BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33**(3) (2010), 487–498

The Partial-Isometric Crossed Products of $\mathbf{c}_0$ by the Forward and the Backward Shifts

[1]SRIWULAN ADJI AND [2]ABBAS HOSSEINI
[1,2]School of Mathematical Sciences, Universiti Sains Malaysia,
11800, USM Penang, Malaysia
[1]wulan@cs.usm.my, [2]hosseini.md08@student.usm.my

Abstract. Let (A, α) be a system consisting of a C^*-algebra A and an extendible endomorphism α on A. We consider the partial-isometric crossed product $A \times_\alpha \mathbb{N}$ generated by a copy of A and a power partial isometry. We show that for an extendible α-invariant ideal I in A, the quotient $(A\times_\alpha \mathbb{N})/(I\times_\alpha \mathbb{N})$ of partial-isometric crossed products is isomorphic to the partial-isometric crossed product $A/I \times_{\tilde{\alpha}} \mathbb{N}$ of the quotient algebra. Then we use this to give concrete descriptions of the partial-isometric crossed products of $\mathbf{c}_0$ by the forward shift and the backward shift.

2010 Mathematics Subject Classification: Primary 46L55; Secondary 06F15, 47B35

Key words and phrases: C^*-algebra, endomorphism, covariant representation, crossed product.

1. Introduction

Let A be a C^*-algebra (not necessarily unital). We refer to [1, 2, 8] to call an endomorphism α on A *extendible* if it extends uniquely to a strictly continuous endomorphism $\overline{\alpha}$ on the multiplier algebra $M(A)$. This happens precisely when there is an approximate identity (a_λ) for A such that $\alpha(a_\lambda)$ converges strictly to a projection in $M(A)$. Thus any endomorphism on a unital C^*-algebra is trivially extendible.

Suppose we have an extendible endomorphism α on A. For any ideal I of A, there always exists a canonical nondegenerate homomorphism $\psi : A \to M(I)$ which satisfies $\psi(a)i = ai$ for $a \in A$ and $i \in I$. Denote by $\overline{\psi}$ the extension of ψ. Then we again refer to [1, 2, 8] to call an ideal I of A *extendible* if I is an α-invariant ideal, i.e. $\alpha(I) \subseteq I$, and it contains an approximate identity (i_λ) such that $\alpha(i_\lambda)$ converges strictly to $\overline{\psi}(\overline{\alpha}(1))$ in $M(I)$. This is equivalent to $\alpha|_I$ extending to a strictly continuous endomorphism $\overline{\alpha|_I}$ on $M(I)$ such that $\overline{\alpha|_I}(1) = \overline{\psi}(\overline{\alpha}(1))$.

Communicated by Rosihan M. Ali, Dato'.
Received: November 9, 2009; *Revised:* May 26, 2010.

Lindiarni and Raeburn in [9] studied the partial-isometric crossed products of semigroup dynamical systems (A, Γ^+, α) where Γ^+ is the positive cone in a totally ordered abelian group Γ and α is an action of Γ^+ by extendible endomorphisms of A. They describe the structure of the crossed product associated to the system $(B_{\Gamma^+}, \Gamma^+, \tau)$ arising in [3] for the analysis of Toeplitz algebras. The crossed product $B_{\Gamma^+} \times_\tau \Gamma^+$ is universal for partial-isometric representation of Γ^+. They show that there is a large commutative diagram of six exact sequences for $B_{\Gamma^+} \times_\tau \Gamma^+$ induced by the ideals I and J such that: the quotients $(B_{\Gamma^+} \times_\tau \Gamma^+)/I$ and $(B_{\Gamma^+} \times_\tau \Gamma^+)/J$ are the Toeplitz algebra $\mathcal{T}(\Gamma)$, the other two quotients $I/(I \cap J)$ and $J/(I \cap J)$ are the commutator ideal $\mathcal{C}_\Gamma$ of $\mathcal{T}(\Gamma)$, and the ideal $I \cap J$ is described by the kernels of homomorphisms of $B_{\Gamma^+} \times_\tau \Gamma^+$ onto the crossed product $B_I \times_\tau \Gamma^+$ of B_I associated to intervals I in Γ^+. When Γ^+ is the additive semigroup $\mathbb{N}$, $B_\mathbb{N}$ is the C^*-algebra $\mathbf{c}$ of convergent sequences, τ is the forward shift on $\mathbf{c}$. The structure theory of $\mathbf{c} \times_\tau \mathbb{N}$ and the crossed product $\mathbf{c} \times_\sigma \mathbb{N}$ by backward shift are described in [9]. The crossed product $\mathbf{c} \times_\tau \mathbb{N}$ is the universal C^*-algebra generated by a power partial isometry. The large commutative diagram for $\mathbf{c} \times_\tau \mathbb{N}$ consists of familiar exact sequences, which includes the exact sequence $0 \longrightarrow \mathcal{A} \longrightarrow \mathbf{c} \times_\tau \mathbb{N} \longrightarrow \mathcal{T}(\mathbb{Z}) \longrightarrow 0$, where $\mathcal{A}$ is the subalgebra $C(\mathbb{N} \cup \{\infty\}, \mathcal{K}(\ell^2(\mathbb{N})))$.

Here we consider systems (A, α) over the additive semigroup $\mathbb{N}$, and our goal is to show that $\mathcal{A}$ is the crossed product $\mathbf{c}_0 \times_\tau \mathbb{N}$ and $C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$ is the crossed product $\mathbf{c}_0 \times_\sigma \mathbb{N}$ by the backward shift σ. If I is an extendible α-invariant ideal of A, then the endomorphism $\tilde{\alpha} : a + I \mapsto \alpha(a) + I$ on the quotient algebra A/I is always extendible, so we can also talk about the system $(A/I, \tilde{\alpha})$. We prove in §2 a version of [1, Theorem 3.1] and [8, Theorem 1.7] for partial-isometric crossed products: the crossed product of the quotient algebra $A/I \times_{\tilde{\alpha}} \mathbb{N}$ is the quotient $(A \times_\alpha \mathbb{N})/(I \times_\alpha \mathbb{N})$. In §3, we apply this theorem to the systems $(\mathbf{c}, \tau)$ and $(\mathbf{c}, \sigma)$ and to the ideal $\mathbf{c}_0$. We show that the ideal $\mathbf{c}_0 \times_\tau \mathbb{N}$ of $\mathbf{c} \times_\tau \mathbb{N}$ is $\ker \varphi_{T^*}$ where $\varphi_{T^*} : \mathbf{c} \times_\tau \mathbb{N} \to \mathcal{T}(\mathbb{Z})$ induced by the Toeplitz representation. For the backward shift we show that the isomorphism $\pi_{F,Q} \times F$ appearing in the structure of $\mathbf{c} \times_\sigma \mathbb{N}$, carries $\mathbf{c}_0 \times_\sigma \mathbb{N}$ onto $C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$.

2. The partial-isometric crossed products

Let (A, α) be a system consisting of a C^*-algebra A which may not have an identity element, and an extendible endomorphism α on A. We take from [9] the definitions of a covariant representation and a partial-isometric crossed product of the system.

