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The Essential Norm and Spectrum of a Weighted Composition Operator on $H^\infty(B_N)$

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Abstract. For φ a holomorphic self-map of the unit ball B_N of $\mathbb{C}^N$, and $u \in H^\infty(B_N)$ (the Banach space of bounded holomorphic functions on B_N), we investigate the essential norm and spectrum of the weighted composition operator uC_φ acting on the space $H^\infty(B_N)$. For φ univalent, not unitary on any slice, and fixing a point of B_N, we obtain a complete characterization of the spectrum of uC_φ.

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1. Introduction

Let $H^\infty(B_N)$ denote the space of bounded holomorphic functions in the unit ball B_N of $\mathbb{C}^N$, endowed with the norm of $\|f\| = \sup_{z\in B_N} |f(z)|$. For φ, a non-constant holomorphic map of the unit ball into itself, the composition operator C_φ with the symbol φ on $H^\infty(B_N)$ is defined by $C_\varphi(f) = f \circ \varphi$. It is easy to see that C_φ is always bounded on $H^\infty(B_N)$ with norm 1. For u holomorphic on B_N, the weighted composition operator uC_φ is defined by $uC_\varphi(f) = u\cdot f\circ\varphi$. Notice that $uC_\varphi 1 = u(z)$, it is obvious that uC_φ is bounded on $H^\infty(B_N)$ if and only if $u \in H^\infty(B_N)$.

Let $\|uC_\varphi\|_e$ and ρ_e denote the essential norm and the essential spectral radius of uC_φ respectively. The essential norm of an operator is the norm of its equivalence class in the Calkin algebra. Similarly, the essential spectrum of an operator is the spectrum of the equivalence class that contains this operator in the Calkin algebra. The essential norms of composition operators on $H^\infty(\mathbb{D})$ were characterised in [15]. The essential norms of weighted composition operators acting on the ball algebras and $H^\infty(B_N)$ were given in [13].

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On Hardy spaces $H^p(B_N)$ when $0 < p < \infty$ and $N > 1$, C_φ is not always bounded. When $p = 2$, spectral information for bounded composition operators on some weighted Hardy spaces was given in [4]. When $N = 1$, that is, on the unit disk, we recommend the interested readers refer to the books by J. H. Shapiro [12] and Cowen and MacCluer [5], which are good sources for information on much of the developments in the theory of composition operators up to the middle of last decade. Recently the spectra of composition operators, both weighted and unweighted ($u \equiv 1$,) have been studied for other spaces of holomorphic functions: See [1, 2, 9, 10, 14, 15], for example.

Motivated by recent works of Aron and Lindström [1], and Zheng [15], we give essential norm estimates and determine the spectra of weighted composition operators uC_φ acting on $H^\infty(B_N)$.

The remainder of the present paper is organized as follows: In the next section, we provide the essential norm estimates of uC_φ acting on $H^\infty(B_N)$. Using these estimates, we find, in Section 3, the essential spectral radius of uC_φ. Then we determine the spectrum of uC_φ. Some techniques are inspired by [4], unlike the weighted Hardy spaces of bounded type in [4], this paper gives a complete characterization of the spectrum, and point out that the same results also hold for the composition operator C_φ (the case $u = 1$).

2. The essential norm

Recall that the essential norm of a bounded linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}.$$

Clearly T is compact if and only if its essential norm is 0.

The estimates of weighted composition operators on $H^\infty(B_N)$ are similar to those of [13], but we obtain different forms from which it is easy to determine the spectral radii.

Proposition 2.1. *If uC_φ is not compact on $H^\infty(B_N)$, then*

$$\|uC_\varphi\|_e \le \min\left\{\sup_{z\in B_N} |u(z)|, 2\lim_{r\to 1}\sup_{z\in E_r} |u(z)|\right\}$$

where $E_r = \{z \in B_N : |\varphi(z)| > r\}$ for $0 < r < 1$.

Proof. First of all $\|uC_\varphi\|_e \le \|uC_\varphi\| = \sup_{z\in B_N} |u(z)|$, we need only to show $\|uC_\varphi\|_e \le 2\lim_{r\to 1}\sup_{z\in E_r} |u(z)|$. The argument is similar to [13], we omit the details here. ∎

Because $u \in H^\infty(B_N)$ can be extremely oscillatory near every boundary point, to give the lower estimate we need the interpolating sequence.

Definition 2.1. *An* ***interpolating sequence*** *$\{z_j\}$ in the ball is the one for which, given any bounded sequence $\{c_j\}$ of complex numbers, there is a bounded analytic function f so that $f(z_j) = c_j$.*

By the proof of Lemma 13 in [4], the following lemma follows.

Lemma 2.1. *Fix $0 < a < 1$, any sequence $\{z_k\}$ in B_N satisfying*

$$\frac{1-|z_k|}{1-|z_{k+1}|} < a < 1$$

is an $H^\infty(B_N)$ interpolating sequence.

This lemma is the finite dimensional case of Theorem 5.1 in [7]. Using it we have the following lower estimate.

Proposition 2.2. *If uC_φ is not compact on $H^\infty(B_N)$, then*

$$2^{-1} \lim_{r\to 1} \sup_{z\in E_r} |u(z)| \le \|uC_\varphi\|_e$$

where E_r is given in Proposition 2.1 above.

