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Entropy for a Pair of Subalgebras via Automorphisms

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Abstract. This is an expository survey article based on a talk for the Special Session (Operator Algebra) in AMC 2009. For two subalgebras A and B of a finite von Neumann algebra M, we discuss a modified version $h(A|B)$ of the Connes-Størmer relative entropy $H(A|B)$. As one of the most elementary examples, we pick two maximal abelian *-subalgebras A, B of the $n \times n$ complex matrices $M_n(\mathbb{C})$, and show relations between $h(A|B)$ and the entropy for unistochastic matrices. Also we show related results in the case of a pair of subfactors of a type II_1 factor.

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1. Introduction

Connes and Størmer [7] introduced the notion of the relative entropy $H(A|B)$ for subalgebras A and B of a finite von Neumann algebra M with a normal tracial state τ. It was given in a step to define the dynamical entropy $H(\alpha)$ for a τ-preserving automorphism α on M. The notion $H(A|B)$ was extended by Connes [6] to the relative entropy $H_\phi(A|B)$ with respect to a state ϕ for subalgebras A, B of a general von Neumann algebra M. If $\phi = \tau$, then $H_\phi(A|B) = H(A|B)$.

Our motivation for this work arises from the following fact: Connes-Størmer entropy $H(\alpha)$ for automorphisms classified a lot of classes of automorphisms on type II_1 factors. However, in the case where M is a type I algebra, Neshveyev-Størmer [12] proved that if α is an inner automorphism of M, then $H(\alpha) = 0$. The most typical examples of finite type I von Neumann algebras are the $n \times n$ matrix algebras $M_n(\mathbb{C})$, and all automorphisms of $M_n(\mathbb{C})$ are inner.

Is it possible to discuss inner automorphisms from the viewpoint of entropy?

As a slight modification of $H(A|B)$ and $H_\phi(A|B)$, we introduce two relative entropies $h(A|B)$ and $h_\phi(A|B)$ for a pair of subalgebras A and B of a finite von

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Neumann algebra M. We show some relations between an inner automorphism Adu of M and $h(A|uAu^*)$ (resp. $h_\phi(A|uAu^*)$). These relations indicate that $h(A|uAu^*)$ gives a kind of entropy with respect to Adu.

The paper is organized as follows. After preliminaries on basic notions in Section 2, in Section 3 we discuss the definitions of $h(A|B)$ and $h_\phi(A|B)$ which are given in [4] and we show several properties of them. In Section 4, we pick a pair of maximal abelian subalgebras A and B of the $n \times n$ complex matrices $M_n(\mathbb{C})$, and we show relations between $h_\phi(A|B)$ and the unitary matrix depending on a pair $\{A, B\}$, which were obtained in the paper [4].

In order to obtain these sharp relations in Section 4, the author considers the modified quantity instead of the Connes-Størmer relative entropy.

In Section 5, we extend the discussion in Section 4 to the case of a pair of subfactors $\{N, uNu^*\}$ where N is a subfactor of a type II_1 factor M and u is a unitary in M, and we show some results in [5], which are related to those in Section 4.

2. Preliminaries

Here we summarize notations, terminologies and basic facts (cf. [13]).

Let M be a finite von Neumann algebra with separable predual, and let τ be a faithful normal tracial state. We consider M as a von Neumann algebra acting on the Hilbert space $L^2(M, \tau)$.

By a von Neumann subalgebra A of M, we mean that A has the same identity as M. A conditional expectation of M onto a von Neumann subalgebra A of M is a completely positive linear map $E : M \to A$ with $E(axb) = aE(x)b$ for all $x \in M$ and $a, b \in A$. In the case of a finite von Neumann algebra M with a faithful normal tracial state τ, there exists always a unique faithful normal conditional expectation E_A of M onto a von Neumann subalgebra A of M such that $\tau(xa) = \tau(E_A(x)a)$ for all $x \in M$ and $a \in A$. It is called the conditional expectation with respect to τ.

2.1. Entropy function η

Entropy function η is defined by

$$\eta(t) = -t \log t \ (0 < t \leq 1) \text{ and } \eta(0) = 0,$$

and we use the following facts (cf. [13]):

(1) η is operator concave on $[0, \infty)$.
(2) If $x, y \in M_+$ and $xy = yx$, then $\eta(xy) = \eta(x)y + x\eta(y)$.
(3) If $x \in M_+$, then $\eta(x) = 0$ iff x is a projection.
(4) If $x, y \in M_+$, then $\tau\eta(x + y) \leq \tau\eta(x) + \tau\eta(y)$ and if $xy = 0$ then we have equality.
(5) For a von Neumann subalgebra A of M and $x \in M_+$, then

$$E_A(\eta(x)) \leq \eta E_A(x),$$

in particular

$$\tau\eta(x) \leq \tau\eta E_A(x) \leq \eta\tau(x).$$

2.2. Relative entropy of positive linear functionals

Let ϕ and ψ be positive linear functionals on a unital C^*-algebra C. The relative entropy $S(\phi, \psi)$ of ϕ and ψ was first defined by Umegaki, extended by Araki and then extended by Uhlmann (cf. [13]), and finally, the following was given by Kosaki [11]:

$$S(\phi,\psi) = \sup_{n\in\mathbb{N}} \sup_{x} \left\{ \phi(1)\log n - \int_{1/n}^{\infty} \left(\phi(y(t)^*y(t)) + \frac{1}{t}\psi(x(t)x(t)^*) \right) \frac{dt}{t} \right\},$$

where the second supremum is taken over all step functions $x : (\frac{1}{n}, \infty) \to C$ with finite range, and $y(t) = 1 - x(t)$. It suffices to consider functions x with values in a fixed weakly dense subspace of C containing the unit.

