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Isomorphism Theorems of Polygroups

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Abstract. A polygroup is a multivalued algebraic system satisfying the group like axioms. In recently, polygroups have been investigated by a number of authors because such groups are related to algebraic combinatorics, color schemes and relations, etc. In this paper, three isomorphism theorems of polygroups will be established and the Fundamental homomorphism theorem of polygroups is also proved. Our results generalize the classical isomorphism theorems of groups to polygroups.

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1. Introduction

The concepts of hyperstructure and hypergroup were first introduced by Marty in 1934 [13]. As a special hypergroup, S. D. Comer considered polygroups and pointed out that polygroups have application in color schemes [3, 4]. He also developed the algebraic theory for polygroups. In recent years, the author and Poursalavati [8] introduced the matrix representations of polygroups over hyperrings and investigated the structure of polygroup hyperrings so that some results of group rings are generalized. Davvaz in [9], using the concept of generalized permutation defined permutation polygroups, also see [10].

In this paper, we consider the normal subpolygroups and strong homomorphisms between polygroups. As a consequence, by using the obtained results, we establish the isomorphism theorems of polygroups. Our results extend the classical results of groups to polygroups. Moreover, by considering the fundamental relation β^* on a polygroup, we prove the fundamental theorem for polygroups.

For notations and terminologies not given in this paper, the reader is referred to the monographs of Corsini and Leoreanu [5, 6].

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2. Subpolygroups and strong homomorphisms

First, we summarize the preliminary definitions and results required in the sequel. Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid.* If A and B are non-empty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \ \text{ and } \ A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, we have $x \circ H = H \circ x = H$.

A polygroup is a special case of a hypergroup. We now give the necessary definitions.

Definition 2.1. *A polygroup is a system $\mathcal{P} =< P, ., e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\cdot$ maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all x, y, z in P:*

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(ii) $e \cdot x = x \cdot e = x$;
(iii) $x \in y \cdot z$ *implies* $y \in x \cdot z^{-1}$ *and* $z \in y^{-1} \cdot x$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ where $A^{-1} = \{a^{-1} | \ a \in A\}$.

Example 2.1. *Double coset algebra.* Suppose that H is a subgroup of a group G. Define a system $G//H =< \{HgH \mid g \in G\}, *, H, ^{-I} >$, where $(HgH)^{-I} = Hg^{-1}H$ and

$$(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H \ |h \in H\}.$$

The algebra of double cosets $G//H$ is a polygroup introduced in Dresher and Ore [12].

Example 2.2. *Prenowitz algebras.* Suppose G is a projective geometry with a set P of points and suppose, for $p \neq q$, $\overline{pq}$ denoted the set of all points on the unique line through p and q. Choose an object $I \notin P$ and form the system

$$P_G =< P \cup \{I\}, \cdot, I, ^{-1} >$$

where $x^{-1} = x$ and $I \cdot x = x \cdot I = x$ for all $x \in P \cup \{I\}$ and for $p, q \in P$,

$$p \cdot q = \begin{cases} \overline{pq} - \{p, q\} & \text{if } p \neq q \\ \{p, I\} & \text{if } p = q. \end{cases}$$

P_G is a polygroup [15].

Example 2.3. *Conjugacy class polygroups.* In dealing with a symmetry group two symmetric operations belong to the same class if they present the same map with respect to (possibly) different coordinate systems where one coordinate system is converted into the other by a member of the group. In the language of group theory this means the elements a, b in a symmetric group G belong to the same class if there exists a $g \in G$ such that $a = gbg^{-1}$, i.e., a and b are conjugate. The collection of all conjugacy classes of a group G is denoted by $\overline{G}$ and the system $< \overline{G}, *, \{e\}, ^{-1} >$ is a polygroup where e is the identity of G and the product $A * B$ of conjugacy classes A and B consists of all conjugacy classes contained in the elementwise product AB. This hypergroup was recognized by Campaigne [2] and Diatzman [11].