But first we shall remind our readers for an operator V on a Hilbert space H to be said a *partial isometry* if $\|Vh\| = \|h\|$ for all $h \in (\ker V)^\perp$, and this is equivalent to $VV^*V = V$. If V is a partial isometry on H then so is V^*, and then the operators VV^* and V^*V are orthogonal projections on the initial space $(\ker V)^\perp$ and the range VH respectively. Accordingly, an element v of a C^*-algebra A is called a partial isometry if $vv^*v = v$. The product of two partial isometries is not in general a partial isometry, and therefore the n product v^n of v is not always a partial isometry for a partial isometry v. A partial isometry v is called a *power partial isometry* if v^n is a partial isometry for every $n \in \mathbb{N}$.

A *partial isometric representation* of $\mathbb{N}$ on a Hilbert space H is a map $V : \mathbb{N} \to B(H)$ such that V_n is a partial isometry and $V_n V_k = V_{n+k}$ for every $n, k \in \mathbb{N}$. Since

$V_n = (V_1)^n$, a partial-isometric representation V of $\mathbb{N}$ is always determined by the single partial isometry V_1. Conversely a single partial isometry W generates a partial isometric representation $V : n \mapsto W^n$ if and only if W is a power partial isometry.

Definition 2.1. *A covariant representation of (A, α) on a Hilbert space H (or in a C^*-algebra B) is a pair (π, V) of a nondegenerate representation π of A on a Hilbert space H and a partial isometric representation V of $\mathbb{N}$ on H such that*

$$\pi(\alpha_m(a)) = V_m \pi(a) V_m^* \quad \text{and} \quad V_m^* V_m \pi(a) = \pi(a) V_m^* V_m \tag{2.1}$$

for all $m \in \mathbb{N}$, $a \in A$.

Every covariant representation (π, V) of (A, α), by Lemma 4.2 of [9], extends to a covariant representation $(\bar{\pi}, V)$ of $(M(A), \overline{\alpha})$, and that the covariance relation in (2.1) is, by Lemma 4.3 of [9], equivalent to

$$\pi(\alpha_m(a))V_m = V_m \pi(a) \text{ and } V_m V_m^* = \overline{\pi}(\overline{\alpha}_m(1)) \text{ for all } a \in A \text{ and } m \in \mathbb{N}.$$

It is shown in [9, Example 4.6] that every system (A, α) admits covariant representations (π, V) with π faithful.

Definition 2.2. *Given a system (A, α), a C^*-algebra B is called a (partial-isometric) crossed product of (A, α), if there exist a nondegenerate homomorphism $i_A : A \to B$ and a homomorphism $i_{\mathbb{N}} : \mathbb{N} \to M(B)$ such that*

(i) *$(i_A, i_{\mathbb{N}})$ is covariant;*
(ii) *for every covariant representation (π, V) of (A, α) on H, there is a nondegenerate representation $\pi \times V$ of B on H such that $(\pi \times V) \circ i_A = \pi$ and $\overline{(\pi \times V)} \circ i_{\mathbb{N}} = V$;*
(iii) *B is generated by $i_A(A)$ and $i_{\mathbb{N}}(\mathbb{N})$.*

If $(j_A, j_{\mathbb{N}})$ is a pair of such homomorphisms for (A, α) in a C^*-algebra C that satisfies (i), (ii) *and* (iii), *then there is an isomorphism of C onto B that takes $(j_A, j_{\mathbb{N}})$ into $(i_A, i_{\mathbb{N}})$.*

Remark 2.1. The crossed product of (A, α) is by [5, Proposition 3.4] the Toeplitz algebra of a Hilbert bimodule. We recall from [7] the definition of this algebra. A Hilbert bimodule over a C^*-algebra A is a right Hilbert A-module X together with a homomorphism $\phi : A \to \mathcal{L}(X)$ that gives a left action $a \cdot x := \phi(a)x$ of A on X. A *Toeplitz representation* of X in a C^*-algebra B is a pair (ψ, π) for which $\psi : X \to B$ is a linear map and $\pi : A \to B$ is a homomorphism that satisfy:

$$\psi(x \cdot a) = \psi(x)\pi(a), \quad \psi(a \cdot x) = \pi(a)\psi(x) \quad \text{and} \quad \psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A)$$

for all $x \in X$ and $a \in A$. The *Toeplitz algebra $\mathcal{T}_X$* of X is the C^*-algebra generated by the range of the universal Toeplitz representation (i_X, i_A) of X, so that whenever (ψ, π) is a Toeplitz representation of X in B, there is a homomorphism $\psi \times \pi : \mathcal{T}_X \to B$ which maps (i_X, i_A) into (ψ, π). For every Hilbert bimodule X, the Toeplitz algebra $\mathcal{T}_X$ always exists and it is unique up to isomorphism.

Given a system (A, α), [5, Proposition 3.4] says there is a partial isometric representation $(i_A, i_{\mathbb{N}})$ of (A, α) in the Toeplitz algebra $\mathcal{T}_X$ of the Hilbert bimodule $X = \overline{\alpha}(1)A$, such that i_A is injective. Let (k_X, k_A) be the universal representation of X in $\mathcal{T}_X$, then for an approximate identity (a_i) in A, $(k_X(\alpha(a_i)))^m$ converges strictly for every $m \in \mathbb{N}$ in $M(\mathcal{T}_X)$ [5, Lemma 3.3], and then the pair $(i_A, i_{\mathbb{N}})$ is defined by

$i_A(a) = k_A(a)$ and $i_{\mathbb{N}}(m) = \lim_{i\to\infty}[(k_X(\alpha(a_i)))^m]^*$. The C^*-algebra $\mathcal{T}_X$ together with this $(i_A, i_{\mathbb{N}})$ is a crossed product for (A,α). Thus $\mathcal{T}_X$ is the partial-isometric crossed product of (A,α), and we use the standard notation $A \times_\alpha \mathbb{N}$ to denote the crossed product of (A,α). Throughout we often use the fact that $(A\times_\alpha \mathbb{N}, i_A, i_{\mathbb{N}})$ is spanned by $\{i_{\mathbb{N}}(s)^* i_A(a) i_{\mathbb{N}}(t) : s,t \in \mathbb{N}, a \in A\}$.

We state the theorem in [9] for faithful representations of $A \times_\alpha \mathbb{N}$.

Theorem 2.1. [9, Theorem 4.8] *A covariant representation (π, V) of (A,α) on H gives a faithful representation $\pi \times V$ of $A \times_\alpha \mathbb{N}$ if and only if π acts faithfully on $(V_n^* H)^\perp$ for every $n > 0$.*

Next, we want to prove a version of [1, Theorem 3.1] and [8, Theorem 1.7] for the partial isometric crossed product of (A,α). We adopt the proof of these theorems and translate into the context of partial-isometric crossed product.

