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Nice Bases for Mixed and Torsion-free Abelian Groups

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Abstract. We prove that the divisible Abelian groups and the global Warfield Abelian groups have a nice basis, that is, they can be represented as a countable ascending union of nice direct sums of cyclic groups. We also show that there exists a mixed Abelian group which does not possess a nice basis as well as we find an unbounded reduced algebraically compact Abelian group which does not satisfy a stronger property. Some related concepts and questions are also considered. This continues our recent investigations in the torsion case published in (Atti Sem. Mat. Fis. Univ. Modena, 2005) and (Ann. Univ. Ferrara – Math., 2007).

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1. Introduction

The notion of *"nice bases"* for Abelian groups arisen absolutely naturally in the study of commutative group rings (see [2]). In fact, it was defined in order to demonstrate one more property of totally projective (= reduced torsion simply presented) Abelian groups, motivated via the Direct Factor Problem for commutative modular group rings. It is well known, mainly by L. Kulikov (see, for example, [7, Corollary 18.4]), that each (torsion) Abelian group may be represented as a countable ascending union of direct sums of cyclic groups. When these subgroups are assumed to be pure in the whole group, which is taken a priory to be primary, then it is a direct sum of cyclic groups too (see, for instance, [8, Theorem 3]). That is why, to avoid these two classical situations, we differ only those subgroups of the chain that are nice in the full group and call all Abelian groups with such a property of its subgroups of the union like *"groups with a nice basis"*.

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In [2, 3, 4] we established some assertions for p-primary Abelian groups with a nice basis as well as we formulated some wide-open problems pertaining to these groups. Some modifications of nice bases are also examined.

The object of this article is to continue the exploration of Abelian groups with nice bases, started by us in [2], but by referring to the more difficult torsion-free and mixed cases. We close the work with a question of interest. It is worthwhile noticing that some of the results are already announced in [4].

We first establish the notation that will be in effect throughout this paper, deferring to [7] for a more detailed discussion. Everywhere in the present paper, suppose that G is an additive Abelian group, possibly either mixed or torsion-free, with p-component of torsion G_p and torsion part $G_t = \oplus_p G_p$. As usual, for any ordinal α, the p^α-th power series of G are defined inductively as follows: $p^0 G = G$ and $p^\alpha G = p(p^{\alpha-1}G)$ if α is unlimit, whereas $p^\alpha G = \cap_{\tau<\alpha} p^\tau G$ otherwise when α is limit. Moreover, we denote by $G^1 = \cap_{n<\omega} nG = \cap_p p^\omega G = \cap_p \cap_{j<\omega} p^j G$ the first Ulm subgroup of G. Even more generally, if τ is an arbitrary ordinal, the τ-th Ulm subgroup G^τ of G can be defined by induction like this: $G^0 = G$, $G^{\tau+1} = \cap_n nG^\tau$ and, if τ is limit, $G^\tau = \cap_{\sigma<\tau} G^\sigma$. Observe that $G^\tau = \cap_p p^{\omega\tau} G$ which gives a major connection between the p^α-th power series and the τ-th Ulm subgroups of G, respectively. All other nomenclatures and terminology are standard and follow essentially those from [7]. Nevertheless, we wish to specify the following two concepts. We shall say that the group G is *separable* if every finite subset of elements of G can be embedded in a direct summand of G which is a direct sum of groups of rank one; such a direct summand is often called *completely decomposable.* Moreover, G is said to be *without elements of infinite height* if $G^1 = 0$. Notice that for reduced primary groups these two notions coincide, but for mixed and torsion-free groups they are totally different.

2. Main results

Before stating the definition of an Abelian group with a nice basis, we recollect the following classical statement of global niceness. The subgroup N of a group G is said to be *nice* in G if the equality

$$p^\alpha(G/N) = (p^\alpha G + N)/N$$

holds for every prime number p and every ordinal number α. This equality is tantamount to the equality $\cap_{\beta<\alpha}(N + p^\beta G) = N + p^\alpha G$ for each prime p and each limit ordinal α.

And so, we are ready with giving up a more general definition than that in [2, 3].