Now, we illustrate constructions using the dihedral group D_4. This group is generated by a counter-clockwise rotation r of 90° and a horizontal reflection h. The group consists of the following 8 symmetries:

$$\{1 = r^0, r, r^2 = s, r^3 = t, h, hr = d, hr^2 = v, hr^3 = f\}.$$

The dihedral groups occur frequently in art and nature. Many of the decorative designs used on floor coverings, pottery, and buildings have one of the dihedral groups as a group of symmetry. In the case of D_4 there are five conjugacy classes: $\{1\}, \{s\}, \{r, t\}, \{d, f\}$ and $\{h, v\}$. Let us denote these classes by $\mathcal{C}_1, \ldots, \mathcal{C}_5$ respectively. Then the polygroup $\overline{D}_4$ is

$*$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}_1$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
$\mathcal{C}_3$	$\mathcal{C}_3$	$\mathcal{C}_3$	$\mathcal{C}_1, \mathcal{C}_2$	$\mathcal{C}_5$	$\mathcal{C}_4$
$\mathcal{C}_4$	$\mathcal{C}_4$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_1, \mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{C}_5$	$\mathcal{C}_5$	$\mathcal{C}_5$	$\mathcal{C}_4$	$\mathcal{C}_3$	$\mathcal{C}_1, \mathcal{C}_2$

As a sample of how to calculate the table entries, consider $\mathcal{C}_3 \cdot \mathcal{C}_3$. To determine this product, compute the elementwise product of the conjugacy classes $\{r, t\}\{r, t\} = \{s, 1\} = \mathcal{C}_1 \cup \mathcal{C}_2$. Thus $\mathcal{C}_3 \cdot \mathcal{C}_3$ consists of the two conjugacy classes $\mathcal{C}_1$, $\mathcal{C}_2$ (see [3]).

For $a, b \in P$, we write the product of a, b by ab instead of $a \cdot b$.

Definition 2.2. *A non-empty subset K of a polygroup P is said to be a subpolygroup of P if, under the hyperoperation in P, K itself forms a polygroup.*

It would be useful to have some criterion for deciding whether a given subset of a polygroup is a subpolygroup. This is the purpose of the next lemma.

Lemma 2.1. *A non-empty subset K of a polygroup P is a subpolygroup of P if and only if*

(i) *$a, b \in K$ implies $ab \subseteq K$;*
(ii) *$a \in K$ implies $a^{-1} \in K$.*

Definition 2.3. *The subpolygroup N of P is normal in P if and only if $a^{-1}Na \subseteq N$ for all $a \in P$.*

The following corollaries are direct consequences of Definitions 2.1–2.3.

Corollary 2.1. *Let N be a normal subpolygroup of P. Then*

(i) *$Na = aN$ for all $a \in P$;*
(ii) *$(Na)(Nb) = Nab$ for all $a, b \in P$;*
(iii) *$Na = Nb$ for all $b \in Na$.*

Corollary 2.2. *Let K and N be subpolygroups of a polygroup P with N normal in P. Then*
(i) *$N \cap K$ is a normal subpolygroup of K;*
(ii) *$NK = KN$ is a subpolygroup of P;*
(iii) *N is a normal subpolygroup of NK.*

Definition 2.4. *If N is a normal subpolygroup of P, then we define the relation $x \equiv y \pmod N$ if and only if $xy^{-1} \cap N \neq \emptyset$. This relation is denoted by $xN_P y$.*