Theorem 2.2. *Suppose α is an extendible endomorphism on a C^*-algebra A, and I is an extendible α-invariant ideal of A. Let $(A \times_\alpha \mathbb{N}, i_A, i_{\mathbb{N}})$ be the crossed product for (A,α). Then there is a short exact sequence*

$$0 \longrightarrow I \times_\alpha \mathbb{N} \xrightarrow{\phi} A \times_\alpha \mathbb{N} \xrightarrow{\psi} A/I \times_{\tilde{\alpha}} \mathbb{N} \longrightarrow 0$$

where ϕ is an isomorphism of $I \times_\alpha \mathbb{N}$ onto the ideal

$$D := \overline{\operatorname{span}}\{i_{\mathbb{N}}(s)^* i_A(a) i_{\mathbb{N}}(t) : a \in I, s,t \in \mathbb{N}\}$$

of $A \times_\alpha \mathbb{N}$. If $(j_I, j_{\mathbb{N}})$ and $(k_{A/I}, k_{\mathbb{N}})$ are the universal covariant pairs for (I,α) and $(A/I, \tilde{\alpha})$, respectively, then

$$\phi \circ j_I = i_A|_I, \quad \overline{\phi} \circ j_{\mathbb{N}} = i_{\mathbb{N}} \quad \text{and} \quad \psi \circ i_A = k_{A/I} \circ q, \quad \overline{\psi} \circ i_{\mathbb{N}} = k_{\mathbb{N}}.$$

Proof. To see D as an ideal of $A \times_\alpha \mathbb{N}$, let $\xi = i_{\mathbb{N}}(s)^* i_A(b) i_{\mathbb{N}}(t)$ where $b \in I$. Since $i_{\mathbb{N}}(m)^*\xi = i_{\mathbb{N}}(m+s)^* i_A(b) i_{\mathbb{N}}(t)$;

$$\begin{aligned} i_A(a)\xi &= i_A(a) i_{\mathbb{N}}(s)^* i_A(b) i_{\mathbb{N}}(t) = (i_{\mathbb{N}}(s) i_A(a^*))^* i_A(b) i_{\mathbb{N}}(t) \\ &= (i_A(\alpha_s(a^*)) i_{\mathbb{N}}(s))^* i_A(b) i_{\mathbb{N}}(t) = i_{\mathbb{N}}(s)^* i_A(\alpha_s(a) b) i_{\mathbb{N}}(t); \end{aligned}$$

and $i_{\mathbb{N}}(m)\xi = i_{\mathbb{N}}(m) i_{\mathbb{N}}(s)^* i_A(b) i_{\mathbb{N}}(t)$ is

$$i_{\mathbb{N}}(s-m)^* i_{\mathbb{N}}(s) i_{\mathbb{N}}(s)^* i_A(b) i_{\mathbb{N}}(t) = i_{\mathbb{N}}(s-m)^* i_A(\overline{\alpha}_s(1) b) i_{\mathbb{N}}(t) \quad \text{for } m < s,$$

$$i_{\mathbb{N}}(m) i_{\mathbb{N}}(m)^* i_A(b) i_{\mathbb{N}}(t) = i_A(\overline{\alpha}_m(1) b) i_{\mathbb{N}}(t) \quad \text{for } m = s,$$

$$\begin{aligned} i_{\mathbb{N}}(m-s) i_{\mathbb{N}}(s)^* i_{\mathbb{N}}(s)^* i_A(b) i_{\mathbb{N}}(t) &= i_{\mathbb{N}}(m-s) i_A(\overline{\alpha}_s(1) b) i_{\mathbb{N}}(t) \\ &= i_A(\alpha_{m-s}(\overline{\alpha}_s(1) b)) i_{\mathbb{N}}(m-s+t) \quad \text{for } m > s, \end{aligned}$$

and which all belong to D. It follows that D is an ideal of $A \times_\alpha \mathbb{N}$.

Because D is an ideal of $A \times_\alpha \mathbb{N}$, there is a canonical homomorphism

$$r : A \times_\alpha \mathbb{N} \to M(D) \quad \text{such that} \quad r(\xi)d = \xi d \ \text{ for } \xi \in A \times_\alpha \mathbb{N} \ \text{ and } d \in D.$$

Denote by $\overline{r}$, the unique extension of r on the multiplier $M(A \times_\alpha \mathbb{N})$, and let $j_I : I \to D$ be the composition

$$I \xrightarrow{i_A|_I} A \times_\alpha \mathbb{N} \xrightarrow{r} M(D),$$

and $j_{\mathbb{N}} : \mathbb{N} \to M(D)$ to be the composition

$$\mathbb{N} \xrightarrow{i_{\mathbb{N}}} M(A \times_\alpha \mathbb{N}) \xrightarrow{\overline{r}} M(D).$$

We claim that the triple $(D, j_I, j_{\mathbb{N}})$ is a crossed product for (I, α). Certainly $j_{\mathbb{N}}$ is a partial isometry representation of $\mathbb{N}$ in $M(D)$, and that

$$j_I((\alpha|_I)_n(i)) = j_{\mathbb{N}}(n) j_I(i) j_{\mathbb{N}}(n)^*$$

and

$$j_{\mathbb{N}}(n)^* j_{\mathbb{N}}(n) j_I(i) = j_I(i) j_{\mathbb{N}}(n)^* j_{\mathbb{N}}(n)$$

for $n \in \mathbb{N}$, $i \in I$. To get j_I nondegenerate, we need the extendibility of ideal I in A: for an approximate identity (i_λ) in I, $\varphi : A \to M(I)$ the canonical homomorphism, we have

$$j_I(i_\lambda)(i_{\mathbb{N}}(n)^* i_A(i) i_{\mathbb{N}}(m)) = i_{\mathbb{N}}(n)^* i_A(\alpha_n(i_\lambda)i) i_{\mathbb{N}}(m)$$

converges to $i_{\mathbb{N}}(n)^* i_A(\overline{\alpha_n|_I}(1_{M(I)}i) i_{\mathbb{N}}(m)$, and the extendibility of I gives

$$i_A(\overline{\alpha_n|_I}(1_{M(I)})i) = i_A(\varphi(\overline{\alpha_n|_I}(1_{M(I)}i)) = \overline{i}_A(\overline{\alpha}_n(1_{M(A)})) i_A(i),$$

so

$$i_{\mathbb{N}}(n)^* i_A(\overline{\alpha_n|_I}(1_{M(I)})i) i_{\mathbb{N}}(m) = i_{\mathbb{N}}(n)^* \overline{i}_A(\overline{\alpha}_n(1_{M(A)})) i_A(i) i_{\mathbb{N}}(m) = i_{\mathbb{N}}(n)^* i_A(i) i_{\mathbb{N}}(m),$$

therefore $j_I(i_\lambda).d$ (and similarly for $d.j_I(i_\lambda)$) converges to d in D for every d, this means that $j_I(i_\lambda)$ converges strictly to $1_{M(D)}$ in $M(D)$, i.e. j_I is nondenegenerate.