Definition 2.1. *The group G has a nice basis if $G = \cup_{n<\omega} G_n, G_n \subseteq G_{n+1} \leq G$ such that each member G_n is a direct sum of cyclic groups and is nice in G.*

As it will be demonstrated in the sequel, the existence of a nice basis is not ever guaranteed as well as if it exists it may not be unique, that is, it is quite possible to exist many nice bases for a given Abelian group. Besides, there is an abundance of classes of mixed and torsion-free Abelian groups which have nice bases. For instance, it is straightforward to see that such is the class of direct sums of cyclic groups.

However, there are larger sorts of Abelian groups which reserve this property; here we shall exhibit a few of them.

Following [9], the subgroup N of a group G is said to be *weakly nice* in G if for every prime p and for every ordinal α, the equality

$$(p^\alpha(G/N)/(p^\alpha G + N)/N)[p] = 0$$

holds, i.e., the co-kernel of the canonical map $(p^\alpha G + N)/N \to p^\alpha(G/N)$ does not contain an element of order p.

Clearly, each nice subgroup is weakly nice, while the converse fails. However, if G/N is torsion, N is weakly nice in G precisely when it is nice in G. Under this new dispensation infinite cyclic groups are weakly nice.

Finally, the subgroup N of G is said to be *strongly nice* in G if for every ordinal τ, the equality

$$(G/N)^\tau = (G^\tau + N)/N$$

holds, which is equivalent to $\cap_{\sigma<\tau}(N + G^\sigma) = N + G^\tau$ for each limit ordinal number τ and $\cap_n n(G^\tau + N) = \cap_n(nG^\tau + nN) = \cap_n nG^\tau + N = G^{\tau+1} + N$.

Notice that finite subgroups are obviously strongly nice. For p-local groups, p is a prime, niceness and strongly niceness do coincide. However, this is not always true in the general case. For example, if G is a free group with a subgroup N such that G/N is a torsion-free group that is reduced but p-divisible for some prime p, then N will be strongly nice in G but not nice.

So, we come to the following formally more weak and more global versions of the previous definition, respectively.

Definition 2.2. *The group G has a weak nice basis if $G = \cup_{n<\omega} G_n, G_n \subseteq G_{n+1} \leq G$ such that each member G_n is a direct sum of cyclic groups and is weakly nice in G.*

Definition 2.3. *The group G has a strong nice basis if $G = \cup_{n<\omega} G_n, G_n \subseteq G_{n+1} \leq G$ such that each member G_n is a direct sum of cyclic groups and is strongly nice in G.*

It is straightforward to see that the following relationship is valid:

$$\text{Definition 2.1} \Longrightarrow \text{Definition 2.2}$$

whereas, by what we have noted above, Definition 2.1 and Definition 2.3 are independent.

In other words, not each group with a strong nice basis is a group with a nice basis and vice versa, while each group with a nice basis is a group with a weak nice basis.

We are now prepared to proceed by proving:

Theorem 2.1. *The Warfield Abelian groups possess a nice basis.*

Proof. The application of [10, Theorem] allows us to the existence of a nice decomposition basis $X = \{x_i : i < \mu\}$, consisting of elements of infinite order, of the mixed group G such that $\langle X\rangle = \oplus_{i<\mu}\langle x_i\rangle$ is nice in G and $G/\langle X\rangle$ is totally projective. Thus, in virtue of [3], one can write that $G/\langle X\rangle = \cup_{n<\omega}(G_n/\langle X\rangle)$, where $G_n \subseteq G_{n+1} \leq G$ with $G_n/\langle X\rangle$ bounded and nice in $G/\langle X\rangle$ for all naturals n. Appealing to a modified variant adapted for the general case of a lemma for niceness of primary Abelian groups (see, for example, [7, Lemma 79.3]), we derive that each G_n is nice in G since $\langle X\rangle$ is nice in G.

On the other hand, $G_n/\langle X\rangle$ being bounded and $\langle X\rangle$ being a direct sum of cyclic groups imply in view of [7, Proposition 18.3] that every G_n is a direct sum of cyclic groups as well. But it is obvious that $G = \cup_{n<\omega} G_n$. So, by Definition 2.1, we have finished the proof. ∎

As an immediate consequence, we yield the following (for the case of primary groups the reader can see [3]).