Lemma 2.2. *The relation N_P is an equivalence relation.*

Proof.
(i) Since $e \in xx^{-1} \cap N$ for all $x \in P$; then $xN_P x$, i.e., N_P is reflexive.
(ii) Suppose that $xN_P y$. Then there exists $z \in xy^{-1} \cap N$ which implies $z^{-1} \in yx^{-1}$ and $z^{-1} \in N$, this means that $yN_P x$, and so N_P is symmetric.
(iii) Let $xN_P y$ and $yN_P z$ where $x, y, z \in P$. Then there exist $a \in xy^{-1} \cap N$ and $b \in yz^{-1} \cap N$. So $x \in ay$ and $z^{-1} \in y^{-1}b$, then $z^{-1}x \subseteq y^{-1}bay$. Since $ba \subseteq N$ and N is a normal subpolygroup, then $y^{-1}bay \subseteq N$. Therefore $z^{-1}x \cap N \neq \emptyset$, which satisfies the condition for $xN_P z$, and so N_P is transitive. ∎

Let $N_P(x)$ be the equivalence class of the element $x \in P$. Suppose that $[P : N] = \{N_P(x) \mid x \in P\}$. On $[P : N]$ we consider the hyperoperation $\odot$ defined as follows: $N_P(x) \odot N_P(y) = \{N_P(z) \mid z \in N_P(x)N_P(y)\}$. For a subpolygroup K of P and $x \in P$, denote the right coset of K by Kx and let P/K be the set of all right cosets of K in P.

Lemma 2.3. *Let N be a normal subpolygroup of P. Then $Nx = N_P(x)$.*

Proof. Suppose that $y \in Nx$. Then there exists $n \in N$ such that $y \in nx$, which implies that $n \in yx^{-1}$, and so $yx^{-1} \cap N \neq \emptyset$. Thus $Nx \subseteq N_P(x)$. Similarly we have $N_P(x) \subseteq Nx$. ∎

Therefore we conclude that $[P : N] = P/N$.

Lemma 2.4. *Let N be a normal subpolygroup of P. Then for all $x, y \in P$, $Nxy = Nz$* for all *$z \in xy$.*

Proof. Suppose that $z \in xy$. Then it is clear that $Nz \subseteq Nxy$. Now, let $a \in Nxy$. Then, by condition (iii) of Definition 2.1, we get $y \in (Nx)^{-1}a$ or $y \in x^{-1}Na$, and so $xy \subseteq xx^{-1}Na$. Since N is a normal subpolygroup, we obtain $xy \subseteq xNx^{-1}a \subseteq Na$. Therefore for every $z \in xy$, we have $z \in Na$ which implies $a \in Nz$. This complete the proof. ∎

Corollary 2.3. *For all $x, y \in P$, we have $N_P(N_P(x)N_P(y)) = N_P(x)N_P(y)$.*

Definition 2.5. [3] *An equivalence relation ρ on a polygroup P is called a conjugation on P if*
(i) *$x\rho y$ implies $x^{-1}\rho y^{-1}$;*
(ii) *$z \in xy$ and $z'\rho z$ implies $z' \in x'y'$ for some $x'\rho x$ and $y'\rho y$.*

Lemma 2.5. [3] *ρ is a conjugation of P if and only if*

(i) $\rho(x)^{-1} = \{y^{-1} \mid y \in \rho(x)\} = \rho(x^{-1})$;
(ii) $\rho(\rho(x)y) = \rho(x)\rho(y)$.

Corollary 2.4. *The equivalence relation N_P is a conjugation on P.*

Proposition 2.1. *$< [P : N], \odot, N_P(e), ^{-I} >$ is a polygroup, where $N_P(a)^{-I} = N_P(a^{-1})$.*