Proof. Since uC_φ is not compact, it is easy to show that $\sup_{z\in B_N} |\varphi(z)| = 1$, that is, whenever r is sufficiently close to 1, E_r is not empty. We want to show that there exists a sequence $\{f_n\} \in H^\infty(B_N)$ with $\|f_n\| = 1$ such that $\{f_n\}$ converges to 0 uniformly on compact subsets of B_N and

$$\lim_{n\to\infty} \|(uC_\varphi)(f_n)\| \ge 2^{-1} \lim_{r\to 1} \sup_{z\in E_r} |u(z)|.$$

Denote $\lim_{r\to 1} \sup_{z\in E_r} |u(z)|$ by A. Without loss of generality we suppose $A > 1$, then for any $\epsilon > 0$, there is a $\delta \in (0,1)$ such that for any $r > \delta$, $\sup_{z\in E_r} |u(z)| > A-\epsilon$, so there exists a $z_\epsilon \in E_r$, $|u(z_\epsilon)| > A - \epsilon$ with $|\varphi(z_\epsilon)| > \delta$.

Now let $\epsilon = 1$, there is a δ_1 and a z_1 such that $|u(z_1)| > A - 1$ and $|\varphi(z_1)| > \delta_1$.

Let $\epsilon = 1/2$, there is a δ_2 such that when $r_2 > \delta_2$, $\sup_{z\in E_{r_2}} |u(z)| > A - 1/2$. Let $r_2' = \max\{r_2, 1 - a + a|\varphi(z_1)|\}$, where a is the fixed number in Lemma 2.1 above, there is a $z_2 \in E_{r_2'}$ such that $|u(z_2)| > A - 1/2$.

By induction we can get a sequence $\{z_j\}_{j=1}^{n+1}$ with $\frac{1-|\varphi(z_j)|}{1-|\varphi(z_{j-1})|} < a < 1$ and $|u(z_j)| > A - \frac{1}{j}$ where $z_j \in E_{r_j'}$.

It follows from Lemma 2.1 that for this sequence $\{z_j\}_{j=1}^{n+1}$, there exists $h_k \in H^\infty(B_N)$ such that $h_k(\varphi(z_j)) = 1$ for $k = j$ and $h_k(\varphi(z_j)) = 0$ for $k \ne j$ with $\|h_k\| = 1$. These h_n's are bounded with norm 1 and $h_n \ne h_m$ if $n \ne m$.

$\{h_n\}$ is a sequence in the unit ball of $H^\infty(B_N)$, so it must have a subsequence which converges to some $h \in H^\infty(B_N)$ weakly by Montel's Theorem. Without loss of generality we also denote this subsequence by $\{h_n\}$. Let $g_n = h_n - h_{n+1}$, then g_n converges to 0 uniformly on compact subsets of B_N as n tends to ∞ with $\|g_n\| \le 2$.

Let $f_n = g_n/2$, then $\|f_n\| \le 1$ and $f_n(\varphi(z_n)) = 1/2$. Then

$$(uC_\varphi)(f_n)(z_n) = u(z_n)f(\varphi(z_n)) = 2^{-1}u(z_n).$$

Thus

$$\begin{aligned}\|uC_\varphi\|_e &\ge \lim_{n\to\infty} \|(uC_\varphi)(f_n)\| \ge \lim_{n\to\infty} \sup_{z\in B_N} |(uC_\varphi)(f_n)(z)| \\ &\ge \lim_{n\to\infty} |u(z_n)| \cdot |f(\varphi(z_n))| = 1/2 \lim_{n\to\infty} |u(z_n)| \\ &\ge 1/2 \lim_{n\to\infty} (A - \frac{1}{n}) = A/2.\end{aligned}$$

This completes the proof. ∎

Combining Proposition 2.1 and Proposition 2.2, we actually have got the following theorem.

Theorem 2.1. *If uC_φ is not compact on $H^\infty(B_N)$, then*

$$1/2 \lim_{r\to 1} \sup_{z\in E_r} |u(z)| \le \|uC_\varphi\|_e \le 2 \lim_{r\to 1} \sup_{z\in E_r} |u(z)|$$

where E_r is given in Proposition 2.1 above.

The constants 1/2 and 2 above may not be sharp, however, we have the following corollary.

Corollary 2.1. *If $u \in H^\infty(B_N)$, then uC_φ acting on $H^\infty(B_N)$ is compact if and only if*

$$\lim_{r\to 1} \sup_{z\in E_r} |u(z)| = 0.$$

If the weight function u belongs to the ball algebra, that is, if $u \in H(B_N) \cap C(\overline{B_N})$, then u is uniformly continuous and can not be extremely oscillatory near the boundary, we have the following lower estimate of the essential norm.

Proposition 2.3. *If uC_φ is not compact on $H^\infty(B_N)$ and $u \in H(B_N) \cap C(\overline{B_N})$, then*

$$\lim_{r\to 1} \sup_{z\in E_r} |u(z)| \le \|uC_\varphi\|_e$$

where E_r is given in Proposition 2.1 above.

Proof. Let $\{f_j\}_{j=1}^\infty$ be a sequence in $H^\infty(B_N)$ with $\|f_j\| = 1$ for all j and $f_n \ne f_m$ if $n \ne m$. Then for any compact operator K on $H^\infty(B_N)$, $\{Kf_j\}$ has a convergent subsequence, without loss of generality we also denote it by $\{Kf_j\}$. Then there exists a $f \in H^\infty(B_N)$ such that a subsequence of $\{f_j\}$ converges to f weakly as j tends to ∞ by Montel's Theorem, denoted the subsequence also by $\{f_j\}$. Without loss of generality we suppose $f \ne f_j$ for all j, then $f_j - f \ne 0$. Let $g_j = f_j - f$, then g_j converges to 0 weakly in $H^\infty(B_N)$ as j tends to ∞. $g_j \ne 0$ implies that $\|g_j\| \ne 0$, thus $\{g_j/\|g_j\|\}$ is a sequence of unit vectors which converges to 0 uniformly on compact subsets of B_N, for the convenient continue to denote it by $\{g_j\}$. For any $N \times N$ unitary matrix U, $\sup_{z\in B_N} |g_j(Uz)| = 1$.