Corollary 2.1. *Simply presented groups possess a nice basis.*

Proof. Each such a group is itself Warfield. Hence Theorem 2.1 works. ∎

Proposition 2.1. *Any completely decomposable group has a nice basis.*

Proof. Every such group splits and is Warfield. Hence Theorem 2.1 can be applied. ∎

Proposition 2.2. *Any countable separable group has a nice basis. In particular, all rational groups have a nice basis.*

Proof. By [11, Corollary 1.6] each such group is completely decomposable, and thus we wish only apply the previous claim. ∎

We continue by showing that other classes of Abelian groups are also equipped with a nice basis. Before doing that, we need a key technical claim (compare with [3] for the torsion case).

Proposition 2.3. *Direct sums of groups with a* (*weak, strong*) *nice basis are also with a* (*weak, strong*) *nice basis.*

Proof. We shall consider only the ordinary niceness, since the remaining two cases are analogous. Write $G = \oplus_{i\in I} G_i$, where each summand G_i has a nice basis, that is, $G_i = \cup_{n<\omega} G_n^{(i)}$, $G_n^{(i)} \subseteq G_{n+1}^{(i)} \leq G_i$ and, for all $n < \omega$, $G_n^{(i)}$ are nice in G_i direct sums of cyclic groups. It is only a routine exercise to check that $\oplus_{i\in I} G_i = \cup_{n<\omega}(\oplus_{i\in I} G_n^{(i)})$ and that, for all $n < \omega$, $\oplus_{i\in I} G_n^{(i)}$ are nice in $\oplus_{i\in I} G_i$ direct sums of cyclic groups. Thus, we conclude that G has a nice basis as asserted. ∎

We are now ready to prove the following.

Proposition 2.4. *Divisible groups have a nice basis.*

Proof. Let G be a divisible group. In accordance with [7, Theorem 23.1] one may write

$$G \cong \oplus_{r_0(G)} Q \oplus \oplus_p [\oplus_{r_p(G)} Z(p^\infty)]$$

where Q is the additive group of all rational numbers, which is countable torsion-free, and $Z(p^\infty)$ is the quasi-cyclic group of type p^∞ where p is a prime. Utilizing [3], all $Z(p^\infty)$ are with nice bases. Moreover, in virtue of [7, v. I, p. 27], we can represent Q like this:

$$Q = \cup_{n<\omega} A_n,$$

where $A_n = \langle 1/n!\rangle$, $n \geq 1$, whence $A_n \subset A_{n+1}$. Observe that $A_1 = \langle 1\rangle = Z$. On the other hand, since Q is divisible, each member A_n of the union is nice in Q, as required. That is why, Q has a nice basis and hereafter we apply Proposition 2.3. ∎

As a direct consequence, we derive the following.

Corollary 2.2. *Direct sums of co-cyclic groups possess a nice basis.*

Proof. Write $G = D \oplus C$ where D is divisible and C is a direct sum of cyclic groups. Henceforth, the assertion follows from Propositions 2.3 and 2.4. ∎

Proposition 2.5. *Any reduced algebraically compact group has a strong nice basis if and only if it is bounded.*

Proof. Write $G = \cup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1} \leq G$ and all G_n are strongly nice in G direct sums of cyclic groups. Since $G^1 = 0$ and $(G/G_n)^1 = (G^1 + G_n)/G_n = 0$, employing [7, Corollary 39.2] we find that G_n are algebraically compact. Hence, [7, v. I, p. 190, Exercise 1 and Corollary 39.10] apply to show that G_n are bounded, whence torsion, and thus G is torsion as well. Furthermore, as observed above, G has to be bounded, as asserted. ∎

We will show now that there exists an uncountable and not torsion algebraically compact group which cannot be endowed with a strong nice basis. Notice that reduced algebraically compact groups are without elements of infinite height since their first Ulm subgroup coincides with the maximal divisible subgroup [7, v. I, p. 191, Exercise 7]. Moreover, countable algebraically compact groups are direct sums of a divisible group and a bounded group [7, v. I, p. 200, Exercise 3(a)]. Likewise, a reduced torsion algebraically compact group is bounded [7, Corollary 40.3]. Hence, in both situations, by what we have shown above the countable or torsion (in particular, bounded) algebraically compact groups possess nice basis. So, we yield the following example.