Proof. For all $a, b, c \in P$, we have

$$\begin{aligned}
(N_P(a) \odot N_P(b)) \odot N_P(c) &= \{N_P(x) \mid x \in N_P(a)N_P(b)\} \odot N_P(c) \\
&= \{N_P(y) \mid y \in N_P(x)N_P(c),\ x \in N_P(a)N_P(b)\} \\
&= \{N_P(y) \mid y \in N_P(N_P(a)N_P(b))N_P(c)\} \\
&= \{N_P(y) \mid y \in (N_P(a)N_P(b))N_P(c)\}, \\
N_P(a) \odot (N_P(b) \odot N_P(c)) &= N_P(a) \odot \{N_P(x) \mid x \in N_P(b)N_P(c)\} \\
&= \{N_P(y) \mid y \in N_P(a)N_P(x),\ x \in N_P(b)N_P(c)\} \\
&= \{N_P(y) \mid y \in N_P(a)N_P(N_P(b)N_P(c))\} \\
&= \{N_P(y) \mid y \in N_P(a)(N_P(b)N_P(c))\}.
\end{aligned}$$

Since $(N_P(a)N_P(b))N_P(c) = N_P(a)(N_P(b)N_P(c))$, we get $(N_P(a) \odot N_P(b)) \odot N_P(c) = N_P(a) \odot (N_P(b) \odot N_P(c))$. Therefore, $\odot$ is associative. It is easy to see that $N_P(e)$ is the unit element in $[P : N]$, and $N_P(x^{-1})$ is the inverse of the element $N_P(x)$. Now, we show that $N_P(c) \in N_P(a) \odot N_P(b)$ implies $N_P(a) \in N_P(c) \odot N_P(b^{-1})$ and $N_P(b) \in N_P(a^{-1}) \odot N_P(c)$.

We have $N_P(c) \in N_P(a) \odot N_P(b)$, and hence $N_P(c) = N_P(x)$ for some $x \in N_P(a)N_P(b)$. Therefore, there exist $y \in N_P(a)$ and $z \in N_P(b)$ such that $x \in yz$, so $y \in xz^{-1}$. This implies that $N_P(y) \in N_P(x) \odot N_P(z^{-1})$, and so $N_P(a) \in N_P(c) \odot N_P(b^{-1})$. Similarly, we get $N_P(b) \in N_P(a^{-1}) \odot N_P(c)$. Therefore $[P : N]$ is a polygroup. ∎

Corollary 2.5. *If N is a normal subpolygroup of P, then $< P/N, \odot, N, ^{-I} >$ is a polygroup, where $Nx \odot Ny = \{Nz \mid z \in xy\}$ and $(Nx)^{-I} = Nx^{-1}$.*

Definition 2.6. *Let $< P_1, \cdot, e_1, ^{-1} >$ and $< P_2, *, e_2, ^{-I} >$ be polygroups. A mapping φ from P_1 into P_2 is said to be a strong homomorphism if for all $a, b \in P_1$,*

i) $\varphi(e_1) = e_2$;
(ii) $\varphi(ab) = \varphi(a) * \varphi(b)$.

Clearly, a strong homomorphism φ is an isomorphism if φ is one to one and onto. We write $P_1 \cong P_2$ if P_1 is isomorphic to P_2.

Because P_1 is a polygroup, $e \in aa^{-1}$ for all $a \in P_1$, then we have $\varphi(e_1) \in \varphi(a) * \varphi(a^{-1})$ or $e_2 \in \varphi(a) * \varphi(a^{-1})$ which implies $\varphi(a^{-1}) \in \varphi(a)^{-1} * e_2$, therefore $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in P_1$. Moreover, if φ is a strong homomorphism from P_1 into P_2, then the kernel of φ is the set $\operatorname{Ker}\varphi = \{x \in P_1 \mid \varphi(x) = e_2\}$. It is trivial that $\operatorname{Ker}\varphi$ is a subpolygroup of P_1 but in general is not normal in P_1.