To show that $\lim_{r\to 1} \sup_{z\in E_r} |u(z)| \le \|uC_\varphi\|_e$, consider $\inf \|uC_\varphi - K\|$ for all compact K.

$$\|uC_\varphi - K\| \ge \lim_{j\to\infty} \|(uC_\varphi - K)g_j(Uz)\| \ge \lim_{j\to\infty} \sup_{z\in B_N} |u(z)g_j(U\varphi(z))|.$$

Let $A = \lim_{r\to 1} \sup_{z\in E_r} |u(z)|$, then for $\epsilon = 1$, there exists $r_1 \in (0,1)$ such that $\sup_{z\in E_r} |u(z)| > A - 1$ for $r > r_1$. Since $|u(z)|$ is continuous on B_N, then there exists $z_1 \in E_{r_1}$ with $|u(z_1)| > A - 1$. By induction, for $\epsilon = \frac{1}{n}$, we get an increasing sequence $\{r_n\}$ with $r_n \in (1 - \frac{1}{n}, 1)$, $z_n \in E_{r_n}$ such that $|u(z_n)| > A - \frac{1}{n}$.

Note that $\{z_n\} \subset E_{r_n}$ such that $|\varphi(z_n)| > r_n$ and $r_n \to 1$, then there exists a subsequence of $\{\varphi(z_n)\}$ converges to some $z_0 \in \partial B_N$. Let $\{\varphi(z_{n_k})\}$ be such a subsequence, then $|u(z_{n_k})| > A - \frac{1}{n_k} > A - \frac{1}{k}$, so without loss of generality we suppose $\{\varphi(z_n)\}$ converges to z_0.

On the other hand, for any fixed j, $\sup_{z\in B_N}|g_j(z)| = 1$ means that for $\epsilon = \frac{1}{n}$, there is $w_n \in B_N$ such that $|g_j(w_n)| > 1 - \frac{1}{n}$. It is clear that $\lim_{n\to\infty}|g_j(w_n)| = 1$. From $w_n \in B_N$ we know $\{w_n\}$ has a subsequence still denoted by $\{w_n\}$, which converges to some $w_0 \in \partial B_N$ because of the maximum modulus principle. So the continuity of $|g_j(w)|$ implies that $|g_j(w_0)| = \lim_{n\to\infty}|g_j(w_n)| = 1$. Let $Uz_0 = w_0$ for a unitary matrix U, then

$$\sup_{z\in B_N}|u(z)g_j(U\varphi(z))| \geq |u(z_{n_k})|\cdot|g_j(U\varphi(z_{n_k}))| \geq (A - \frac{1}{k})|g_j(U\varphi(z_{n_k}))|$$

from which, let $k \to \infty$,

$$\sup_{z\in B_N}|u(z)g_j(U\varphi(z))| \geq A|g_j(Uz_0)| = A|g_j(w_0)|$$

the lower estimate follows by letting $j \to \infty$. ∎

Combining Proposition 2.2 and Proposition 2.3, we have the following theorem.

Theorem 2.2. *If uC_φ is not compact on $H^\infty(B_N)$ and $u \in H(B_N) \cap C(\overline{B_N})$, then*

$$\lim_{r\to 1}\sup_{z\in E_r}|u(z)| \leq \|uC_\varphi\|_e \leq \min\{\sup_{z\in B_N}|u(z)|, 2\lim_{r\to 1}\sup_{z\in E_r}|u(z)|\}$$

where $E_r = \{z \in B_N : |\varphi(z)| > r\}$ for $0 < r < 1$.

If we let $u = 1$, by Theorem 2.2, we have the following corollary.

Corollary 2.2. *The essential norm of C_φ on $H^\infty(B_N)$ is either 1 or 0.*

The estimates of the essential norm of uC_φ acting on the ball algebra or $H^\infty(B_N)$ can also be found in [13] with different forms. More generally, the essential norm of a composition operator on a uniform algebra has recently been characterized in [6].

3. The essential spectral radius and spectrum

For uC_φ acting on $H^\infty(B_N)$, we denote its spectral radius by $\rho(uC_\varphi)$. Then

$$\rho(uC_\varphi) = \lim_{n\to\infty}\|(uC_\varphi)^n\|^{\frac{1}{n}}$$

and the essential spectral radius is given by

$$\rho_e(uC_\varphi) = \lim_{n\to\infty}\|(uC_\varphi)^n\|_e^{\frac{1}{n}}.$$

Throughout the remainder of this paper, φ_n will denote the n^{th} iterate of φ, that is, $\varphi_1 = \varphi$ and $\varphi_n = \varphi \circ \varphi_{n-1}$ for all $n > 1$. For any $f \in H^\infty(B_N)$,

$$(uC_\varphi)^n(f(z)) = u(z)u(\varphi(z))\cdots u(\varphi_{n-1}(z))\cdot C_{\varphi_n}f(z).$$

So $(uC_\varphi)^n$ is a weighted composition operator with symbol φ_n and weight $u(z)u(\varphi(z))\cdots u(\varphi_{n-1}(z))$. Using Theorem 2.1 and Theorem 2.2, the essential spectral radius follows immediately.