Next, since any covariant representation (π, V) of $(I, \alpha|_I)$ on H extends to the covariant representation $(\overline{\pi}, V)$ of $(M(I), \overline{\alpha|_I})$ such that if $\varphi : A \to M(I)$ denotes the canonical homomorphism, then the pair $(\overline{\pi} \circ \overline{\varphi}, V)$ is a covariant representation of (A, α). Consequently we have a nondegenerate representation ρ of $A \times_\alpha \mathbb{N}$ which satisfies $\rho \circ i_A = \overline{\pi} \circ \varphi$ and $\overline{\rho} \circ i_{\mathbb{N}} = V$. Moreover $\rho|_D$ is nondegenerate, hence it extends to the representation $\overline{\rho|_D}$ of $M(D)$ such that $\overline{\rho|_D} \circ \overline{r} = \overline{\rho}$. So $\rho|_D \circ j_I = \rho|_D \circ i_A|_I = \overline{\pi} \circ \varphi|_I = \pi$ and $\overline{\rho|_D} \circ j_{\mathbb{N}} = \overline{\rho|_D} \circ (\overline{r} \circ i_{\mathbb{N}}) = \overline{\rho} \circ i_{\mathbb{N}} = V$. This completes the proof of our claim.

Finally to get a surjective homomorphism $\psi : A \times_\alpha \mathbb{N} \longrightarrow A/I \times_{\tilde{\alpha}} \mathbb{N}$ with $\ker \psi = D$, we note that $(k_{A/I} \circ q, k_{\mathbb{N}})$ is a covariant representation of (A, α) in $A/I \times_{\tilde{\alpha}} \mathbb{N}$. Hence there is a nondegenerate representation $\psi := k_{A/I} \circ q \times k_{\mathbb{N}}$ of $A \times_\alpha \mathbb{N}$ such that $\psi \circ i_A = k_{A/I} \circ q$ and $\overline{\psi} \circ i_{\mathbb{N}} = k_{\mathbb{N}}$. So the range of ψ is all of $A/I \times_{\tilde{\alpha}} \mathbb{N}$. The ideal D is certainly contained in $\ker \psi$. To see that $\ker \psi \subset D$, take a representation ρ of $A \times_\alpha \mathbb{N}$ with $\ker \rho = D$. Then $(\rho \circ i_A, \overline{\rho} \circ i_{\mathbb{N}})$ is a covariant representation of (A, α), and $I \subset \ker \rho \circ i_A$. So $\tilde{\rho} : a + I \in A/I \mapsto \rho \circ i_A(a)$ is a well-defined representation of A/I, which together with $\rho \circ i_{\mathbb{N}}$ form a covariant representation of $(A/I, \tilde{\alpha})$. Consequently there is a nondegenerate representation Φ of $A/I \times_{\tilde{\alpha}} \mathbb{N}$ that satisfies $\overline{\Phi} \circ k_N = \overline{\rho} \circ i_{\mathbb{N}}$. We then check that $\Psi \circ \psi$ agrees with ρ on their spanning elements, hence $\Phi \circ \psi = \rho$ on the two algebras. Therefore $\ker \psi = D$. ∎

Remark 2.2. Theorem 2.2 can also be proved using the theory of Toeplitz algebras of Hilbert bimodules, as is in Example 3.12 [6] for the case of isometric crossed products. The crossed product of (A, α) is the Toeplitz algebra $\mathcal{T}_X$ of the Hilbert bimodule $X := \overline{\alpha}(1)A$. An α-extendibly invariant ideal I of A gives the bimodule $XI = \overline{\alpha}(1_{M(I)})I$ associated to $\alpha|_I$, such that $X/XI = \overline{\tilde{\alpha}}(1_{M(A/I)})(A/I)$. So $\mathcal{T}_{XI}$ is

$I \times_\alpha \mathbb{N}$, and [6, Corollary 3.2] implies that $\mathcal{T}_X/\mathcal{T}_{XI} \simeq \mathcal{T}_{X/XI}$, which is the crossed product for $(A/I, \tilde{\alpha})$.

3. The crossed products $\mathbf{c}_0 \times_\tau \mathbb{N}$ and $\mathbf{c}_0 \times_\sigma \mathbb{N}$

3.1. The crossed product $\mathbf{c}_0 \times_\tau \mathbb{N}$

Consider the unital C^*-algebra $\mathbf{c}$ of convergent sequences, and the action τ of $\mathbb{N}$ on $\mathbf{c}$ generated by the forward shift:

$$\tau_1(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots).$$

Viewing sequences in $\mathbf{c}$ as functions on $\mathbb{N}$, each function

$$1_n(m) = \begin{cases} 1 & \text{if } m \geq n \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\mathbf{c}$, and $\overline{\text{span}}\{1_n : n \in \mathbb{N}\}$ is all of $\mathbf{c}$. The unit in $\mathbf{c}$ is $1 := 1_0$, and the action τ of $\mathbb{N}$ on $\mathbf{c} = \overline{\text{span}}\{1_n : n \in \mathbb{N}\}$ satisfies $\tau_m(1_n) = 1_{m+n}$ which is trivially extedible.

Any partial isometric representation V of $\mathbb{N}$, by [9, §5], induces a representation π_V of the algebra $\mathbf{c}$ given by $\pi_V(1_n) = V_n V_n^*$, such that (π_V, V) is a covariant representation of $(\mathbf{c}, \tau)$, and it is follows from [9, Proposition 5.4] that the representation $\pi_V \times V$ of $\mathbf{c} \times_\tau \mathbb{N}$ is faithful if and only if

$$(1 - V_r^* V_r)(V_u V_u^* - V_t V_t^*) \neq 0 \quad \text{for every } r > 0, u < t \text{ in } \mathbb{N}. \tag{3.1}$$

The crossed product $(\mathbf{c} \times_\tau \mathbb{N}, i)$ is the universal C^*-algebra generated by a power partial isometry by [7, Proposition 5.3]: if B is a unital C^*-algebra and $w \in B$ is a power partial isometry, then there is a unital homomorphism $h : (\mathbf{c} \times_\tau \mathbb{N}, i) \to B$ which satisfies $h(i_\mathbb{N}(1)) = w$.

The ideal $\mathbf{c}_0 = \overline{\text{span}}\{1_m - 1_n : m < n \in \mathbb{N}\}$ of sequences convergent to 0 is an extendible τ-invariant ideal of $\mathbf{c}$. So by Theorem 2.2 we obtain a short exact sequence

$$0 \longrightarrow \mathbf{c}_0 \times_\tau \mathbb{N} \longrightarrow \mathbf{c} \times_\tau \mathbb{N} \xrightarrow{q} \mathbf{c}/\mathbf{c}_0 \times_{\tilde{\tau}} \mathbb{N} \longrightarrow 0. \tag{3.2}$$

We show in the Lemma 3.1 that $\mathbf{c}/\mathbf{c}_0 \times_{\tilde{\tau}} \mathbb{N}$ is isomorphic to the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$, the unital C^*-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by $\{T_n : n \in \mathbb{N}\}$, where T_n is the nonunitary isometry defined on the usual basis $\{e_m : m \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$ by $T_n(e_m) = e_{n+m}$ for all $m \in \mathbb{N}$. We recall from [3] for the readers that every isometric representation of $\mathbb{N}$ gives a unital representation ρ_V of the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$ such that $\rho_V(T_n) = V_n$, and if each of V_n is nonunitary then ρ_V is faithful. So $\mathcal{T}(\mathbb{Z})$ is the universal C^*-algebra generated by a nonunitary isometry [4], and with the homomorphism $\psi_T : T_m \mapsto \epsilon_m \in C(\mathbb{T})$ (ϵ_m is the evaluation map in $C(\mathbb{T})$), there is an exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T}(\mathbb{Z}) \xrightarrow{\psi_T} C(\mathbb{T}) \longrightarrow 0.$$

Murphy extends this theorem in [10] from $(\mathbb{Z}, \mathbb{N})$ to the pair (Γ, Γ^+) of partially ordered abelian group Γ and its positive cone Γ^+.