Example 2.1. Unbounded reduced algebraically compact groups do not have a strong nice basis.

Before we continue, we need one more crucial technicality.

Lemma 2.1. *Let $N \leq G$ be (weakly, strongly) nice in G and let $F \leq M \leq G$ be finite. Then*

(a) *$N + F$ is (weakly, strongly) nice in G;*
(b) *$(N + F)/F$ is (weakly, strongly) nice in G/F;*
(c) *$N \cap G^1$ is (weakly, strongly) nice in G^1;*
(d) *M/F is (weakly, strongly) nice in G/F if and only if M is (weakly, strongly) nice in G.*

Proof. Since the claims on ordinary niceness are either elementary or well known (compare with [7]), we will omit their verification. So, we will be concentrated only on strong niceness and weak niceness.

(a) What suffices to prove is that $\cap_{\sigma<\tau}(N + F + G^\sigma) = N + F + G^\tau$ for each limit ordinal τ and $\cap_n n(G^\tau + N + F) = G^{\tau+1} + N + F$. To this aim, choose x in the first intersection. So, we write $x = g_\sigma + a + f = g_{\sigma'} + a' + f' = \cdots$ for some σ': $\sigma < \sigma' < \tau$. Since F is finite whereas the intersection is infinite owing to the fact that $\tau \geq \omega$, and so the number of equalities is also infinite, we may assume that $f = f'$. Hence $g_\sigma + a = g_{\sigma'} + a'$ and thus $x \in \cap_{\sigma<\tau}(N + G^\sigma) + F = N + G^\tau + F$ as required.

The second intersection is analogous. The same trick works and for weak niceness.

(b) Since finite subgroups are always strongly, respectively weakly, nice in the containing group, we shall use (a) like this: $\cap_{\sigma<\tau}((N+F)/F+(G/F)^{\sigma}) = \cap_{\sigma<\tau}((N+F)/F+(G^{\sigma}+F)/F) = \cap_{\sigma<\tau}[(N+F+G^{\sigma})/F] = [\cap_{\sigma<\tau}(N+F+G^{\sigma})]/F = (N+F+G^{\tau})/F = (N+F)/F+(G^{\tau}+F)/F = (N+F)/F+(G/F)^{\tau}$, when τ is limit. The other intersection can be processed identically.

The weak niceness is similar.

(c) It is enough to show that $N\cap G^1$ is strongly, respectively weakly, nice in G. For this purpose, we observe with the aid of the modular law from [7] that $\cap_{\sigma<\tau}(N\cap G^1+G^{\sigma}) \subseteq \cap_{\sigma<\tau}(N+G^{\sigma})\cap G^1 = (N+G^{\tau})\cap G^1 = N\cap G^1+G^{\tau}$ because $\tau \geq 1$ and $\cap_n n(N\cap G^1+G^{\tau}) \subseteq \cap_n n(N+G^{\tau})\cap G^1 = (N+G^{\tau+1})\cap G^1 = N\cap G^1+G^{\tau+1}$.

The weak niceness is identical.

(d) We shall be concerned only with the strong niceness because the weak niceness is similar.

($\Longrightarrow$). Since $\cap_{\sigma<\tau}(M/F+(G/F)^{\sigma}) = \cap_{\sigma<\tau}(M/F+(G^{\sigma}+F)/F) = \cap_{\sigma<\tau}[(M+G^{\sigma})/F] = [\cap_{\sigma<\tau}(M+G^{\sigma})]/F$ and $(G/F)^{\tau}+M/F = (G^{\tau}+F)/F+M/F = (G^{\tau}+M)/F$, we deduce that $[\cap_{\sigma<\tau}(M+G^{\sigma})]/F = (G^{\tau}+M)/F$, i.e., $\cap_{\sigma<\tau}(M+G^{\sigma}) = M+G^{\tau}$ as required.

The second relationship uses the same idea.

