq l+1$ at $\zeta = 0$. Thus

$F^{(k)}(\zeta) - 1/\zeta^l \neq 0$ on $\mathbb{C}$. Then Lemma 2.3 implies that $F^{(k)}(\zeta) - 1/\zeta^l$ is a rational function. Furthermore, we have

$$F^{(k)}(\zeta) - \frac{1}{\zeta^l} = \frac{1}{p(\zeta)},$$

where $p(\zeta)$ is a polynomial with a zero of order at least $l+1$ at $\zeta = 0$. By using the Laurent expansion of $F^{(k)}(\zeta)$ around $\zeta = \infty$, we have

$$F^{(k)}(\zeta) = \frac{1}{\zeta^l} + O\left(\frac{1}{\zeta^{l+1}}\right), \quad \zeta \to \infty.$$

Repeated integrations give

$$F^{(k-l+1)}(\zeta) = \frac{(-1)^{l-1}}{(l-1)!\zeta} + q(\zeta) + O\left(\frac{1}{\zeta^2}\right), \quad \zeta \to \infty,$$

where $q(\zeta)$ is a polynomial of degree $\leq l-2$. The residue theorem yields

$$\frac{1}{2\pi i}\int_{|\zeta|=R} F^{(k-l+1)}(\zeta)d\zeta = \frac{(-1)^{l-1}}{(l-1)!}$$

for $R > 0$ large enough. On the other hand, $F^{(k-l+1)}(\zeta)$ has the primitive function $F^{(k-l)}(\zeta)$, and thus its integral on closed paths must vanish, which is a contradiction.

Therefore, by Lemma 2.3 and 2.4, we deduce that $F(\zeta)$ is a constant, a contradiction. This finally completes the proof of Theorem 1.5. ∎

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References

[1] H. Chen, Yosida functions and Picard values of integral functions and their derivatives, *Bull. Austral. Math. Soc.* **54** (1996), no. 3, 373–381.
[2] H. H. Chen and Y. X. Gu, Improvement of Marty's criterion and its application, *Sci. China Ser. A* **36** (1993), no. 6, 674–681.
[3] M. Fang, Picard values and normality criterion, *Bull. Korean Math. Soc.* **38** (2001), no. 2, 379–387.
[4] M. Fang and L. Zalcman, Normal families and shared values of meromorphic functions, *Ann. Polon. Math.* **80** (2003), 133–141.
[5] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
[6] C. Meng, Normal families and shared values of meromorphic functions, *Bull. Malays. Math. Sci. Soc. (2)* **31** (2008), no. 1, 85–90.
[7] X. Pang and L. Zalcman, Normal families and shared values, *Bull. London Math. Soc.* **32** (2000), no. 3, 325–331.
[8] J. L. Schiff, *Normal Families*, Springer, New York, 1993.
[9] W. Schwick, Sharing values and normality, *Arch. Math. (Basel)* **59** (1992), no. 1, 50–54.
[10] Y. Wang and M. Fang, Picard values and normal families of meromorphic functions with multiple zeros, *Acta Math. Sinica (N.S.)* **14** (1998), no. 1, 17–26.
[11] Y. Xu, On the value distribution of derivatives of meromorphic functions, *Appl. Math. Lett.* **18** (2005), no. 5, 597–602.
[12] Y. Xu, Normality criterion concerning sharing functions, *Houston J. Math.* **32** (2006), no. 3, 945–954 (electronic).
[13] Y. Xu, Normal families and exceptional functions, *J. Math. Anal. Appl.* **329** (2007), no. 2, 1343–1354.

[14] L. Yang, *Value Distribution Theory*, Translated and revised from the 1982 Chinese original, Springer, Berlin, 1993.
[15] L. Zalcman, Normal families: new perspectives, *Bull. Amer. Math. Soc.* (*N.S.*) **35** (1998), no. 3, 215–230.
[16] R. Zhu and Y. Xu, On the normal meromorphic functions, *Bull. Malays. Math. Sci. Soc.* (*2*) **30** (2007), no. 2, 129–133.