Lemma 3.1. *There is an isomorphism* $\phi : (\mathbf{c}/\mathbf{c}_0 \times_{\tilde{\tau}} \mathbb{N}, k) \longrightarrow \mathcal{T}(\mathbb{Z})$ *such that* $\phi(k_\mathbb{N}(m)) = T_m^*$ *for all* $m \in \mathbb{N}$.

Proof. First we consider the system $(\mathbb{C}, \mathrm{id})$. We claim the crossed product $(\mathbb{C}\times_{\mathrm{id}}\mathbb{N}, \iota)$ is isomorphic to the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$. To justify this, we let

$$j : \lambda \in \mathbb{C} \mapsto \lambda I \in \mathcal{T}(\mathbb{Z}) \quad \text{and} \quad w : n \in \mathbb{N} \mapsto T_n^* \in \mathcal{T}(\mathbb{Z}). \tag{3.3}$$

Then we want to show that $(\mathcal{T}(\mathbb{Z}), j, w)$ is a crossed product for $(\mathbb{C}, \mathrm{id})$. One can see from (3.3) that (j, w) is a covariant representation of $(\mathbb{C}, \mathrm{id})$ in $\mathcal{T}(\mathbb{Z})$, and that $\{w_n = T_n^* : n \in \mathbb{N}\} \cup j(\mathbb{C})$ generates the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$.

Next suppose (π, V) is a covariant representation of $(\mathbb{C}, \mathrm{id})$ on H. Then $\pi(\lambda) = \lambda I_{B(H)}$ and $V_n V_n^* = I_{B(H)}$ for all n. Therefore $V^* : n \mapsto V_n^*$ is an isometric representation of $\mathbb{N}$ on H, so we have a unital representation ρ_{V^*} of $\mathcal{T}(\mathbb{Z})$ on H such that $\rho_{V^*}(w_n) = V_n^*$ for $n \in \mathbb{N}$ and $\rho_{V^*}(j(\lambda)) = \lambda I_{B(H)}$ for $\lambda \in \mathbb{C}$. Thus $(\mathcal{T}(\mathbb{Z}), j, w)$ is a crossed product for $(\mathbb{C}, \mathrm{id})$ as we claimed. Consequently there is an isomorphism $\Phi : (\mathbb{C} \times_{\mathrm{id}} \mathbb{N}, \iota) \to \mathcal{T}(\mathbb{Z})$ such that $\Phi(\iota_{\mathbb{C}}(\lambda)) = \lambda I$ and $\Phi(\iota_{\mathbb{N}}(n)) = T_n^*$ for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$.

By viewing $\mathbf{c}$ as the algebra of all functions f on $\mathbb{N}$ that have limits as $n \to \infty$, we let the map $\ell : \mathbf{c} \to \mathbb{C}$ defined by $\ell(f) = \lim_{n\to\infty} f(n)$. It is a surjective homomorphism with $\ker \ell = \mathbf{c}_0$, which therefore induces an isomorphism $\tilde{\ell} : f + \mathbf{c}_0 \mapsto \ell(f)$ of the quotient $\mathbf{c}/\mathbf{c}_0$ onto $\mathbb{C}$ such that $\tilde{\ell} \circ \tilde{\tau} = \mathrm{id} \circ \tilde{\ell}$. So the system $(\mathbf{c}/\mathbf{c}_0, \tilde{\tau})$ is equivariant to $(\mathbb{C}, \mathrm{id})$, and hence we have an isomorphism $\rho : (\mathbf{c}/\mathbf{c}_0 \times_{\tilde{\tau}} \mathbb{N}, k) \longrightarrow (\mathbb{C} \times_{\mathrm{id}} \mathbb{N}, \iota)$ that satisfies $\rho(k_{\mathbb{N}}(m)) = \iota_{\mathbb{N}}(m)$ and $\rho(k_{\mathbf{c}/\mathbf{c}_0}(f)) = \iota_{\mathbb{C}}(\tilde{\ell}(f)) = \lim_{n\to\infty} f(n)$.

Finally let ϕ be the composition

$$(\mathbf{c}/\mathbf{c}_0 \times_{\tilde{\tau}} \mathbb{N}, k) \xrightarrow{\rho} (\mathbb{C} \times_{\mathrm{id}} \mathbb{N}, \iota) \xrightarrow{\Phi} \mathcal{T}(\mathbb{Z}).$$

Then ϕ is the isomorphism which satisfies the requirement. ∎

We now consider the algebra $\mathcal{A}$ defined by authors of [9] as

$$\mathcal{A} = \{f : \mathbb{N} \to K(\ell^2(\mathbb{N})) : f(n) \in P_n K(\ell^2(\mathbb{N})) P_n \text{ and } \lim_{n\to\infty} f(n) \text{ exists}\} \tag{3.4}$$

where $P_n := 1 - T_{n+1}T_{n+1}^*$ is the projection onto $\mathrm{span}\{e_i : 0 \le i \le n\}$. The isometric representation $T^* : m \in \mathbb{N} \mapsto T_m^*$ of $\mathbb{N}$ in $\mathcal{T}(\mathbb{Z})$ gives a surjective homomorphism $\varphi_{T^*} : \mathbf{c} \times_\tau \mathbb{N} \to \mathcal{T}(\mathbb{Z})$, and its kernel $\ker \varphi_{T^*}$ is, by [9, Proposition 6.9], isomorphic to $\mathcal{A}$. We shall recall the construction of this isomorphism.