2.3. Relative entropy of Connes-Størmer

Here we review the relative entropy defined by Connes and Størmer in [7] (cf. [17, 13]).

Let S be the set of all finite families (x_i) of positive elements in M such that $1 = \sum_i x_i$. Let A and B be two von Neumann subalgebras of M. The relative entropy $H(A|B)$ is given by

$$H(A \mid B) = \sup_{(x_i)\in S} \sum_i (\tau\eta E_B(x_i) - \tau\eta E_A(x_i)).$$

Let ϕ be a normal state on M. Let Φ be the set of all finite families (ϕ_i) of positive linear functionals on M with $\phi = \sum_i \phi_i$. The relative entropy $H_\phi(A|B)$ of A and B with respect to ϕ was given by Connes [6] (cf. [13]) as

$$H_\phi(A|B) = \sup_{(\phi_i)\in\Phi} \sum_i (S(\phi_i \mid_A, \phi \mid_A) - S(\phi_i \mid_B, \phi \mid_B)),$$

and if ϕ is a finite trace τ then

$$H_\tau(A|B) = H(A \mid B).$$

3. Conditional relative entropy

Let M be a finite von Neumann algebra, and let τ be a faithful normal tracial state of M. We modify the Connes-Størmer relative entropy $H(A|B)$ for a pair of subalgebras A, B as follows:

Definition 3.1. [4] *Let A and B be two von Neumann subalgebras of M. Let S be the set of all finite families (x_i) of positive elements in M with $1 = \sum_i x_i$. The conditional relative entropy $h(A \mid B)$ of A and B is*

$$h(A \mid B) = \sup_{(x_i)\in S} \sum_i (\tau\eta E_B(E_A(x_i)) - \tau\eta E_A(x_i)).$$

Let $S(A) \subset S$ be the set of all finite families (x_i) of positive elements in A with $1 = \sum_i x_i$. Then it is clear that

$$h(A \mid B) = \sup_{(x_i)\in S(A)} \sum_i (\tau\eta E_B(x_i) - \tau\eta(x_i)).$$

The following lemma is useful to compute $h(A|B)$.

Lemma 3.1. *Let $S'(A) \subset S(A)$ be the set of all finite families (x'_i) with each x'_i a scalar multiple of a projection in A. Then*

(1)
$$h(A \mid B) = \sup_{(x'_i)\in S'(A)} \sum_i (\tau\eta E_B(x'_i) - \tau\eta(x'_i)).$$

(2) *If $u \in M$ is a unitary, then*
$$h(A \mid uAu^*) = \sup_{(\lambda_i p_i)\in S'(A)} \sum_i \lambda_i \tau\eta E_A(u^* p_i u),$$
where $p_i \in A$ is a projection and λ_i is a positive number.

Proof.
(1) Let $S(A) \subset S$ be the set of all finite families $(x_i)_i$ of positive elements in A with $1 = \sum_i x_i$. By the spectral decomposition for x_i, for a given $\epsilon > 0$ there exists $x'_i \in A$ such that $0 \le x'_i \le x_i,\ x_i - x'_i \le \epsilon$ and x'_i has finite spectrum, i.e.
$$x'_i = \sum_j \alpha_{ij} p^i_j$$
for some scalars $\alpha_{ij} \ge 0$ and for some mutually orthogonal projections $(p^i_j)_j$ in A. As η is continuous we get
$$\sum(\tau\eta E_B(x_i) - \tau\eta x_i) \le \sum(\tau\eta E_B(x'_i) - \tau\eta x'_i) + \delta(\epsilon),$$
where $\delta(\epsilon) \to 0$ when $\epsilon \to 0$. Then we have that
$$\tau\eta E_B(x'_i) = \tau\eta(\sum_j \alpha_{ij} E_B(p^i_j)) \le \sum_j \tau\eta E_B(\alpha_{ij} p^i_j)$$
and $\tau\eta(x'_i) = \sum_j \eta(\alpha_{ij})\tau(p^i_j) = \sum_j \tau\eta(\alpha_{ij} p^i_j)$, so that
$$\begin{aligned}\sum_i(\tau\eta E_B(x_i) - \tau\eta(x_i)) &\le \sum_i(\tau\eta E_B(x'_i) - \tau\eta(x'_i)) + \delta(\epsilon)\\ &\le \sum_{i,j}(\tau\eta(E_B(\alpha_{ij}p^i_j)) - \tau\eta(\alpha_{ij}p^i_j)) + \delta(\epsilon)\\ &\le \sup_{(y_i)\in S'(A)} \sum_j(\tau\eta(E_B(y_i)) - \tau\eta(y_i)) + \delta(\epsilon).\end{aligned}$$
The above proof is similar to the proof of [17, Lemma 3.1].