Theorem 3.1. *If uC_φ is not compact on $H^\infty(B_N)$, then*

$$\rho_e(uC_\varphi) = \lim_{n\to\infty}\left(\lim_{r\to 1}\sup_{z\in E_r}|u(z)u(\varphi(z))\cdots u(\varphi_{n-1}(z))|\right)^{\frac{1}{n}}$$

where E_r is given in Proposition 2.1 in the last section.

If $u = 1$, we obtain the essential spectral radius of the composition operator on the $H^\infty(B_N)$.

Corollary 3.1. *The essential spectral radius of C_φ on $H^\infty(B_N)$ is either 1 or 0. If $C_{\varphi_n}(= C_\varphi^n)$ is compact for some $n \geq 1$, then $\rho_e(C_\varphi) = 0$, otherwise $\rho_e(C_\varphi) = 1$.*

Unlike the finite dimensional case, if X is an infinite dimensional Banach space, then for $H^\infty(B_X)$, it can occur that $0 < \rho_e(C_\varphi) < 1$. See [6] for more details. For the spectrum we have the following theorem.

Theorem 3.2. *Suppose $u \in H^\infty(B_N)$ and φ is a holomorphic map of B_N into B_N that is univalent with $\varphi(a) = a$ for some $a \in B_N$, and $\varphi_a \circ \varphi \circ \varphi_a$ is not unitary on any slice where φ_a is the involution which interchanges a and 0, then*

$$\sigma(uC_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leq \rho_e(uC_\varphi)\} \cup \{0, u(a), u(a)\mu\}$$

where μ is all products of eigenvalues of $\varphi'(a)$.

From the above theorem, the spectrum for the composition operator on $H^\infty(B_N)$ follows immediately if we let $u = 1$.

Corollary 3.2. *Suppose φ is a holomorphic map of B_N into B_N under the condition of Theorem 3.2 above, then*

$$\sigma(C_\varphi) = \overline{\mathbb{D}}, \ if \ \|\varphi_n\|_\infty = 1 \ for \ all \ n \in \mathbb{N},$$

and if $\|\varphi_n\|_\infty < 1$ for some $n \in \mathbb{N}$,

$$\sigma(C_\varphi) = \{all \ products \ of \ eigenvalues \ of \ \varphi'(a)\} \cup \{0, 1\}.$$

This corollary is a special case of Theorem 7.1 in [7].

The proof of Theorem 3.2 will be given after some lemmas. For the proof, we also need some complex calculation skills.

Now we introduce two subspaces.

Definition 3.1. *For $f \in H^\infty(B_N)$, the homogeneous expansions of f is denoted by $f(z) = \sum_{s=0}^\infty f_s(z)$. Then, for a non-negative integer m, the subspaces L_m and H_m of $H^\infty(B_N)$ are given by*

$$H_m = \left\{ f(z) = \sum_{s=0}^\infty f_s(z) \in H^\infty(B_N) : f_s(z) = 0 \text{ for all } s \geq m \right\}$$

and

$$L_m = \left\{ f(z) = \sum_{s=0}^\infty f_s(z) \in H^\infty(B_N) : f_s(z) = 0 \text{ for all } s < m \right\}.$$

According to Lemma 7 and its argument in [4], it is convenient to order the monomials z^α by ordering the multi-indices. When $|\alpha| < |\beta|$, we say $\alpha < \beta$; when $|\alpha| = |\beta|$, we say $\alpha < \beta$ if there is j_0 so that $\alpha_j = \beta_j$ for $j < j_0$ and $\alpha_{j_0} > \beta_{j_0}$. This ordering has the convenient property that if z^α precedes $z^{\alpha'}$ and z^β precedes $z^{\beta'}$, then $z^\alpha z^\beta$ precedes $z^{\alpha'} z^{\beta'}$. Similar to Lemma 7 in [15] we get the following lemma.

Lemma 3.1. *Suppose $\varphi(0) = 0$. Then H_m is an invariant subspace of uC_φ and $\sigma(C_m) \subset \sigma(uC_\varphi)$ where $C_m = uC_\varphi|_{H_m}$.*

Proof. For any $f \in H_m \subset H^\infty(B_N)$, $uC_\varphi(f) = u \cdot f \circ \varphi \in H^\infty(B_N)$. Let

$$f(z) = \sum_{|\gamma|=m}^{\infty} c_\gamma z^\gamma$$

be the homogeneous expansion of f. Based on the argument given after Lemma 6 in [4], there is no loss of generality to assume that $\varphi'(0)$ is lower triangular. Let ϵ_j denote the multi-index corresponding to the monomial z_j for $j = 1, 2, \cdots, N$, then if $\varphi = (\varphi_{(1)}, \varphi_{(2)}, \cdots, \varphi_{(N)})$,

$$\varphi_{(j)}(z) = \sum_\alpha a_{(j),\alpha} z^\alpha$$

where $a_{(j),\alpha} = 0$ for $\alpha < \epsilon_j$. Now the multiplicative property of the ordering implies that

$$z^\beta \circ \varphi = \varphi_{(1)}^{\beta_1} \varphi_{(2)}^{\beta_2} \cdots \varphi_{(N)}^{\beta_N} = \sum_\alpha b_\alpha z^\alpha,$$

where $b_\alpha = 0$ for $\alpha < \beta$. This means that

$$\begin{aligned} u(z) f(\varphi(z)) &= u(z) \sum_{|\gamma|=m}^{\infty} c_\gamma \varphi(z)^\gamma = u(z) \sum_{|\gamma|=m}^{\infty} c_\gamma \varphi_{(1)}^{\gamma_1} \varphi_{(2)}^{\gamma_2} \cdots \varphi_{(N)}^{\gamma_N} \\ &= u(z) \sum_{|\gamma|=m}^{\infty} c_\gamma \sum_\alpha b'_\alpha z^\alpha \end{aligned}$$

where $b'_\alpha = 0$ for $\alpha < \gamma$. So $u \cdot f \circ \varphi \in H_m$ and thus H_m is invariant under uC_φ.