For every n, consider the operator $P_n T P_n := P_n T_1 P_n$ on $\ell^2(\mathbb{N})$. It is a power partial isometry:

$$(P_n T_k P_n)(P_n T_k P_n)^*(P_n T_k P_n) = P_n T_k T_k^* T_k P_n = P_n T_k P_n$$

for $k \le n$, and $P_n T_k P_n = 0$ for all $k > n$. So, by the universality of $\mathbf{c} \times_\tau \mathbb{N}$, there is a unital representation π_n of $(\mathbf{c} \times_\tau \mathbb{N}, i)$ on $\ell^2(\mathbb{N})$ such that $\pi_n(i_{\mathbb{N}}(1)) = P_n T P_n$ and $\pi_n(i_{\mathbf{c}}(1_m)) = P_n T_m T_m^* P_n$. We note that each of the representation π_n is not faithful: For an arbitrary $n \in \mathbb{N}$ choose $u, t \in \mathbb{N}$ such that $n < u < t$, then $P_n T_u = 0$ and $P_n T_t = 0$, therefore

$$(1 - P_n T_r^* P_n T_r P_n)(P_n T_u T_u^* P_n - P_n T_t T_t^* P_n) = 0 \quad \text{for any } r,$$

so π_n is not faithful by [9, Proposition 5.4]. The key is that for every $a \in \ker \varphi_{T^*}$, the sequence $\{\pi_n(a)\}_{n\in\mathbb{N}}$ belongs to $\mathcal{A}$ [9, §6], and therefore $a \mapsto \pi(a) := \{\pi_n(a)\}$ is a well-defined map of $\ker \varphi_{T^*}$ into $\mathcal{A}$, and is then proved in [9, Proposition 6.9] that the map $\pi : \ker \varphi_{T^*} \to \mathcal{A}$ is an isomorphism.

We show in the next corollary that the crossed product $\mathbf{c}_0 \times_\tau \mathbb{N}$ is the ideal $\ker \varphi_{T^*}$ of $\mathbf{c} \times_\tau \mathbb{N}$.

Corollary 3.1. *The ideal* $\ker \varphi_{T^*}$ *of* $\mathbf{c} \times_\tau \mathbb{N}$ *is the crossed product* $\mathbf{c}_0 \times_\tau \mathbb{N}$*. Thus we have an isomorphism* $\pi : \mathbf{c}_0 \times_\tau \mathbb{N} \longrightarrow \mathcal{A}$ *such that, for* $i, j, s < t \in \mathbb{N}$*,*

(3.5) $\pi[i_{\mathbb{N}}(i)^* i_{\mathbf{c}}(1_s - 1_t) i_{\mathbb{N}}(j)]$ *is the sequence* $\{P_n T_i^* P_n (T_s T_s^* - T_t T_t^*) P_n T_j P_n\}_{n \in \mathbb{N}}$

which converges to $T_i^*(T_s T_s^* - T_t T_t^*) T_j$*.*

Proof. Consider the quotient map q in (3.2) and the isomorphism ϕ in Lemma 3.1. We see that $\phi \circ q(i_{\mathbb{N}}(n)) = \phi(k_{\mathbb{N}}(n)) = T_n^* = \varphi_{T^*}(i_{\mathbb{N}}(n))$ for all n. So we get the commutative diagram:

$$\begin{array}{ccc} \mathbf{c} \times_\tau \mathbb{N} & \xrightarrow{q} & \mathbf{c}/\mathbf{c}_0 \times_{\tilde{\tau}} \mathbb{N} \\ & \searrow^{\varphi_{T^*}} & \downarrow \phi \\ & & \mathcal{T}(\mathbb{Z}). \end{array} \tag{3.6}$$

Consequently $\mathbf{c}_0 \times_\tau \mathbb{N} = \ker q = \ker \phi \circ q = \ker \varphi_{T^*}$, and is isomorphic to $\mathcal{A}$ by [9, Proposition 6.9]. The map π on every spanning element of $\mathbf{c}_0 \times_\tau \mathbb{N}$ is

$$\begin{aligned} \pi[i_{\mathbb{N}}(i)^* i_{\mathbf{c}}(1_s - 1_t) i_{\mathbb{N}}(j)] &= \{\pi_n[i_{\mathbb{N}}(i)^* i_{\mathbf{c}}(1_s - 1_t) i_{\mathbb{N}}(j)]\}_n \\ &= \{P_n T_i^* P_n (T_s T_s^* - T_t T_t^*) P_n T_j P_n\}_{n \in \mathbb{N}} \\ &= \{T_i^* P_n (T_s T_s^* - T_t T_t^*) P_n T_j\}_{n \in \mathbb{N}}, \end{aligned} \tag{3.7}$$

and since

$$P_n (T_s T_s^* - T_t T_t^*) P_n = T_s T_s^* - T_t T_t^*$$

for $n > t > s$, the sequence in (3.7) converges to $T_i^*(T_s T_s^* - T_t T_t^*) T_j \in K(\ell^2(\mathbb{N}))$. ∎

3.2. The crossed product $\mathbf{c}_0 \times_\sigma \mathbb{N}$

Now consider the system $(\mathbf{c}, \sigma)$ where the action σ is given by the backward shift:

$$\sigma_k(1_n) = \begin{cases} 1_{n-k} & \text{if } n \geq k \\ 1 & \text{otherwise.} \end{cases}$$

Each of σ_k is an extendible endomorphism of $\mathbf{c}$. The ideal $\mathbf{c}_0$ is a σ-invariant ideal of $\mathbf{c}$. It is an extendible ideal because for the approximate identity $(1 - 1_n)_{n \in \mathbb{N}}$ in $\mathbf{c}_0$, the sequence $\sigma_k(1 - 1_n)_{n \in \mathbb{N}} = (1 - 1_{n-k})_{n \in \mathbb{N}}$ converges strictly to $1 = \sigma_k(1)$ in $M(\mathbf{c}_0)$. So by Theorem 2.2 there is a short exact sequence

$$0 \longrightarrow \mathbf{c}_0 \times_\sigma \mathbb{N} \longrightarrow \mathbf{c} \times_\sigma \mathbb{N} \xrightarrow{q} \mathbf{c}/\mathbf{c}_0 \times_{\tilde{\sigma}} \mathbb{N} \longrightarrow 0. \tag{3.8}$$

The same proof of Lemma 3.1 is valid for the system $(\mathbf{c}, \sigma)$, and we can therefore have an isomorphism $\phi : (\mathbf{c}/\mathbf{c}_0 \times_{\tilde{\sigma}} \mathbb{N}, k) \longrightarrow \mathcal{T}(\mathbb{Z})$ such that

$$\phi(k_{\mathbb{N}}(i)^* k_{\mathbf{c}/\mathbf{c}_0}([1_m]) k_{\mathbb{N}}(j)) = T_i T_j^* \text{ for all } i, j, m \in \mathbb{N}.$$

We shall now remind our readers the universal property of $\mathbf{c} \times_\sigma \mathbb{N}$ described in [9, Proposition 7.1]. Every covariant representation (π, v) of $(\mathbf{c}, \sigma)$ always satisfies

$$v_n v_n^* = v_n \pi(1) v_n^* = \pi(\sigma_n(1)) = \pi(1) = 1 \text{ for every } n \in \mathbb{N}.$$

Thus v represents $\mathbb{N}$ as coisometry operators. Let

$$Q_0 = 1 - v^*v \quad \text{and} \quad Q_n = \pi(1_n) - v^*\pi(\sigma_n(1))v \text{ for } n > 0,$$

then every Q_n is a projection in which $\cdots \leq Q_{n+1} \leq Q_n \leq Q_{n-1} \leq \cdots \leq Q_0$. From this sequence $\{Q_n\}$ and the coisometry v_1, we can recover the representation π by the following equation

$$\pi(1_n) = (v_1^*)^n(v_1)^n + \sum_{k=0}^{n-1}(v_1^*)^n Q_{n-k}(v_1)^n \quad \text{for all } n > 0. \tag{3.9}$$

Conversely for any coisometry w on a Hilbert space H and a sequence of decreasing projections $\{Q_n\}$, there is a covariant representation $(\pi_{w,Q}, w)$ of $(\mathbf{c}, \sigma)$ on H such that $\pi_{w,Q}$ satisfies the equation (3.9). Thus covariant representations of $(\mathbf{c}, \sigma)$ is in bijective correspondence to pairs of coisometries and decreasing sequences of projections.