We remark that in the above computation
$$\begin{aligned}\sum_{i,j}\tau\eta(E_B(\alpha_{ij}p^i_j)) - \tau\eta(\alpha_{ij}p^i_j) &= \sum_i(\eta(\alpha_{ij})\tau(p^i_j) + \alpha_{ij}\tau\eta E_B(p^i_j) - \eta(\alpha_{ij})\tau(p^i_j))\\ &= \sum_{i,j}\alpha_{ij}\tau\eta(E_B(p^i_j)).\end{aligned}$$

(2) Let $(\lambda_i p_i) \in S'(A)$, where $p_i \in A$ is a projection and λ_i is a positive number. Since $E_{uAu^*}(x) = uE_A(u^*xu)u^*$ for all x in M, we have that
$$h(A \mid uAu^*) = \sup_{(\lambda_i p_i)\in S'(A)} \sum_i(\tau\eta E_{uAu^*}(\lambda_i p_i) - \tau\eta(\lambda_i p_i))$$

$$= \sup_{(\lambda_i p_i)\in S'(A)} \sum_i \lambda_i \tau\eta E_A(u^* p_i u).$$ ∎

Lemma 3.2. *Let θ be an automorphism of M. Then*

$$h(A \mid B) = h(\theta(A) \mid \theta(B)).$$

Proof. Since $E_{\theta(B)}E_{\theta(A)} = \theta E_B E_A \theta^{-1}$ and $\tau\eta\theta = \tau\eta$, we have that

$$\begin{aligned} h(\theta(A) \mid \theta(B)) &= \sup_{(x_i)\in S} \sum_i (\tau\eta E_{\theta(B)}(E_{\theta(A)}(x_i)) - \tau\eta E_{\theta(A)}(x_i)) \\ &= \sup_{(y_i)\in S} \sum_i (\tau\eta\theta E_B(E_A(y_i)) - \tau\eta\theta E_A(y_i)) \\ &= \sup_{(y_i)\in S} \sum_i (\tau\eta E_B(E_A(y_i)) - \tau\eta E_A(y_i)) \\ &= h(A|B). \end{aligned}$$ ∎

Next, we modify the Connes relative entropy $H_\phi(A|B)$ for two subalgebras A and B with respect to a normal state ϕ as follows:

Definition 3.2. [4] *Let M be a finite von Neumann algebra, and let τ be a faithful normal tracial state. Let ϕ be a normal state of M. We denote by Φ the set of all finite families (ϕ_i) of positive linear functionals on M with $\phi = \sum_i \phi_i$. Let A and B be two von Neumann subalgebras of M.*

The conditional relative entropy of A and B with respect to ϕ is

$$h_\phi(A|B) = \sup_{(\phi_i)\in\Phi} \sum_i (S(\phi_i \mid_A, \phi \mid_A) - S((\phi_i \circ E_A) \mid_B, (\phi \circ E_A) \mid_B).$$

If we let $\Phi(A) \subset \Phi$ be the set of all finite families (ϕ_i') in Φ such that $\phi_i' = \phi_i' \circ E_A$ for each i, then

$$h_\phi(A|B) = \sup_{(\phi_i')\in\Phi(A)} \sum_i (S(\phi_i' \mid_A, \phi \mid_A) - S((\phi_i' \circ E_A) \mid_B, (\phi \circ E_A) \mid_B).$$

Note 3.1. The finiteness of M in the definition of $h_\phi(A|B)$ is only used for the existence of a conditional expectation E_A from M onto A.

We have the following properties for $h(A|B)$ and $h_\phi(A|B)$:

(1) $h_\phi(A|B) \geq 0$, and if $A \subset B$ then $h_\phi(A|B) = 0$. Moreover, if $h(A|B) = 0$, then $A \subset B$.

(2) If $E_B E_C = E_C E_B$, then

$$\tau\eta E_B(x) \leq \tau\eta E_B(E_C(x))$$

so that

$$h(A|B) \leq h(A|C) + h(C|B).$$

(3) $h_\phi(A|B)$ is increasing in A and decreasing in B.

(4) If $\{A_n\}$ and $\{B_n\}$ are increasing sequences of von Neumann subalgebras of A and B such that $A = (\cup_n A_n)''$ and $B = (\cup_n B_n)''$, then

$$h_\phi(A_n|B) \nearrow h_\phi(A|B) \quad \text{and} \quad h_\phi(A|B_n) \searrow h_\phi(A|B);$$

furthermore

$$h_\phi(A|B) = \lim_{n\to\infty} h_\phi(A_n|B_n).$$

(5) The conditional relative entropy of A and B with respect to a tracial state τ is $h(A|B)$, that is,

$$h_\tau(A|B) = h(A|B).$$

(6) If $B \subset A$ and if there exists a ϕ-preserving faithful normal conditional expectation $\mathcal{E}_B : A \to B$, then

$$h_\phi(A|B) = \sup_{(\phi_i)\in\Phi} \sum_i S(\phi_i|_A, \phi_i \circ \mathcal{E}_B).$$

(7) If M is abelian, then for all subalgebras A and B, $h_\phi(A|B) = H_\phi(A|B)$ which coincides with the conditional entropy in the classical case.

In fact, these properties (1)–(7) are shown by a similar method to the proof in [13], which we give here for the sake of completeness as follows:

Proof.
(1) The trivial decomposition $(\phi) \in \Phi$ gives that $S(\phi\,|_A, \phi\,|_A) = 0$ and $S((\phi\circ E_A)\,|_B, (\phi \circ E_A)\,|_B) = 0$ so that $h_\phi(A|B) \geq 0$.

Assume that $A \subset B$. Then by [13, Theorem 2.3.1 (vi)],

$$S(\phi_i|_A, \phi|_A) = S(\phi_i \circ E_A|_A, \phi \circ E_A|_A) \leq S(\phi_i \circ E_A|_B, \phi \circ E_A|_B)$$

so that $h_\phi(A|B) = 0$.

If A is not contained in B, then there exists a projection p in A which is not contained in B. If $E_B(p)$ is a projection, then $E_B(pE_B(p)) = E_B(p)$, and

$$\tau((p - E_B(p))^*(p - E_B(p))) = 2(\tau(p) - \tau(pE_B(p))) = 0,$$

so that, by the faithfulness of τ, we have that $p = E_B(p)$, which contradicts that $p \notin B$.