($\Longleftarrow$). Observe that $\cap_{\sigma<\tau}(M/F+(G/F)^{\sigma}) = \cap_{\sigma<\tau}(M/F+(G^{\sigma}+F)/F) = \cap_{\sigma<\tau}[(G^{\sigma}+M)/F] = [\cap_{\sigma<\tau}(G^{\sigma})+M]/F = (G^{\tau}+M)/F = (G^{\tau}+F)/F+M/F = (G/F)^{\tau}+M/F$ as required.

The second relationship exploits the same method. ∎

Proposition 2.6. *If G has a (weak, strong) nice basis, then G^1 has a (weak, strong) nice basis.*

Proof. Write $G = \cup_{n<\omega}G_n \subseteq G_{n+1} \leq G$ and, for each $n<\omega$, G_n is a (weak, strong) nice direct sum of cyclic groups. Furthermore, $G^1 = \cup_{n<\omega}(G_n\cap G^1)$ where using [7, Theorem 18.1] the intersection $G_n\cap G^1$ is a direct sum of cyclic groups. Hereafter, owing to Lemma 2.1(c), the assertion follows. ∎

Problem 2.1. *If G has a (weak, strong) nice basis, is then G/G^1 also with a (weak, strong) nice basis?*

So, we come to the following.

Example 2.2. If G is a group such that G^1 is unbounded reduced algebraically compact, then G does not possess a strong nice basis.

Proof. Indeed, suppose the contrary. Hence, in view of Proposition 2.6, G^1 has a nice basis. Invoking to Proposition 2.5, G^1 must be bounded which is the desired contradiction. ∎

Certainly, it is also not realistic to happen that each mixed or torsion-free Abelian group will possess a nice basis. The following example demonstrably shows this. Before formulating it, we need a bit of technicalities.

Lemma 2.2. *Suppose that A is an Abelian group such that $A_p = 0$. Then $p^\omega A = p^{\omega+1}A$.*

Proof. Since $p^\omega A = \cap_{n<\omega} p^n A$, we observe that any $x \in p^\omega A$ can be written as $x = pa_1 = p^n a_n$ with $a_1 \in A$ and $a_n \in A$ for all $n \geq 2$. The lack of p-elements in A leads to $a_1 = p^{n-1}a_n \in p^{n-1}A$ for all $n \geq 2$, whence $a_1 \in p^\omega A$. Thus $x \in p^{\omega+1}A$ and hence $p^\omega A \subseteq p^{\omega+1}A$. Because this inclusion is tantamount to the wanted equality, we are done. ∎

Proposition 2.7. *Suppose $A = B \oplus C$ is an Abelian group.*

(1) *If $p^\omega A = p^\omega B$ for every prime p and A has a nice basis, then B has a nice basis.*
(2) *If $A^1 = B^1$ and A has a strong nice basis, then B has a strong nice basis.*

Proof. (1) Write $A = \cup_{n<\omega} A_n$, where $A_n \subseteq A_{n+1} \leq A$ such that each member of the union is a direct sum of cyclic groups and is nice in A. Consequently, $B = \cup_{n<\omega}(A_n \cap B)$, where, in conjunction with [7, Theorem 18.1], $A_n \cap B$ is a direct sum of cyclic groups. Moreover, in order to prove that $A_n \cap B$ is nice in B for every index n, it is enough to illustrate that $\cap_{\tau<\alpha}(A_n \cap B + p^\tau B) = A_n \cap B + p^\alpha B$ for any limit ordinal α. Indeed, with the aid of the modular law from [7], we compute that $\cap_{\tau<\alpha}(A_n \cap B + p^\tau B) \subseteq \cap_{\tau<\alpha}(A_n + p^\tau A) \cap B = (A_n + p^\alpha A) \cap B = (A_n + p^\alpha B) \cap B = A_n \cap B + p^\alpha B$. Since the last inclusion is equivalent to the desired equality, we are finished.

(2) As for the second half-part, we can apply the same idea. ∎

Corollary 2.3. *Let $G = H \oplus K$ such that K is torsion without elements of infinite height. Then G has a (weak, strong) nice basis if and only if H has a (weak, strong) nice basis.*

Proof. About the necessity, it easily follows that $G^1 = H^1$ and hence Proposition 2.7 is applicable to infer the implication.