Corollary 2.6. *Let φ be a strong homomorphism from P_1 into P_2. Then φ is injective if and only if $\operatorname{Ker}\varphi = \{e_1\}$.*

Proof. Let $y, z \in P_1$ be such that $\varphi(y) = \varphi(z)$. Then $\varphi(y) * \varphi(y^{-1}) = \varphi(z) * \varphi(y^{-1})$. It follows that $\varphi(e_1) \in \varphi(yy^{-1}) = \varphi(zy^{-1})$, and so there exists $x \in yz^{-1}$ such that $e_2 = \varphi(e_1) = \varphi(x)$. Thus, if $\operatorname{Ker}\varphi = \{e_1\}$, $x = e_1$, whence $y = z$. Now, let $x \in \operatorname{Ker}\varphi$. Then $\varphi(x) = e_2 = \varphi(e_1)$. Thus, if φ is injective, we conclude that $x = e_1$. ∎

We are now in a position to state and review the fundamental theorems in polygroup theory.

Theorem 2.1 (First Isomorphism Theorem)**.** *Let φ be a strong homomorphism from P_1 into P_2 with kernel K such that K is a normal subpolygroup of P_1. Then $P_1/K \cong Im\varphi$.*

Proof. We define $\psi : P_1/K \longrightarrow Im\varphi$ by setting $\psi(Kx) = \varphi(x)$ for all $x \in P_1$. It is easy to see that ψ is an isomorphism. ∎

Theorem 2.2 (Second Isomorphism Theorem)**.** *If K and N are subpolygroups of a polygroup P, with N normal in P, then $K/N \cap K \cong NK/N$.*

Proof. Since N is a normal subpolygroup of P, $NK = KN$. Consequently NK is a subpolygroup of P. Further $N = Ne \subseteq NK$ given that N is a normal subpolygroup of NK; consequently NK/N is defined. Define $\varphi : K \longrightarrow NK/N$ by $\varphi(k) = Nk$. φ is a strong homomorphism. Consider any $Na \in NK/N$, $a \in NK$. Now, $a \in NK$ given $a \in nk$ for some $n \in N$, $k \in K$. Thus, by Lemma 2.4, $Na = Nnk = Nk = \varphi(k)$. This shows that φ is also onto. If we can establish that $\operatorname{Ker}\varphi = N \cap K$, since $N \cap K$ is a normal subpolygroup of K, we shall get that $K/N \cap K \cong NK/N$. For any $k \in K$, $k \in \operatorname{Ker}\varphi \iff \varphi(k) = N \iff Nk = N \iff k \in N \iff k \in N \cap K$ (since $k \in K$), i.e., $k \in \operatorname{Ker}\varphi \iff k \in N \cap K$. This yields $\operatorname{Ker}\varphi = N \cap K$. Hence that results follows. ∎

Theorem 2.3 (Third Isomorphism Theorem)**.** *If K and N are normal subpolygroups of a polygroup P such that $N \subseteq K$, then K/N is a normal subpolygroup of P/N and $(P/N)/(K/N) \cong P/K$.*

Proof. We leave it to reader to verify that K/N is a normal subpolygroup of P/N. Further $\varphi : P/N \longrightarrow P/K$ defined by $\varphi(Nx) = Kx$ is a strong homomorphism of P/N onto P/K such that $\operatorname{Ker}\varphi = K/N$. ∎

Let $< P_1, \cdot, e_1, ^{-1} >$ and $< P_2, *, e_2, ^{-I} >$ be two polygroups. Then on $P_1 \times P_2$ we can define a hyperproduct as follows: $(x_1, y_1)o(x_2, y_2) = \{(x, y) \mid x \in x_1x_2,\ y \in y_1 * y_2\}$. We recall this as the direct hyperproduct of P_1 and P_2. Clearly, $P_1 \times P_2$ equipped with the usual direct hyperproduct becomes a polygroup.