The crossed product $\mathbf{c} \times_\sigma \mathbb{N}$ which, by definition, is the universal C^*-algebra generated by the canonical covariant representation $(k_{\mathbf{c}}, k_{\mathbb{N}})$, is generated by the coisometry $k_{\mathbb{N}}(1)$ and by elements

$$q_n := k_{\mathbf{c}}(1_n) - k_{\mathbb{N}}(1)^* k_{\mathbf{c}}(\sigma_n(1)) k_{\mathbb{N}}(1),$$

such that whenever we have a pair $(w, \{Q_n\})$ of a coisometry w and a sequence of projections $\{Q_n\}$ in a C^*-algebra B with $\cdots \leq Q_{n+1} \leq Q_n \leq Q_{n-1} \leq \cdots \leq Q_0$, there is a homomorphism $\pi_{w,Q} \times w : \mathbf{c} \times_\sigma \mathbb{N} \to B$ that satisfies $\pi_{w,Q} \times w(k_{\mathbb{N}}(1)) = w$ and $\pi_{w,Q} \times w(q_m) = Q_m$ for all m. [9, Proposition 7.3] says that $\pi_{w,Q} \times w$ is faithful if and only if $Q_n \neq Q_{n+1}$ for all $n \geq 0$.

Authors in [9] prove that there is a faithful representation of $\mathbf{c} \times_\sigma \mathbb{N}$, in the C^*-algebra $C_b(\mathbb{N}, B(\ell^2(\mathbb{N})))$. We shall now recall the construction. Let T be the unilateral shift on $\ell^2(\mathbb{N})$ and T^* its adjoint. Then the coisometry element of $C_b(\mathbb{N}, B(\ell^2(\mathbb{N})))$ is given by the constant function $F : n \mapsto T^*$. Each projection $Q_m \in C_b(\mathbb{N}, B(\ell^2(\mathbb{N})))$ is defined by

$$Q_m(n) = \begin{cases} 1 - TT^* & \text{for } n \geq m \\ 0 & \text{otherwise.} \end{cases}$$

It can be seen from this definition that $\{Q_m\}$ is a decreasing sequence of projections in which $Q_m \neq Q_{m+1}$ for all $m \geq 0$. So by [9, Proposition 7.1] there is a covariant representation $(\pi_{F,Q}, F)$ of $(\mathbf{c}, \sigma)$, and the representation $\pi_{F,Q} \times F$ of $(\mathbf{c} \times_\sigma \mathbb{N}, k_{\mathbf{c}}, k_{\mathbb{N}})$ satisfies $\pi_{F,Q} \times F(k_{\mathbb{N}}(1)) = F$ and $\pi_{F,Q} \times F(q_m) = Q_m$. Moreover [9, Proposition 7.3] says that $\pi_{F,Q} \times F$ is a faithful representation of $\mathbf{c} \times_\sigma \mathbb{N}$ in $C_b(\mathbb{N}, B(\ell^2(\mathbb{N})))$. Note that $\pi_{F,Q} \times F$ maps every spanning element

$$\xi := k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_m) k_{\mathbb{N}}(j)$$

of $(\mathbf{c} \times_\sigma \mathbb{N}, k_{\mathbf{c}}, k_{\mathbb{N}})$ into the function $\pi_{F,Q} \times F(\xi)$ given by

$$(\pi_{F,Q} \times F(\xi))(n) = \begin{cases} T_{i+(m-n)} T^*_{j+(m-n)} & \text{for } n \leq m \\ T_i T_j^* & \text{for } n > m. \end{cases}$$

So $(\pi_{F,Q} \times F(\xi))(n)$ belongs to $\mathcal{T}(\mathbb{Z})$ for all n, and $\lim_{n\to\infty}(\pi_{F,Q} \times F(\xi))(n) = T_i T_j^*$. It is shown in [9, Theorem 7.4] that the range of $\pi_{F,Q} \times F$ is the C^*-algebra

$$\mathcal{B} := \{f \in C(\mathbb{N} \cup \{\infty\}, \mathcal{T}(\mathbb{Z})) : \psi_T(f(n)) \text{ is constant}\},$$

where ψ_T is the homomorphism in the exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T}(\mathbb{Z}) \xrightarrow{\psi_T} C(\mathbb{T}) \to 0.$$

Thus $\pi_{F,Q} \times F$ is an isomorphism of $\mathbf{c} \times_\sigma \mathbb{N}$ onto $\mathcal{B}$. We describe the C^*-algebra $\pi_{F,Q} \times F(\mathbf{c}_0 \times_\sigma \mathbb{N})$ in the next proposition.

Proposition 3.1. *The isomorphism $\pi_{F,Q} \times F : \mathbf{c} \times_\sigma \mathbb{N} \longrightarrow \mathcal{B}$ in* [9, Theorem 7.4] *restricts to an isomorphism of the crossed product $\mathbf{c}_0 \times_\sigma \mathbb{N}$ onto $C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$.*

Proof. By applying Theorem 2.2 to the system $(\mathbf{c}, \sigma)$ and the extendible ideal $\mathbf{c}_0$ of $\mathbf{c}$, the crossed product $\mathbf{c}_0 \times_\sigma \mathbb{N}$ is isomorphic to the ideal

$$D = \overline{\operatorname{span}}\{k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_s - 1_t) k_{\mathbb{N}}(j) : s < t \in \mathbb{N}, i, j \in \mathbb{N}\}$$

of $(\mathbf{c} \times_\sigma \mathbb{N}, k_{\mathbf{c}}, k_{\mathbb{N}})$. We show that $\pi_{F,Q} \times F(D)$ and $C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$ contain each other. So we write the spanning families for these two algebras

$$\pi_{F,Q} \times F(D) = \overline{\operatorname{span}}\{\pi_{F,Q} \times F(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_s - 1_t) k_{\mathbb{N}}(j)) : i, j \in \mathbb{N},\ s < t \in \mathbb{N}\},$$

and the ideal $C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$ is spanned by the functions $\{e_{ij}^m : i, j, m \in \mathbb{N}\}$ in which

$$e_{ij}^m(n) = \begin{cases} T_i(1 - TT^*)T_j^* & \text{for } m = n \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$e_{ij}^m = \pi_{F,Q} \times F(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_m - 1_{m+1}) k_{\mathbb{N}}(j) - k_{\mathbb{N}}(i+1)^* k_{\mathbf{c}}(1_{m-1} - 1_m) k_{\mathbb{N}}(j+1)) \tag{3.10}$$

it follows that every e_{ij}^m belongs to $\pi_{F,Q} \times F(D)$, which therefore gives the first inclusion

$$C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N}))) \subset \pi_{F,Q} \times F(D).$$