As for the sufficiency, since torsion separable groups have by [3] nice bases, it follows directly from Proposition 2.3. ∎

Remark 2.1. When $K^1 \neq 0$, an example was given in [6] which illustrates that the necessity in Corollary 2.3 fails provided that G is p-torsion.

As an immediate consequence, we yield

Corollary 2.4. *If G_t is bounded, then G has a (weak, strong) nice basis if and only if G/G_t has a (weak, strong) nice basis.*

Proof. In virtue of [7, Theorem 27.1] we may write $G \cong G_t \oplus (G/G_t)$. Henceforth, Corollary 2.3 works. ∎

An Abelian group A is called *p-splitting* if A_p is its direct summand as well as A is called *p-reduced* if its maximal p-divisible subgroup is zero. So, we come to the following concrete example.

Example 2.3. Let G be a p-splitting and p-reduced Abelian group whose p-primary component G_p is of length $\omega \cdot 2$ such that both $p^\omega G_p$ and $G_p/p^\omega G_p$ are torsion-complete. Then G does not have a nice basis.

Proof. In fact, write $G \cong G_p \oplus G/G_p$. Furthermore, according to Lemma 2.2, $p^\omega(G/G_p)$ is p-divisible. Hence, $p^\omega(G/G_p) = 0$. Likewise, it was argued in [3] that G_p is without a nice basis. Therefore, because of $G^{p^\omega} = G_p^{p^\omega}$, in virtue of Proposition 2.7 we deduce that so does G, as claimed. ∎

Note 2.1. In contrast with the primary case, probably not every mixed or torsion-free group without elements of infinite height has a nice basis.

Conjecture 2.1. *Each algebraically compact Abelian group has a nice basis if and only if it is the direct sum of a divisible group and a bounded group.*

It was proved in [2] and [3] that any countable Abelian p-group possesses a nice basis. However, this is not longer true for torsion-free groups.

Example 2.4. There is a torsion-free countable Abelian group which has a nice basis but is not a Warfield group.

Suppose G is a finite rank (greater than 1) countable torsion-free Abelian group which is not completely decomposable, but which has a full-rank free subgroup F for which the torsion group G/F is a direct sum of cyclic groups – there are such groups G even of rank 2. Furthermore, appealing to [3], $G/F = \cup_{i<\omega}(G_i/F)$ where G_i/F are bounded nice subgroups of G/F with $G_i \subseteq G_{i+1} \leq G$. Thus, $G = \cup_{i<\omega} G_i$ and since F is nice in G it easily follows that G_i is nice in G. Moreover, [7, Proposition 18.3] insures that all G_i are direct sums of cyclic groups. On the other hand, G need not be Warfield because it is not completely decomposable.

We now proceed by proving some affirmations about nice and weak (respectively, strong) nice bases of Abelian groups.

Proposition 2.8. *If $N \leq G$, N is a nice direct sum of cyclic groups and G/N has a nice basis such that $(G/N)_t$ is bounded, then G has a nice basis.*

Proof. Write $G/N = \cup_{n<\omega}(G_n/N)$, where $G_n \subseteq G_{n+1} \leq G$, whence $G = \cup_{n<\omega} G_n$, and all G_n/N are nice in G/N direct sums of cyclic groups. Thus, G_n are nice in G (see [7, Lemma 79.3]). Moreover, since $(G_n/N)_t$ is bounded, [7, v. I, p. 112, Exercise 2] implies that G_n are direct sums of cyclic groups. ∎

Proposition 2.9. *Let $F \leq G$ be finite. Then G is a direct sum of cyclic groups if and only if G/F is a direct sum of cyclic groups and G is without elements of infinite height.*

Proof. ($\Longrightarrow$). Observe that G can be written as $G = B \oplus C$ for some subgroup C with B a direct sum of finitely many cyclic groups, thus it is bounded, and $F \leq B$. Now, with the aid of the modular law from [7, v. I], we have that $G/F = (B/F) \oplus (C \oplus F)/F \cong (B/F) \oplus C$ is clearly a direct sum of cyclic groups since B/F is bounded.