Corollary 2.7. *If N_1, N_2 are normal subpolygroups of P_1, P_2 respectively, then $N_1 \times N_2$ is a normal subpolygroup of $P_1 \times P_2$ and $(P_1 \times P_2)/(N_1 \times N_2) \cong P_1/N_1 \times P_2/N_2$.*

Let P be a polygroup. We define the relation β^* as the smallest equivalence relation on P such that the quotient P/β^*, the set of all equivalence classes, is a group. In this case β^* is called the fundamental equivalence relation on P and P/β^* is called the fundamental group. The product $\otimes$ in P/β^* is defined as follows: $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$ for all $z \in \beta^*(x)\beta^*(y)$. This relation is studied by Corsini [6] concerning hypergroups, see also [16, 17]. Let $\mathcal{U}_P$ be the set of all finite products of elements of P. We define the relation β as follows: $x\beta y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}_P$. We have $\beta^* = \beta$ for hypergroups. Since polygroups are certain subclasses of hypergroups, we have $\beta^* = \beta$ [6, Theorem 81]. The kernel of the canonical map $\varphi : P \longrightarrow P/\beta^*$ is called the core of P and is denoted by ω_P. Here we also denote by ω_P the unit of P/β^*. It is easy to prove that the following statements: $\omega_P = \beta^*(e)$ and $\beta^*(x)^{-1} = \beta^*(x^{-1})$ for all $x \in P$.

Theorem 2.4. [9, Theorem 5.9] *Let β_1^*, β_2^* and β^* be fundamental equivalence relations on polygroups P_1, P_2 and $P_1 \times P_2$ respectively, then $(P_1 \times P_2)/\beta^* \cong P_1/\beta_1^* \times P_2/\beta_2^*$.*

Corollary 2.8. *If N_1, N_2 are normal subpolygroups of P_1, P_2 respectively, and β_1^*, β_2^* and β^* fundamental equivalence relations on $P_1/N_1,\ P_2/N_2$ and $(P_1 \times P_2)/(N_1 \times N_2)$ respectively, then*

$$((P_1 \times P_2)/(N_1 \times N_2))/\beta^* \cong (P_1/N_1)/\beta_1^* \times (P_2/N_2)/\beta_2^*.$$

Definition 2.7. *Let f be a strong homomorphism from P_1 into P_2 and let β_1^*, β_2^* be fundamental relations on P_1, P_2 respectively. Then we define*

$$\overline{\operatorname{Ker} f} = \{\beta_1^*(x) \mid x \in P_1,\ \beta_2^*(f(x)) = \omega_{P_2}\}.$$

Lemma 2.6. $\overline{\operatorname{Ker} f}$ *is a normal subgroup of the fundamental group P_1/β_1^*.*

Proof. Assume that $\beta_1^*(x), \beta_1^*(y) \in \overline{\operatorname{Ker} f}$ then for every $z \in xy^{-1}$ we have $\beta_1^*(z) = \beta_1^*(x) \otimes \beta_1^*(y^{-1})$. On the other hand, we have

$$\beta_2^*(f(z)) = \beta_2^*(f(x)f(y^{-1})) = \beta_2^*(f(x)) \otimes \beta_2^*(f(y^{-1})) = \omega_{P_2} \otimes \omega_{P_2} = \omega_{P_2}.$$

Therefore $\beta_1^*(z) \in \overline{\operatorname{Ker} f}$. Now, let $\beta_1^*(a) \in P_1/\beta_1^*$ and $\beta_1^*(x) \in \overline{\operatorname{Ker} f}$ then for every $z \in axa^{-1}$ we have $\beta_1^*(z) = \beta_1^*(a) \otimes \beta_1^*(x) \otimes \beta_1^*(a^{-1})$. On the other hand, we have

$$\begin{aligned}\beta_2^*(f(z)) &= \beta_2^*(f(a)f(x)f(a^{-1}))\\ &= \beta_2^*(f(a)) \otimes \beta_2^*(f(x)) \otimes \beta_2^*(f(a^{-1}))\\ &= \beta_2^*(f(a)) \otimes \omega_{P_2} \otimes \beta_2^*(f(a^{-1}))\\ &= \beta_2^*(f(aa^{-1})) = \beta_2^*(f(e_1)) = \beta_2^*(e_2) = \omega_{P_2}.\end{aligned}$$