Since L_m is finite dimensional, the second statement follows by Lemma 7.17 in [5] or as Lemma 7 in [15]. ∎

Lemma 3.2. *Suppose φ is the same as in Theorem 3.2 with $\varphi(0) = 0$, if $\lambda \neq 0$ is an eigenvalue of uC_φ, then $\lambda \in \{u(0), u(0)\mu\}$. Moreover, $\{u(0), u(0)\mu\} \subset \sigma(uC_\varphi)$ where μ denotes all possible products of eigenvalues for $\varphi'(0)$.*

Proof. If λ is an eigenvalue of uC_φ and f is a corresponding eigenvector of λ, then $u(z)f(\varphi(z)) = \lambda f(z)$. Upon differentiating both sides, we arrived at the first statement, for the detail we refer the readers to check Lemma 2.1 in [16]. To prove the second statement, without loss of generality, we may assume that $\varphi'(0)$ is lower triangular, and $\mu = \lambda_1^{s_1} \cdots \lambda_N^{s_N}$ where $\lambda_1, \cdots, \lambda_N$ are eigenvalues of $\varphi'(0)$. First of all, $u(0) \in \sigma(uC_\varphi)$ since for any $f \in H^\infty(B_n)$, $(u(0)I - uC_\varphi)f \neq 1$. Indeed, if $u(0)f(z) - u(z)f(\varphi(z)) = 1$, then $u(0)f(0) - u(0)f(\varphi(0)) = 1$. This is a contradiction.

Similarly, $u(0)\lambda_1 \in \sigma(uC_\varphi)$. Without loss of generality we suppose $\lambda_1 \neq 1$. If there exists $f \in H^\infty(B_n)$ such that $(u(0)\lambda_1 I - uC_\varphi)f = z_1$, then $f(0) = 0$ and

$$u(0)\lambda_1 \frac{\partial f(z)}{\partial z_1} - \frac{\partial u(z)}{\partial z_1} f(\varphi(z)) - u(z)\varphi'(z) \frac{\partial f(z)}{\partial \varphi_{(j)}} = 1.$$

Since $f(0) = 0$ and $\varphi'(0)$ is a lower triangular matrix,

$$u(0)\lambda_1 \frac{\partial f(0)}{\partial z_1} - u(0)\lambda_1 \frac{\partial f(0)}{\partial z_1} = 1.$$

This contradiction implies that $u(0)\lambda_1 \in \sigma(uC_\varphi)$.

To show $u(0)\lambda_2 \in \sigma(uC_\varphi)$, assume $f \in H^\infty(B_n)$ such that $(u(0)\lambda_2 I - uC_\varphi)f = z_2$. Then $f(0) = 0$ and

$$u(0)\lambda_2 \frac{\partial f(0)}{\partial z_1} - u(0)\lambda_1 \frac{\partial f(0)}{\partial z_1} = 0,$$

which means $\frac{\partial f(0)}{\partial z_1} = 0$. Because

$$u(0)\lambda_2 \frac{\partial f(z)}{\partial z_2} - \frac{\partial u(z)}{\partial z_2} f(\varphi(z)) - u(z)\varphi'(z)\frac{\partial f(z)}{\partial \varphi_{(i)}} = 1,$$

we have

$$u(0)\lambda_2 \frac{\partial f(0)}{\partial z_2} - u(0)\lambda_2 \frac{\partial f(0)}{\partial z_2} = 1$$

where we used $f(0) = 0$, $\frac{\partial f(0)}{\partial z_1} = 0$ and the fact that $\varphi'(0)$ is lower triangular. By induction we claim that for $\lambda_j \neq 1$, $j = 1, \cdots, N$, $u(0)\lambda_i \in \sigma(uC_\varphi)$.

Similarly, $u(0)\lambda_1^2 \in \sigma(uC_\varphi)$. Indeed, as we have shown above, assume $f \in H^\infty(B_N)$ such that $(u(0)\lambda_1^2 I - uC_\varphi)f = z_1^2$, we get $f(0) = 0$ and $\frac{\partial f(0)}{\partial z_j} = 0$. We also have

$$u(0)\lambda_1^2 \frac{\partial f(z)}{\partial z_1} - \frac{\partial u(z)}{\partial z_1} f(\varphi(z)) - u(z)\varphi'(z)\frac{\partial f(z)}{\partial \varphi_{(i)}} = 2z_1$$

and

$$u(0)\lambda_1^2 \frac{\partial^2 f(0)}{\partial z_1^2} - u(0)\lambda_1^2 \frac{\partial^2 f(0)}{\partial z_1^2} = 2.$$

This is a contradiction which means $u(0)\lambda_1^2 \in \sigma(uC_\varphi)$. By induction we have $u(0)\mu \in \sigma(uC_\varphi)$ if μ is a possible product of eigenvalues of $\varphi'(0)$. ∎