If $x \in M^+$ is not a projection, then $\eta(x) > 0$, and so, we have

$$h(A|B) \geq \tau\eta E_B(p) + \tau\eta E_B(1-p) > 0.$$

(2) If $E_B E_C = E_C E_B$, then

$$\tau\eta E_B(x) \leq \tau\eta E_C(E_B(x)) = \tau\eta E_B(E_C(x))$$

so that

$$\begin{aligned}
h(A|B) &= \sup_{(x_i)\in S(A)} \sum_i (\tau\eta E_B(x_i) - \tau\eta(x_i)) \\
&\leq \sup_{(x_i)\in S(A)} \sum_i (\tau\eta E_B(E_C(x_i)) - \tau\eta(x_i)) \\
&= \sup_{(x_i)\in S(A)} \sum_i (\tau\eta E_B(E_C(x_i)) - \tau\eta E_C(x_i) + \tau\eta E_C(x_i) - \tau\eta(x_i)) \\
&\leq \sup_{(y_i)\in S(C)} \sum_i (\tau\eta E_B(y_i)) - \tau\eta(y_i)) + \sup_{(x_i)\subset A} \sum_i (\tau\eta E_C(x_i) - \tau\eta(x_i)) \\
&= h(C|B) + h(A|C).
\end{aligned}$$

(3) It is clear that $h_\phi(A|B)$ is increasing in A by the definition. In order to prove that $h_\phi(A|B)$ is decreasing in B, let B_1, B_2 be von Neumann subalgebras with $B_1 \subset B_2$ and let $\iota : B_1 \hookrightarrow B_2$ be the inclusion map, then $\psi|_{B_1} = \psi|_{B_2} \circ \iota$ for any positive linear functional ψ of M so that

$$\begin{aligned} S((\phi_i \circ E_A) \mid_{B_1}, \phi \circ E_A \mid_{B_1}) &= S((\phi_i \circ E_A) \mid_{B_2} \circ \iota, \phi \circ E_A \mid_{B_2} \circ \iota) \\ &\leq S((\phi_i \circ E_A) \mid_{B_2}, \phi \circ E_A \mid_{B_2}). \end{aligned}$$

This implies that $h_\phi(A|B)$ is decreasing in B.
(4) Let $\gamma_n : A_n \hookrightarrow A$ be the embedding map, then $\psi|_{A_n} = \psi \circ \gamma_n$ for any functional ψ of A. Let E_n be the τ-conditional expectation of M onto A_n, then $\psi \circ \gamma_n \circ E_n \to \psi$ weakly* for any positive linear functional ψ of A. So by [13, Corollary 2.3.5],

$$S(\phi_i|_{A_n}, \phi|_{A_n}) \nearrow S(\phi_i|_A, \phi|_A).$$

Since by the definition

$$h_\phi(A_n|B) = \sup_{(\phi_i)\in\Phi(A)} \sum_i (S(\phi_i \circ \gamma_n, \phi \circ \gamma_n) - S((\phi_i \circ E_A) \mid_B, \phi \circ E_A \mid_B)),$$

we have $h_\phi(A_n|B) \nearrow h_\phi(A|B)$. Similarly, $h_\phi(A|B_n) \searrow h_\phi(A|B)$.
(5) Assume $\phi = \tau$. Let $(\phi_i) \in \Phi$, then $\phi_i \leq \tau$ so that there is an $a_i \in A$ with $\phi_i|_A(x) = \tau(a_i x)$ for all $x \in A$. Then $(a_i) \in S(A)$, and by [13, Theorem 2.3.1(x)]

$$S(\phi_i|_A, \tau|_A) = \tau(a_i \log(a_i)).$$

On the other hand, for every $y \in B$, we have that

$$\phi_i \circ E_A(y) = \tau(a_i E_A(y)) = \tau(a_i y) = \tau(E_B(a_i)y),$$

where E_B is also τ-conditional expectation onto B. This shows that for all $y \in B$

$$(\phi_i \circ E_A)|_B(y) = \tau(E_B(a_i)y)$$

and again by [13, Theorem 2.3.1 (x)]

$$S(\phi_i \circ E_A|_B, \tau \circ E_A|_B) = \tau(E_B(a_i) \log E_B(a_i)).$$

Hence

$$h_\tau(A|B) = \sup_{(a_i)\in S(A)} \sum_i (\tau\eta E_B(a_i) - \tau\eta(a_i))$$

which is $h(A|B)$.
(6) By [13, 2.2.1], we have that

$$S(\phi_i \mid_A, \phi \mid_A) = S(\phi_i \mid_B, \phi \mid_B) + S(\phi_i \mid_A, \phi_i \mid_A \circ \mathcal{E}_B).$$

By the assumption that $B \subset A$ it is clear that $(\phi_i \circ E_A) \mid_B = \phi_i \mid_B$. Hence we have that

$$\begin{aligned} h_\phi(A|B) &= \sup_{(\phi_i)\in\Phi} \sum_i (S(\phi_i \mid_A, \phi \mid_A) - S((\phi_i \circ E_A) \mid_B, (\phi \circ E_A) \mid_B)) \\ &= \sup_{(\phi_i)\in\Phi} S(\phi_i \mid_B, \phi \mid_B) + S(\phi_i \mid_A, \phi_i \mid_A \circ \mathcal{E}_B) - S(\phi_i \mid_B, \phi \mid_B) \\ &= \sup_{(\phi_i)\in\Phi} S(\phi_i|_A, \phi_i \circ \mathcal{E}_B). \end{aligned}$$