Hence, we get $\beta_1^*(z) \in \overline{\operatorname{Ker} f}$. This completes the proof. ∎

Theorem 2.5. *Let P be a polygroup, M, N two normal subpolygroups of P with $N \subseteq M$ and $\phi : P/N \longrightarrow P/M$ canonical map. Suppose that β_M^*, β_N^* are the fundamental equivalence relations on $P/M,\ P/N$, respectively. Then $((P/N)/\beta_N^*)/\overline{\operatorname{Ker} \phi} \cong (P/M)/\beta_M^*$.*

Proof. We define the map $\psi : (P/N)/\beta_N^* \longrightarrow (P/M)/\beta_M^*$ by $\psi : \beta_N^*(Nx) \longmapsto \beta_M^*(Mx)$ (for all $x \in P$). We must check that ψ is well-defined, that is, that if $x, y \in P$ and $\beta_N^*(Nx) = \beta_N^*(Ny)$ then $\beta_M^*(Mx) = \beta_M^*(My)$. Now $\beta_N^*(Nx) = \beta_N^*(Ny)$ if and only if $\{Nx, Ny\} \subseteq u$ for some $u \in \mathcal{U}_{P/N}$. By Lemma 2.4 and Corollary 2.5, we have $u = Nx_1 \odot Nx_2 \odot \ldots \odot Nx_n = \{Nz \mid z \in \prod_{i=1}^n x_i\}$. Therefore for some $z_1 \in \prod_{i=1}^n x_i,\ z_2 \in \prod_{i=1}^n x_i$ we have $Nx = Nz_1$ and $Ny = Nz_2$. So there exist $a \in xz_1^{-1} \cap N$ and $b \in yz_2^{-1} \cap N$, then $x \in az_1$ and $y \in bz_2$. Hence $Mx \in Ma \odot Mz_1$ and $My \in Mb \odot Mz_2$. Since $a, b \in N \subseteq M$, then $Ma = M$, $Mb = M$. Since

$M \odot Mz_1 = Mz_1$ and $M \odot Mz_2 = Mz_2$, we have $Mx = Mz_1$ and $My = Mz_2$. From $\{Mz_1, Mz_2\} \subseteq \{Mz \mid z \in \prod_{i=1}^n x_i\}$, we get $\{Mx, My\} \subseteq \{Mz \mid z \in \prod_{i=1}^n x_i\} = Mx_1 \odot Mx_2 \odot \ldots \odot Mx_n$. Therefore, $\beta_M^*(Mx) = \beta_M^*(My)$. This follows that ψ is well-defined. Moreover, ψ is a strong homomorphism, for if $x, y \in P_1$ then

$$\begin{aligned} \psi(\beta_N^*(Nx) \otimes \beta_N^*(Ny)) &= \psi(\beta_N^*(Nxy)) = \beta_M^*(Mxy) \\ &= \beta_M^*(Mx) \otimes \beta_M^*(My) \\ &= \psi(\beta_N^*(Nx)) \otimes \psi(\beta_M^*(My)), \end{aligned}$$

and $\psi(\omega_{P/N}) = \psi(\beta_N^*(N)) = \beta_M^*(M) = \omega_{P/M}$. Clearly, ψ is onto. Now, we show that $\operatorname{Ker}\psi = \overline{\operatorname{Ker}\phi}$. We have

$$\begin{aligned} \operatorname{Ker}\psi &= \{\beta_N^*(Nx) \mid \psi(\beta_N^*(Nx)) = \omega_{P/N}\} \\ &= \{\beta_N^*(Nx) \mid \beta_M^*(Mx) = \omega_{P/N}\} \\ &= \{\beta_N^*(Nx) \mid \beta_M^*(\phi(Nx)) = \omega_{P/N}\} \\ &= \overline{\operatorname{Ker}\phi}. \end{aligned}$$

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