($\Longleftarrow$). If G is a p-group, the assertion follows directly from Dieudonné's criterion (see, e.g., [5]). However, for the general mixed case, we shall suggest the following more direct approach. Write $G/F = (G/F)_t \oplus M$ for some subgroup M of G/F. But $(G/F)_t = G_t/F$ and so $G/F \cong (G_t/F) \oplus G/G_t$. Furthermore, since G/G_t is a direct sum of cyclic groups, it follows from [7] that G splits, i.e., $G = G_t \oplus R$ for some subgroup R of G. Moreover, G_t/F is a direct sum of cyclic groups and G_t

is separable, hence by the aforementioned criterion of Dieudonné applied for every prime p, we deduce that G_t is a direct sum of cyclic groups. Finally, the same holds for G, as desired. ∎

Proposition 2.10. *Suppose $F \leq G$ is finite. If G has a (weak, strong) nice basis, then G/F has a (weak, strong) nice basis. Moreover, if G is a p-group such that G/F has a nice basis, then G is a countable union (not necessarily ascending) of nice subgroups which are direct sums of cyclic groups.*

Proof. Write $G = \cup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1} \leq G$ and all G_n are (weakly, strongly) nice in G direct sums of cyclic groups. Therefore, $G/F = \cup_{n<\omega}(G_n + F)/F$. Since $(G_n + F)/F \cong G_n/(G_n \cap F)$ we may utilize Proposition 2.9 to infer that $(G_n + F)/F$ are direct sums of cyclic groups. On the other hand, referring to Lemma 2.1(b), we derive that $(G_n + F)/F$ are (weakly, strongly) nice in G/F.

Conversely, write $G/F = \cup_{n<\omega}(A_n/F)$, where $A_n \subseteq A_{n+1} \leq G$ and all A_n/F are (weakly, strongly) nice in G/F and are direct sums of cyclic groups. Utilizing [1], A_n is the direct sum of a direct sum of cyclic groups and a countable group, say $A_n = B_n \oplus C_n$. Thus, $A_n = \cup_{i<\omega}(B_n \oplus F_{in})$ where F_{in} are finite with exponent p^n such that $C_n = \cup_{i<\omega} F_{in}$ and $F_{in} \subseteq F_{i+1n}$. Therefore, $G = \cup_{n<\omega} A_n = \cup_{n<\omega}(B_n + F_n)$ for some finite subgroups F_n of G. Observe also that $B_n + F_n$ is a direct sum of cyclic groups since $p^n(B_n + F_n) = p^n B_n$ is so (see [7, Proposition 18.3]) taking into account that $p^n F_n = 0$.

On the other hand, Lemma 2.1(d) gives that A_n are (weakly, strongly) nice in G. Since B_n being balanced in A_n is (weakly, strongly) nice in G, Lemma 2.1(a) insures that $B_n + F_n$ is (weakly, strongly) nice in G, as required. ∎

It is worth emphasizing that, although $A_n \subseteq A_{n+1}$, the inclusions $B_n \subseteq B_{n+1}$ and $F_{in} \subseteq F_{in+1}$ may not hold always. In fact, if F is a finite subgroup of the Abelian p-group G such that G/F is a direct sum of cyclic groups, it is well known that $G = A \oplus B$ where $A \supseteq F$ is countable and B is a direct sum of cyclic groups (see, e.g., [1]). If now K is another group with $K \supseteq G$ such that K/F is a direct sum of cyclic groups, it does not follow that $K = C \oplus D$ where $C \supseteq A$ is countable and $D \supseteq B$ is a direct sum of cyclic groups. Indeed, this is not true even if all the groups are finite. For instance, let $K = \langle x \rangle \oplus \langle y \rangle$ where $order(x) = p$ and $order(y) = p^2$. Let $F = \langle x \rangle$ and $B = \langle x + py \rangle$. So $G = K[p]$. Then we cannot write $K = C \oplus D$ where C contains A and D contains B because $py = (x + py) - x$ has height one in K while $x + py$ and x both have height zero in K.

The following statement is a complement to results in the primary case from [3] (see [6] too) extending Example 3 from [3].

Theorem 2.2. *Suppose A is an Abelian p-group with $A/p^\omega A$ a direct sum of cyclic groups. Then A has a nice basis if and only if $p^\omega A$ has a nice basis.*

Proof. The necessity was proved in [3] (compare with Proposition 2.6).