For any $f \in H^\infty(B_N)$, we consider the pairing

$$\langle f, g\rangle = \int_{B_N} f(z)\overline{g(z)}dv(z), \;\; f \in H^\infty(B_N), \;\; g \in A^1(B_N)$$

where dv is the normalized Lebesgue measure of B_N and $A^1(B_N)$ is the Bergman space on B_N, see [17]. Straightforward computation shows that for

$$K_w(z) = \frac{1}{(1 - \langle z, w\rangle)^{N+1}}\,,$$

we have for any $f \in H^\infty(B_N)$,

$$\langle f, K_w\rangle = f(w).$$

So the norm of evaluation functional at $w \in B_N$ can be considered as the $\|\cdot\|_{A^1}$ of K_w. Similarly, for any $f \in H_m$, let

$$K_w^m(z) = \sum_{s=m}^{\infty} \frac{(N-1+s)!(s+1)}{(N-1)!s!}\langle z, w\rangle^s,$$

then

$$\langle f, K_w^m\rangle = f(w) \tag{3.1}$$

for f in H_m. To give the $\|\cdot\|_{A^1}$ of K_w^m, we must first compute that

$$\int_{B_N} |\langle z, w\rangle|^s dv(z) = 2N\int_0^1 r^{2N-1+s}dr \int_{S_N} |\langle \zeta, w\rangle|^s \, d\sigma(\zeta)$$

$$= \frac{\Gamma(N-1)\Gamma(\frac{s+2}{2})}{\Gamma(N+s)} \frac{2N(N-1)}{2N+s}|w|^s.$$

Then by Lemma 3 of [4], we have the following lemma.

Lemma 3.3. *Suppose m is a positive integer greater than $N-1$. If w and z are points in B_N with $|\langle z,w\rangle| < 1/(N3^{N+1})$, then*

$$\frac{5}{6N} P(m)(m+1)|\langle z,w\rangle|^m \leq |K_w^m(z)| \leq \left(\frac{3}{2}\right)^N P(m)(m+1)|\langle z,w\rangle|^m$$

and

$$\frac{5}{6N} Q(N,m)|w|^m \leq \|K_w^m\|_{A_1} \leq \left(\frac{3}{2}\right)^N Q(N,m)|w|^m$$

where

$$P(m) = 1 - \sum_{k=0}^{N-2} \frac{(m+N-1)\cdots(m+1+k)}{(N-1-k)!}$$

which is given in [4] *and*

$$Q(N,m) = P(m)(m+1)\frac{\Gamma(N-1)\Gamma\left(\frac{m+2}{2}\right)}{\Gamma(N+m)} \frac{2N(N-1)}{2N+m}.$$

We say the sequence of points $\{z_k\}_{k=-K}^{\infty}$ in B_N is an *iteration sequence* for φ if $\varphi(z_k) = z_{k+1}$ for $k \geq -K$.

To prove Theorem 3.2 we also need some other lemmas which have been proved by Carl Cowen and Barbara MacCluer in [4].

Lemma 3.4. [4, Lemma 11] *Given ζ and η in ∂B_N, for each positive integer m there exist a homogeneous polynomials p_m of degree $2m$ such that $|p_m(\zeta)| = |p_m(\eta)| = 1$ and $\|p_m\|_\infty = 1$.*

Lemma 3.5. [4, Lemma 12–14] *If φ is an analytic self map of the unit ball with $\varphi(0) = 0$ which is not unitary on any slice in B_N, then for $0 < r < 1$, there is $A > 1$ so that*

$$\frac{1-|\varphi(z)|}{1-|z|} > A$$

for all z with $|z| \geq r$. If $\{z_k\}_{-K}^{\infty}$ is an iteration sequence with $|z_n| \geq r$ for some $n \geq 0$ and if $\{w_k\}_{-K}^{n}$ is arbitrary, then there is an $M < \infty$ and an h in $H^\infty(B_N)$ such that $h(z_k) = w_k$ for $-K \leq k \leq n$ and $\|h\|_\infty \leq M \sup\{|w_k| : -K \leq k \leq n\}$. Further there exists $c < 1$ such that $\frac{|z_{k+1}|}{|z_k|} \leq c$ whenever $|z_k| \leq c$.

Now we return to the proof of Theorem 3.3.

The argument is essentially the same as [1] and [7]. All proofs are based on showing the fact that the adjoint on an invariant space is not bounded from below. However, unlike the unit disc case in [1] and the unweighted case in [7], the argument has a longer form for the complete proof. On the bounded analytic function space of the infinite dimensional ball, the essential spectral radius is unknown, so it is an open problem to characterize the spectra of weighted composition operators in the infinite dimensional case.

Without loss of generality, suppose $a = 0$. In fact, if $a \neq 0$, let $\varphi_a(z)$ be the automorphism of B_N which interchanges a and 0, $u_1 = u \circ \varphi_a$ and $\phi = \varphi_a \circ \varphi \circ \varphi_a$. Then $\phi(0) = 0$, $u_1(0) = u(a)$, $C_{\varphi_a} \circ C_{\varphi_a} = I$, so

$$C_{\varphi_a} \circ u_1 C_\phi \circ C_{\varphi_a} = uC_\varphi.$$

Hence $u_1 C_\phi$ and uC_φ are similar and they have the same spectrum and essential spectral radius.