(7) First we assume that A and B are finite dimensional subalgebras of M. Then A (resp. B) is generated by mutually orthogonal minimal projections $\{e_i\}_{i=1}^m$ (resp. $\{f_j\}_{j=1}^n$). Then the computation in the proof of Lemma 3.1 shows

$$h_\phi(A|B) = \sum_i \phi(\eta(E_B(e_i))) = \sum_i \sum_j \eta(\frac{\phi(f_j e_i)}{\phi(f_j)})\phi(f_j).$$

This corresponds to the conditional entropy for two finite measurable partitions of X, for which $M = L^\infty(X, \mu)$ and ϕ is the state arising from μ, so that $h_\phi(A|B) = H_\phi(A|B)$ for finite-dimensional subalgebras A and B.

The property (4) holds also for Connes relative entropy $H_\phi(A|B)$. Hence, by using the property (4), we have that $h_\phi(A|B) = H_\phi(A|B)$ for all subalgebras A and B of M. ∎

Here, we modify the form $H_\phi(A)$ of the *mutual entropy* for a von Neumann subalgebra A of M (cf. [13]) and let

$$h_\phi(A) = \sup_{(\phi_i)\in\Phi(A)} \sum_i (\eta(\phi_i(1)) + S(\phi_i|_A, \phi|_A)).$$

Clearly, we have that $0 \le h_\phi(A) \le H_\phi(A)$.

In the case that ϕ is the trace τ, we denote $h_\phi(A)$ by $h(A)$ which is given by

$$h(A) = \sup_{(x_i)\in S(A)} \sum_i (\eta(\tau(x_i)) - \tau\eta(x_i)).$$

The following relations hold among $h_\phi(A|B)$, $h_\phi(A)$ and $H_\phi(A|B)$:

Proposition 3.1. [5] *Let ϕ be a normal state of a finite von Neumann algebra M with a fixed normal normalized trace τ. Let A and B be von Neumann subalgebras of M. Then*

(1)

$$h_\phi(A|B) \le h_\phi(A|\mathbb{C}\mathbf{1}) = h_\phi(A).$$

(2) *Assume that $E_A E_B = E_B E_A$. Then*

$$h_\phi(A|B) = h_\phi(A|A\cap B);$$

in particular, if $A \cap B = \mathbb{C}$, then

$$h_\phi(A|B) = h_\phi(A).$$

(3) *Let A and B be von Neumann subalgebras of M with $B \subset A$. Then*

$$H_\phi(A|B) = h_\phi(A|B).$$

(4) *Assume that $E_A E_B = E_B E_A$. Then*

$$H(A|B) = h(A|B).$$

Moreover, if $\phi = \phi \circ E_A$, then

$$H_\phi(A|B) = H_\phi(A|A\cap B) \quad \textit{and} \quad H_\phi(A|B) = h_\phi(A|B).$$

We need the following lemma in proving the above proposition.

Lemma 3.3. [5] *Let A and B be von Neumann subalgebras of M, and let ψ, ϕ be positive linear functionals on M. If $E_A E_B = E_B E_A$, then*

$$S((\psi \circ E_A) \mid_B, (\phi \circ E_A) \mid_B) = S((\psi \circ E_A) \mid_{A \cap B}, (\phi \circ E_A) \mid_{A \cap B}).$$

4. Relative entropy for maximal abelian subalgebras of matrices and the entropy of unistochastic matrices

In this section, we restrict our investigation to the maximal abelian subalgebras (abbreviated as MASA's) of the $n \times n$ complex matrices $M_n(\mathbb{C})$.

If A and B are two MASA's of $M_n(\mathbb{C})$, then there exists a unitary matrix u with $B = uAu^*$, which we denote by $u(A, B)$.

The aim of this section is to discuss the notion of the entropy for unistochastic matrices from the operator algebraic viewpoint.

4.1. Unistochastic matrices and the entropy

When a matrix $x \in M_n(\mathbb{C})$ is given, we denote the (i, j)-component of x by $x(i, j)$. A matrix $b \in M_n(\mathbb{C})$ is called *bistochastic* if

$$b(i, j) \geq 0, \quad \text{and} \quad \sum_i b(i, j) = \sum_j b(i, j) = 1, \ (\forall i, j).$$

Let $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ be a probability vector.

The weighted entropy $H_\lambda(b)$ of a bistochastic matrix b with respect to λ and the entropy $H(b)$ for a bistochastic matrix b is defined by Życzkowski, Kuś, Słomczyński and Sommers ([21]) in the following forms, respectively:

$$H_\lambda(b) = \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} \eta(b(j, k)),$$

and

$$H(b) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(b(i, j)).$$

A typical example of bistochastic matrix b is the unistochastic matrix $b(u)$ defined from a unitary matrix u by

$$(b(u))(i, j) = |u(i, j)|^2, \ (\forall i, j).$$

It is clear that $b(u)$ is bistochastic.

Relation to Schur's Lemma

What is the meaning of the entropy $H(b(u))$ from the theory of operator algebras?