For the other inclusion, let $i, j, s \in \mathbb{N}$, then we have

$$\begin{aligned} \pi_{F,Q} \times F\left(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_s - 1_{s+1}) k_{\mathbb{N}}(j)\right)(n) &= \begin{cases} T_{i+s-n}(1 - TT^*)T_{j+s-n}^* & \text{for } s \geq n \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{k=1}^{s} e_{i+s-k, j+s-k}^k(n). \end{aligned}$$

If $s < t$ in $\mathbb{N}$, then $1_s - 1_t = \sum_{u=1}^{t-s} 1_{s+(u-1)} - 1_{s+u}$. Therefore

$$\begin{aligned} \pi_{F,Q} \times F(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_s - 1_t) k_{\mathbb{N}}(j)) &= \sum_{u=1}^{t-s} \pi_{F,Q} \times F(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_{s+(u-1)} - 1_{s+u}) k_{\mathbb{N}}(j)) \\ &= \sum_{u=1}^{t-s} \sum_{k=1}^{n} e_{x-k, y-k}^k, \end{aligned}$$

for $x = i + s + (u - 1) - k$ and $y = j + s + (u - 1) - k$. Thus

$$\pi_{F,Q} \times F(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_s - 1_t) k_{\mathbb{N}}(j)) \in C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N}))),$$

and hence $\pi_{F,Q} \times F(D) \subset C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$. ∎

Consider the map $\epsilon_\infty : \mathcal{B} \to \mathcal{T}(\mathbb{Z})$ defined by $\epsilon_\infty(f) = f(\infty)$. It is a $*$-homomorphism, which satisfies

$$\epsilon_\infty \circ (\pi_{F,Q} \times F)\ (k_{\mathbb{N}}(i)k_{\mathbf{c}}(1_m)k_{\mathbb{N}}(j)) = T_iT_j^* = \phi \circ q\ (k_{\mathbb{N}}(i)k_{\mathbf{c}}(1_m)k_{\mathbb{N}}(j),$$

where q is the quotient map in the sequence

$$0 \to \mathbf{c}_0 \times_\sigma \mathbb{N} \to \mathbf{c} \times_\sigma \mathbb{N} \xrightarrow{q} \mathbf{c}/\mathbf{c}_0 \times_{\tilde{\sigma}} \mathbb{N} \to 0,$$

and $\phi : (\mathbf{c}/\mathbf{c}_0 \times_\sigma \mathbb{N}, \iota) \longrightarrow \mathcal{T}(\mathbb{Z})$ is the isomorphism such that

$$\phi(\iota_{\mathbb{N}}(i)^*\iota_{\mathbf{c}/\mathbf{c}_0}([1_m])\iota_{\mathbb{N}}(j)) = T_iT_j^* \quad \text{for all } i, j, m \in \mathbb{N}.$$

So we have

$$\epsilon_\infty \circ (\pi_{F,Q} \times F) = \phi \circ q, \tag{3.11}$$

and therefore ϵ_∞ is surjective. We claim that $\ker \epsilon_\infty = C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$. From (3.10) we know that every $f \in C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$ belongs to $\pi_{F,Q} \times F(\mathbf{c} \times_\sigma \mathbb{N}) = \mathcal{B}$, and which $\epsilon_\infty(f) = 0$. So

$$C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N}))) \subset \ker \epsilon_\infty.$$

If $g \in \ker \epsilon_\infty$ then $\phi^{-1}(\epsilon_\infty(g)) = 0$. From (3.11), we have $\phi^{-1} \circ \epsilon_\infty = q \circ (\pi_{F,Q} \times F)^{-1}$. It then follows that $q \circ (\pi_{F,Q} \times F)^{-1}(g) = 0$. Thus $(\pi_{F,Q} \times F)^{-1}(g) \in \ker q = \mathbf{c}_0 \times_\sigma \mathbb{N}$, and hence

$$g \in \pi_{F,Q} \times F(\ker q) = \pi_{F,Q} \times F(\mathbf{c}_0 \times_\sigma \mathbb{N}) = C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N}))).$$

So $\ker \epsilon_\infty \subset C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$, and we have proved the claim.

We can now conclude this diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{c}_0 \times_\sigma \mathbb{N} & \longrightarrow & \mathbf{c} \times_\sigma \mathbb{N} & \xrightarrow{q} & \mathbf{c}/\mathbf{c}_0 \times_{\tilde{\sigma}} \mathbb{N} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{F,Q} \times F} & & \downarrow{\scriptstyle \pi_{F,Q} \times F} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N}))) & \longrightarrow & \mathcal{B} & \xrightarrow{\epsilon_\infty} & \mathcal{T}(\mathbb{Z}) & \longrightarrow & 0 \end{array} \tag{3.12}$$

Acknowledgement. The first author would like to thank Prof. Iain Raeburn for his valuable suggestions. This research is supported by the Universiti Sains Malaysia Fundamental Research Grant Scheme, and the Universiti Sains Malaysia Graduate Fellowship.

References

[1] S. Adji, Invariant ideals of crossed products by semigroups of endomorphisms, in *Functional Analysis and Global Analysis* (*Quezon City, 1996*), 1–8, Springer, Singapore.

[2] S. Adji, Semigroup crossed products and the structure of Toeplitz algebras, *J. Operator Theory* **44** (2000), no. 1, 139–150.

[3] S. Adji, M. Laca, M. Nilsen and I. Reburn, Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups, *Proc. Amer. Math. Soc.* **122** (1994), no. 4, 1133–1141.

[4] L. A. Coburn, The C^*-algebra generated by an isometry, *Bull. Amer. Math. Soc.* **73** (1967), 722–726.

[5] N. J. Fowler, Discrete product systems of Hilbert bimodules, *Pacific J. Math.* **204** (2002), no. 2, 335–375.

[6] N. J. Fowler, P. S. Muhly and I. Raeburn, Representations of Cuntz-Pimsner algebras, *Indiana Univ. Math. J.* **52** (2003), no. 3, 569–605.

[7] N. J. Fowler and I. Raeburn, The Toeplitz algebra of a Hilbert bimodule, *Indiana Univ. Math. J.* **48** (1999), no. 1, 155–181.

[8] N. S. Larsen, Non-unital semigroup crossed products, *Math. Proc. R. Ir. Acad.* **100A** (2000), no. 2, 205–218.

[9] J. Lindiarni and I. Raeburn, Partial-isometric crossed products by semigroups of endomorphisms, *J. Operator Theory* **52** (2004), no. 1, 61–87.

[10] G. J. Murphy, Ordered groups and Toeplitz algebras, *J. Operator Theory* **18** (1987), no. 2, 303–326.