By Lemma 3.2 we have that $\{u(0), u(0)\mu\} \subset \sigma(uC_\varphi)$. For $\lambda \in \sigma(uC_\varphi)$ with $|\lambda| > \rho_e(uC_\varphi)$, it follows that λ is an eigenvalue (that is true for all bounded operators, see Proposition 2.2 in [3]). If $\lambda \neq 0$ is an eigenvalue, Lemma 3.2 gives that $\lambda \in \{u(0), u(0)\mu\}$, so it remains to show that

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \rho_e(uC_\varphi)\} \subset \sigma(uC_\varphi).$$

If $\rho_e(uC_\varphi) = 0$, the argument is proved since $0 \in \sigma(uC_\varphi)$ when uC_φ is not invertible. Now assume that $\rho_e(uC_\varphi) > 0$ and denote $\rho_e(uC_\varphi)$ by ρ, because the spectrum of uC_φ is closed, we can fix a λ with $0 < |\lambda| < \rho$. By Lemma 3.1 it is sufficient to show that $\lambda \in \sigma(C_m)$ for some m. We find a positive integer m such that $(C_m - \lambda I)^*$ is not bounded from below, which means $C_m - \lambda I$ is not invertible.

For $r = 1/N3^{N+1}$ it follows from Lemma 3.5 that there exists $c < 1$ so that $|\varphi(z)|/|z| \leq c$ where $|z| \leq r$. Since φ is univalent, $J_c\varphi'(0) \neq 0$, $\varphi'(0)$ is invertible. Let $z = \psi(w) = \varphi^{-1}(w)$, note that $\psi(0) = 0$ and $\psi'(0) = (\varphi'(0))^{-1}$, $\|\psi'(0)\| = \|(\varphi'(0))^{-1}\|$, then $\psi(w) = \psi(0) + \psi'(0)w + o(|w|) = \psi'(0)w + o(|w|)$. From which we have $|\psi(w)| \leq 2\|\psi'(0)\||w|$ for w approaching zero, that is $|z| \leq 2\|\psi'(0)\||\varphi(z)|$, $|\varphi(z)| \geq \frac{1}{2\|(\varphi'(0))^{-1}\|}|z|$ for z approaching zero. On the other hand, since φ is an open map and $\varphi(0) = 0$, the univalence of φ guarantees $\varphi(z)$ is not near zero when z is not near zero. Thus, there is c_0 (with $0 < c_0 < c$) so that $|\varphi(z)|/|z| \geq c_0$ for $0 < |z| < 1$. Now let $\{z_k\}_{-K}^\infty$ be any iteration sequence with $c > |z_0| > 1/N3^{N+1}$. Since $\{|z_k|\}$ is not increasing, $\{k : |z_k| > c_0/N3^{N+1}\}$ is a finite subset of $\mathbb{N}$, and let $n = \max\{k : |z_k| > c_0/N3^{N+1}\}$. Note that $|z_1|/|z_0| \geq c_0$ implies $|z_1| > c_0/N3^{N+1}$ so that $n \geq 1$. Let M be the interpolation constant for Lemma 3.5 with $r = c_0/N3^{N+1}$. Since $u \in H^\infty(B_N)$ is continuous, $0 < C := \max\{\sup_{|z| \leq c_0/N3^{N+1}} |u(z)|, |u(z_n)|\} < \infty$. Choose m greater than $N - 1$ so that

$$\frac{c^m C}{|\lambda|} < 1 \quad \text{and} \quad \frac{c^{2m} C}{|\lambda|} < \frac{1}{1 + \gamma M}, \tag{3.2}$$

where $\gamma = 2N(3/2)^N$. Next we will show $C_m^* - \bar{\lambda} I$ is not bounded below on H_m.

If $\{z_k\}_{k=-K}^\infty$ is an iteration sequence for φ with n defined as above, let us define the linear fractional $L_{\lambda,u}$ on H_m by

$$L_{\lambda,u}(f) = \sum_{k=-K}^{\infty} \lambda^{-k} u(z_{-K}) \cdots u(z_{k-1}) f(z_k), \quad f \in H_m$$

where we agree that $u(z_{-K})u(z_{-K-1}) = 1$ in the first term of the sum.

For $k > n$, since $|z_k|^2 < |z_k| < \frac{1}{N3^{N+1}} \leq \frac{1}{3^{N+1}}$, Lemma 3.3 implies

$$\|K_{z_k}^m\|_{A^1} \leq (3/2)^N Q(N, m)|z_k|^m$$

where $Q(N,m)$ is given in Lemma 3.3, and the choice of n gives

$$|z_k| = |z_n|\left(\frac{|z_{n+1}|}{|z_n|}\right)\cdots\left(\frac{|z_k|}{|z_{k-1}|}\right) \le |z_n|c^{k-n}. \tag{3.3}$$

It follows that

$$\begin{aligned}&\sum_{k=n+1}^{\infty}|\lambda|^{-k}|u(z_{-K})|\cdots|u(z_{k-1})|\|K^m_{z_k}\|_{A^1}\\ &\le \frac{|z_n|^m}{|\lambda|^n}\left(\frac{3}{2}\right)^N Q(N,m)|u(z_{-K})|\cdots|u(z_{n-1})|\sum_{k=n+1}^{\infty}\left(\frac{c^mC}{|\lambda|}\right)^{k-n} < \infty.\end{aligned}$$

Then the series

$$\sum_{k=-K}^{\infty}\bar{\lambda}^{-k}\overline{u(z_{-K})}\cdots\overline{u(z_{k-1})}K^m_{z_k}$$

converges in $A^1(B_N)$, which means $L_{\lambda,u}(f)$ gives a bounded linear functional on H_m by (3.1).