We would like to consider that it should be an invariant related to inner automorphisms. For example, we just remember Schur's Lemma. Let

$$\lambda = (\lambda_1, \cdots, \lambda_n) \quad (\lambda_i \in \mathbb{R})$$

and let

$$d = (d_1, \cdots, d_n) \quad (d_i \in \mathbb{R}).$$

Then Schur's Lemma says that if λ is the eigenvalue sequence of a self-adjoint matrix $a \in M_n(\mathbb{C})$, and if d is the diagonal sequence of a, then λ majorizes d, that is, for each k with $1 \leq k \leq n$,

$$\sum \text{ the } k \text{ largest } \lambda' s \;\geq \sum \text{ the } k \text{ largest } d' s,$$

and

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i.$$

These were shown by using the fact that there exists a unitary matrix $u = (u(i,j))_{ij}$ such that

$$d_i = \sum_j |u(i,j)|^2 \lambda_j, \quad j = 1, \cdots, n.$$

Thus a unitary u and the unistochastic matrix $b(u)$ defined by u appeared in the step to get a diagonal matrix from d with $d = d^*$.

If we put $H(Adu) = H(b(u))$ for the automorphism $Adu(x) = uxu^*$, then we have the following:

Lemma 4.1. *Let M be the $M_n(\mathbb{C})$. Consider automorphisms $\theta = Adu$ for unitary matrices u in M. Then*

$$H(\theta^{-1}) = H(\theta),$$

Proof. Let $w \in M$ be a unitary with $Adw = \theta^{-1}$. Then $w = \gamma u^*$ for some $\gamma \in \mathbb{T}$. Hence $w(i,j) = \gamma \overline{u(j,i)}$ for each $i, j = 1, \cdots, n$. Here $w(i,j)$ is the (i,j)-component of the matrix w. It is clear that

$$H(\theta^{-1}) = \frac{1}{n} \sum_{i,j} \eta(|w(i,j)|^2) = \frac{1}{n} \sum_{i,j} \eta(|u(i,j)|^2) = H(\theta). \quad ∎$$

4.2. Notions of entropy in finite dimensional cases

Several kinds of notions for entropy in the above section are given in the case of finite-dimensional algebras as follows, which we use for our computations in this section.

Let M be a finite-dimensional von Neumann algebra, so M is a finite direct sum of full matrix algebras. We denote by Tr the standard trace on M, that is, $\mathrm{Tr}(p) = 1$ for every minimal projection $p \in M$. Let ϕ be a positive linear functional on M. We denote Q_ϕ the density operator of ϕ, that is, $Q_\phi \in M$ is a unique positive operator with

$$\phi(x) = \mathrm{Tr}(Q_\phi x) \quad (x \in M).$$

Von Neumann entropy $S(\phi)$.
The von Neumann entropy $S(\phi)$ of a positive linear functional ϕ is given by

$$S(\phi) = \mathrm{Tr}(\eta(Q_\phi)),$$

Lemma 4.2. *Let A be a von Neumann subalgebra of M. Then*

(1) *The density matrix for $\phi \circ E_A$ is $E_A(Q_\phi)$.*
(2) *$\phi = \phi \circ E_A$ if and only if $E_A(Q_\phi) = Q_\phi$.*

Proof.
(1) Since $\mathrm{Tr}(E_A(Q_\phi)x) = \mathrm{Tr}(E_A(Q_\phi)E_A(x)) = \mathrm{Tr}(Q_\phi E_A(x)) = \phi \circ E_A(x) = \mathrm{Tr}(Q_{\phi\circ E_A}x)$ for all $x \in M$, we have that $Q_{\phi\circ E_A}$ is $E_A(Q_\phi)$.
(2) It is obvious from (1). ∎

$S(\phi, \psi)$ in finite dimensional cases.
Let ψ and φ be two positive linear functionals on C. If $\psi \leq \lambda\varphi$ for some $\lambda > 0$, then the relative entropy of ψ and φ is given as

$$S(\psi, \varphi) = \mathrm{Tr}(Q_\psi(\log Q_\psi - \log Q_\varphi)),$$

(cf. [13, 14]).

Assume that M is finite dimensional and that the density matrix of ϕ is contained in A. Let $\Phi(A) \subset \Phi$ be the set of all finite families (ϕ_i) whose density operators $(Q_i)_i$ are contained in A. Since the density matrix of $\phi_i \circ E_A$ is $E_A(Q_i)$, we have

$$h_\phi(A|B) = \sup_{(\phi_i)\in\Phi(A)} \sum_i (S(\phi_i\mid_A, \phi\mid_A) - S(\phi_i\mid_B, \phi \circ E_A\mid_B)).$$

By using this formula, we have the results when $M = M_n(\mathbb{C})$ in the following section.

4.3. Relations between $h_\phi(A|uAu^*)$ and $H(b(u))$

Let ϕ be a positive linear functional on $M_n(\mathbb{C})$. We set the eigenvalues of Q_ϕ as $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. Let us decompose Q_ϕ into the form $Q_\phi = \sum_{i=1}^n \lambda_i e_i$, where $\{e_1, \cdots, e_n\}$ are mutually orthogonal minimal projections in $M_n(\mathbb{C})$, which we fix. Let D_ϕ be the MASA generated by the projections $\{e_1, \cdots, e_n\}$.

Let $\{e_{kl}\}_{k,l=1,\cdots,n}$ be a system of matrix units of $M_n(\mathbb{C})$ such that $e_{ii} = e_i$ for all $i = 1, \cdots, n$. We give the matrix representation for each $x \in M_n(\mathbb{C})$ depending on these matrix units $\{e_{kl}\}_{k,l=1,\cdots,n}$ so that D_ϕ is the diagonal algebra.

Theorem 4.1. [4] *Let ϕ be a state of $M_n(\mathbb{C})$, and let $u \in M_n(\mathbb{C})$ be a unitary. Then*

$$h_\phi(D \mid uDu^*) = H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uDu^*}).$$

The following holds from the relation of $h_\tau(A|B) = h(A|B)$:

Corollary 4.1. [4] *Let D be the algebra of diagonal $n \times n$ matrices, and let $u \in M_n(\mathbb{C})$ be a unitary. Then*

$$\begin{aligned} h(D \mid uDu^*) = H(b(u)) &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \eta(\mid u(i,j)\mid^2) \\ &= \max_\phi h_{\phi_v}(D \mid uDu^*), \end{aligned}$$

where the maximum is taken over all states ϕ of $M_n(\mathbb{C})$ and ϕ_v is the state given by the inner perturbation of ϕ by v : $\phi_v(x) = \phi(vxv^*)$.

Corollary 4.2. [4] *Let A and B be maximal abelian subalgebras of $M_n(\mathbb{C})$. Then*

$$h(A \mid B) = H(b(u)) = \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \eta(\mid u(i,j)\mid^2)$$

where the $(u(i,j))_{ij}$ *for* $u = u(A,B)$ *is given in the matrix units whose minimal projections generate* A.

In [18], Popa showed five equivalent conditions for a pair $\{A, B\}$ of von Neumann subalgebras of a finite von Neumann algebra M. If one of these five equivalent conditions is satisfied then the $\{A, B\}$ is called an *orthogonal* pair. Such a pair is also called *complementary* subalgebras in [16].

One of the most algebraic condition for "orthogonality" is that two conditional expectations E_A and E_B commute and that $A \cap B$ is trivial.

The following corollary shows that two maximal abelian subalgebras of $M_n(\mathbb{C})$ are most separated if they are orthogonal pair.

Corollary 4.3. [4] *Let* $\{A_0, B_0\}$ *be a pair of maximal abelian subalgebras of* $M_n(\mathbb{C})$. *Then* $\{A_0, B_0\}$ *is an orthogonal pair if and only if*

$$h(A_0 \mid B_0) = \log n = \max\ h(A \mid B),$$

where the maximum is taken over the set of pairs $\{A, B\}$ *of maximal abelian subalgebras of* $M_n(\mathbb{C})$.

Remark 4.1. We remark on the above relation that $h(D|uDu^*) = H(b(u))$.

(1) In general, the relation $H(D|uDu^*) = H(b(u))$ does not hold (see [16, Appendix] for example).

(2) Our relation is generalized by Okayasu [15] as follows: For $i = 1, 2$, let $(e^i_j)_{j\in I_i}$ be a finite set of mutually orthogonal projections with $\sum_j e^i_j = 1$ in a finite von Neumann algebra M. Put $A_i = \{e^i_j\}''$. Then

$$h(A_1|A_2) = \sum_{j,k} \eta\tau(e^1_j e^2_k) - \eta\tau(e^2_k)$$

and $\{A_1, A_2\}$ is orthogonal iff $h(A_1|A_2) = H(A_1) = \sum_i \eta\tau(e^1_i)$.

5. Entropy for Conjugate Pairs of Subfactors

In this section, we replace $M_n(\mathbb{C})$ in Section 4 by II_1 factors. The results in Section 4 suggest that if A, B are maximal abelian subalgebras of a II_1 factor, then $h(A|B) = \infty$. We would like to discuss pairs $\{A, B\}$ of a II_1 factor with finite value $h(A|B)$.

For a *-endomorphism σ of a finite von Neumann algebra N, some relation between the entropy $H(\sigma)$ for σ and the relative entropy $H(N \mid \sigma(N))$ was obtained in the papers [1, 2, 3, 9, 19]. The relation is, roughly speaking, that

$$H(\sigma) = H(N \mid \sigma(N))$$

under a certain condition. Many classes of such endomorphisms σ can be extended to an automorphism α of a finite von Neumann algebra M which contains N as a von Neumann subalgebra. Some examples of such endomorphisms appeared in relation to Jones index theory of subfactors. In [2], we studied a nice class of such endomorphisms σ extendable to an automorphism α (which we called *basic* *-endomorphism) and showed that

$$H(\alpha) = H(N|\sigma(N)).$$

Now, by Proposition 3.1, we have that

$$H(\alpha) = H(N|\sigma(N)) = h(N|\sigma(N)) = h(N|\alpha(N)).$$

This means that, for an automorphism α of a II_1 factor M, we may be able to choose a subfactor $N \subset M$ such that the entropy for α is given as $h(N|\alpha(N))$.

Our discussion in this section is motivated by these results, and we develop the discussions in Section 4 by replacing type I_n factors $M_n(\mathbb{C})$ with type II_1 factors and maximal abelian subalgebras to subfactors with finite Jones index.

Let M be a type II_1 factor. The minimal value of the indices for non-trivial subfactor N of M is 2. If N is a subfactor of M with index 2, then M is decomposed into the crossed product of N by an outer automorphism with period 2.

Hence, for such an inclusion of factor-subfactor $N \subset M$, we would like to study the set of $h(N|uNu^*)$ where u is a unitary in M. Also, we study the inclusion of factor-subfactor $N \subset M$, where M is given as the crossed product of N by a finite group G.