We also need a lower bound for $L_{\lambda,u}$. By Lemma 3.4, there exists a homogeneous polynomial p_m of degree $2m$ with $|p_m(z_0)| = 1$, $|p_m(Rz_n)| = 1$ and $\|p_m\|_\infty = 1$ on $|z_0|\partial B_N$, where $R|z_n| = |z_0|$ and $R > 1$ since $|z_0| = R|z_n| < R|z_0|$. Using Lemma 3.5 with $r = \frac{c_0}{N3^{N+1}}$, we can find h in $H^\infty(B_N)$ so that $\|h\|_\infty \le M$, $h(z_k) = 0$ for $k < n$ if $k \ne 0$, $|h(z_0)| = 1$, $|h(z_n)| = \frac{1}{p_m(z_n)\gamma R^{2m}}$, $u(z_{-K})\cdots u(z_{-1})h(z_0)p_m(z_0) \ge 0$ and $u(z_{-K})\cdots u(z_{n-1})\frac{h(z_n)p_m(z_n)}{\lambda^n} \ge 0$.

Let $g(z) = h(z)p_m(z) \in H_{2m} \subset H_m$, it is easy to check $\|g\|_\infty \le M$ and

$$\begin{aligned}L_{\lambda,u}(g) &= |u(z_{-K})|\cdots|u(z_{-1})| + \frac{|u(z_{-K})|\cdots|u(z_{n-1})|}{\lambda^n\gamma R^{2m}}\\ &\quad + \sum_{k=n+1}^{\infty}\lambda^{-k}u(z_{-K})\cdots u(z_{k-1})h(z_k)p_m(z_k).\end{aligned}$$

The construction of p_m implies $|(\frac{|z_0|}{|z_k|})^{2m}p_m(z_k)| = |p_m(\frac{z_k}{|z_k|}|z_0|)| \le 1$ so that $|p_m(z_k)| \le \frac{|z_k|^{2m}}{|z_0|^{2m}}$. Using this inequality, the norm estimate $\|h\|_\infty \le M$, and then by (3.2) and (3.3), it follows that

$$\begin{aligned}&\left|\sum_{k=n+1}^{\infty}\lambda^{-k}u(z_{-K})\cdots u(z_{k-1})h(z_k)p_m(z_k)\right|\\ &\le \sum_{k=n+1}^{\infty}|u(z_{-K})|\cdots|u(z_{k-1})|\frac{|z_k|^{2m}}{|z_0|^{2m}}|\lambda|^{-k}|h(z_k)|\\ &\le \frac{M|u(z_{-K})|\cdots|u(z_{n-1})||z_n|^{2m}}{|z_0|^{2m}|\lambda|^n}\sum_{k=n+1}^{\infty}\left(\frac{c^{2m}C}{|\lambda|}\right)^{k-n}\\ &\le \frac{|u(z_{-K})|\cdots|u(z_{n-1})||z_n|^{2m}}{\gamma|z_0|^{2m}|\lambda|^n}.\end{aligned}$$

Finally, remembering that $R|z_n| = |z_0|$, we obtain

$$\left|\sum_{k=n+1}^{\infty} \lambda^{-k} u(z_{-K}) \cdots u(z_{k-1}) h(z_k) p_m(z_k)\right| \leq \frac{|u(z_{-K})| \cdots |u(z_{n-1})|}{\gamma R^{2m} |\lambda|^n}.$$

Then

$$\|L_{\lambda,u}\| \geq \frac{|L_{\lambda,u}(g)|}{\|g\|_\infty} \geq \frac{|u(z_{-K})| \cdots |u(z_{-1})|}{M}. \tag{3.4}$$

For $f \in H_m$, straightforward computation shows that

$$\langle f, (C_m - \lambda I)^* L_{\lambda,u}\rangle = -\lambda^{K+1} f(z_{-K}).$$

From which we obtain that

$$\begin{aligned} \|(C_m - \lambda I)^* L_{\lambda,u}\| &= \sup_{0 \neq f \in H_m} \frac{\left|\left\langle f, (C_m - \lambda I)^* L_{\lambda,u}\right\rangle\right|}{\|f\|_\infty} \\ &= \sup_{0 \neq f \in H_m} \frac{|\lambda|^{K+1} |f|}{\|f\|_\infty} = |\lambda|^{K+1}. \end{aligned}$$

Since $|\lambda| < \rho$, given $0 < |\lambda| < \rho$ we can pick μ so that $|\lambda| < \mu < \rho$. By Theorem 3.1, there exists n_0 such that for all $s \geq n_0$,

$$\|(uC_\varphi)^s\|_e > \mu^s.$$

Hence for any $K \geq n_0$ we can find a $w \in B_N$ so that $|u(w)||u(\varphi(w))| \cdots |u(\varphi_{K-1}(w))| \geq \frac{\mu^K}{2} > 0$ and $|\varphi_K(w)| \geq \frac{1}{N3^{N+1}}$.

For every $K \geq n_0$ define the iteration sequence $\{z_k\}_{k=-K}^{\infty}$ by letting $z_{-K} = w$ and $z_{k+1} = \varphi(z_k)$ for $k \geq -K$. Then $|z_0| = |\varphi_K(w)| \geq \frac{1}{N3^{N+1}}$ and $|u(z_{-K})||u(z_{-K+1})| \cdots |u(z_{-1})| \geq \frac{\mu^K}{2} > 0$, and

$$\frac{\|(C_m - \lambda I)^* L_{\lambda,u}\|}{\|L_{\lambda,u}\|} \leq \frac{M|\lambda|^{K+1}}{|u(z_{-K})| \cdots |u(z_{-1})|} \leq 2M|\lambda| \frac{|\lambda|^K}{\mu^K}.$$

Choosing $K \geq n_0$, where K is sufficiently large, it follows that $(C_m - \lambda I)^*$ is not bounded from below as desired, which completes the proof.

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