Let N be a type II_1 factor, and let τ the canonical trace state. Let G be a finite group, and let α be an action of G on N such that α_g is outer for all $g \in G, g \neq 1$ the unit of G, so that if $\alpha_g(x)a = ax$ for all $x \in N$, then $a = 0$. Let

$$M = N \times_\alpha G,$$

the crossed product of N by G with respect to α. We identify N with the von Neumann subalgebra embedded in M, and denote by v the unitary representation of G in M such that every v_g is a unitary in M such that

$$\alpha_g(x) = v_g x v_g^*, \quad (x \in N, g \in G).$$

Then every $x \in M$ is written in the unique form:

$$x = \sum_{g \in G} x_g v_g, \quad (x_g \in N),$$

and $u = \sum_{g \in G} u_g v_g \in M$ is a unitary if and only if

$$\sum_{g \in G} u_{hg} \alpha_h(u_g^*) = \delta_{h,1} \quad \text{and} \quad \sum_{g \in G} \alpha_g^{-1}(u_g^* u_{gh}) = \delta_{h,1},$$

where we denote the identity of G by 1.

5.1. Entropy for Inner Automorphisms

Let $u \in M$ be a unitary. We define the entropy of the inner automorphism $\mathrm{Ad}u$ of M with respect to N.

Definition 5.1. [5]. *Let $M = N \times_\alpha G$, where N is a type II_1 factor, G is a finite group and the action α is outer. Then entropy of the inner automorphism* Adu *of M with respect to N is given by*

$$H_N(\mathrm{Ad}u) = \sum_{g \in G} \eta\tau(u_g u_g^*).$$

To understand the value $\sum_g \eta\tau(u_g u_g^*)$, we consider the case of $G = \mathbb{Z}_n$. Then M is the crossed product of N by the cyclic group generated by an outer automorphism α with the outer period n. The matrix representation of $u \in M$ is given by $u = (u(i,j))_{ij}$ where $u(i,j) \in N$ and

$$u(i,j) = \alpha^{i-1}(u(i-1, j-1)) \quad (1 \leq i, j \leq n).$$

Hence $\sum_g \eta\tau(u_g u_g^*)$ is considered as a generalization of the entropy of the unistochastic matrix.

The quantity $H_N(\mathrm{Ad}u)$ gives that:

If θ is an inner automorphism $\mathrm{Ad}u$, then

$$H_N(\theta^{-1}) = H_N(\theta).$$

Theorem 5.1. [5] *The* $H_N(\mathrm{Adu})$ *gives an upper bound for* $h(N|uNu^*)$ *:*

$$h(N|uNu^*) \leq H_N(\mathrm{Adu}) = \sum_{\mathrm{g}} \eta\tau(\mathrm{u_g u_g^*}).$$

5.2. Case $G = \mathbb{Z}_n$

Let N be a II_1 factor with the canonical tracial state τ, and let α be an automorphism on N such that α^n is the identity and α^i is outer for $i = 1, \cdots, n-1$. Let $M = N \times_\alpha \mathbb{Z}_n$.

Remark 5.1. If an inclusion $N \subset M$ of type II_1 factors with finite index is irreducible (for example $N' \cap M = \mathbb{C}1$), then $H(M|N) = \log[M : N]$ by [17]. Hence, under the assumption of this subsection, we have that for a unitery $u \in M$

$$h(N|uNu^*) \leq H(N|uNu^*) \leq H(M|uNu^*) = H(M|N) = \log n.$$

Is it possible for some unitary to satisfy the equality?

The following shows the existence of a unitary u such that $h(N|uNu^*)$ of the pair $\{N, uNu^*\}$ is able to reach $\log n$, that is, the set $\{\{N, uNu^*\}; u \text{ unitary} \in M\}$ is rich.

Theorem 5.2. [5] *There exists a unitary* $u \in M$ *which satisfies the following:*

(1) $h(N|uNu^*) = \log n$.
(2) *The conditional expectations* E_N *and* E_{uNu^*} *commute.*
(3) $\sqrt{n}E_N(u)$ *is a unitary* w *with* $\alpha(w) = \gamma w$, *where* γ *is a primitive* n*-th root.*

5.3. The special case $G = \mathbb{Z}_2$

Let us consider the special case of $G = \mathbb{Z}_2$ (which is our main purpose). Then M is the crossed product of a II_1 factor N by an outer automorphism α on N with period 2. Here, we give a continuous family of conjugate pairs of subfactors with index 2 induced by inner automorphisms as follows:

Theorem 5.3. [5] *Let* $N \subset M$ *be the above. Then for every* $\lambda \in [0,1]$, *we have a unitary* $u(\lambda) \in M$ *such that*

$$h(N \mid u(\lambda)\ N\ u(\lambda)^*) = H_N(\mathrm{Adu}(\lambda)) = \eta(\lambda) + \eta(1-\lambda).$$

The square

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ N \cap u(\lambda)Nu(\lambda)^* & \subset & u(\lambda)Nu(\lambda)^* \end{array}$$

is a commuting square in the sence of [8] *(i.e., $E_N E_{u(\lambda)Nu(\lambda)^*} = E_{u(\lambda)Nu(\lambda)^*} E_N$) if and only if $\lambda = 1/2$.*

Corollary 5.1. [5] *Let $N \subset M$ and the unitary $u(\lambda)$ be as above. Then*

$$h(N \mid u(1/2)\ N\ u(1/2)^*) = \max_\lambda h(N \mid u(\lambda)\ N\ u(\lambda)^*) = \log 2.$$

We remark that the above results in 5.4 correspond to the relations between $h(D|uDu^*)$ and $H(b(u))$ in Section 4.4.

Note 5.1. Finally we remark that in the case where M is given as the crossed product $N \times_\alpha G$, it is possible for $h(A|B)$ to attain the maximum by “mall A”.

For example we have:

Example 5.1. [5]. If A a von Neumann subalgebra of M contains the unitary group u_G and if B is a von Neumann subalgebra of M contained in N, then

$$h(A|B) = \log|G| = H(M|N) = \log[M : N].$$

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