As for the sufficiency, suppose that $\{N_n\}_{n<\omega}$ is a nice basis for $p^\omega A$ and $\{a_j\}_{j\in J}$ is a collection of elements of A such that $A/p^\omega A = \oplus_{j\in J}\langle a_j + p^\omega A\rangle$, and let $a_j + p^\omega A$ have order p^{e_j}. For $n < \omega$, let $J_n = \{j \in J : e_j \leq n \text{ and } p^{e_j} a_j \in N_n\}$ and define

$$M_n = N_n + \langle a_j : j \in J_n \rangle.$$

We shall prove four things about M_n that are:

(1) $M_n \subseteq M_{n+1}$.
By definition, $N_n \subseteq N_{n+1}$ and, moreover, it is clear that $J_n \subseteq J_{n+1}$. These two inclusions ensure the desired relation.

(2) $A = \cup_{n<\omega} M_n$.
Choose $a \in p^\omega A$, hence $a \in N_k$ for some $k < \omega$. Since $N_k \subseteq M_k$ we have $a \in M_k$. Next, assume that $a \in A \setminus p^\omega A$ whence $a + p^\omega A \in A/p^\omega A$ and thus $a + p^\omega A = s_1 a_{j_1} + \cdots + s_t a_{j_t} + p^\omega A$ for some indices $j_1, \cdots, j_t$, integers $s_1, \cdots, s_t$ and $t \in \mathbb{N}$. But $j_1, \cdots, j_t \in J_m$ for some $m < \omega$. Since $a = b + s_1 a_{j_1} + \cdots + s_t a_{j_t}$ for some $b \in p^\omega A$, hence $b \in N_k$ for some $k < \omega$, we conclude that $a \in N_l + \langle a_j : j \in J_l \rangle = M_l$ for some $l < \omega$, as required.

(3) M_n *are direct sums of cyclic groups.*
It is self-evident that $p^n \langle a_j : j \in J_n \rangle \subseteq N_n$, whence $p^n M_n = p^n N_n$ is a direct sum of cyclic groups. Therefore, invoking [7, Proposition 18.3], we derive that M_n is a direct sum of cyclic groups for any index n.

(4) M_n *are nice in* A.
Since $A/(M_n + p^\omega A) = A/(\langle a_j : j \in J_n \rangle + p^\omega A) \cong A/p^\omega A/(\langle a_j : j \in J_n \rangle + p^\omega A)/p^\omega A = \oplus_{j \in J} \langle a_j + p^\omega A \rangle / \oplus_{j \in J_n} \langle a_j + p^\omega A \rangle \cong \oplus_{j \notin J_n} \langle a_j + p^\omega A \rangle = \oplus_{j \in J \setminus J_n} \langle a_j + p^\omega A \rangle$ is separable, it follows that $M_n + p^\omega A$ is nice in A, so that $(M_n + p^\omega A)/p^\omega A$ is nice in $A/p^\omega A$ (see [7, Lemma 79.3]). On the other hand, because it is readily checked that $\langle a_j : j \in J_n \rangle \cap p^\omega A \subseteq N_n$, we have by the modular law that $M_n \cap p^\omega A = N_n$ is nice in $p^\omega A$. But it is well known that (e.g., [7, v. II, p. 93, Exercise 10]) a subgroup L of a group K is nice if and only if $L \cap p^\alpha K$ is nice in $p^\alpha K$ and $(L + p^\alpha K)/p^\alpha K$ is nice in $K/p^\alpha K$ for some arbitrary ordinal α. That is why, M_n is nice in A for every index n.

Finally, in view of points (1)–(4), the sequence $\{M_n\}_{n<\omega}$ forms a nice basis for A as required. ∎

Of some interest is also the question under which additional conditions on a subgroup C of an Abelian p-group A, A/C equipped with a nice basis implies the same property for A and visa versa. The following answers this in some partial cases.

Proposition 2.11. *Let C be a countable nice subgroup of the Abelian p-group A such that $C \cap p^\omega A = 0$ and A/C is a group with a nice basis. Then A has a nice basis.*

Proof. Write $A/C = \cup_{n<\omega}(A_n/C)$ where $C \leq A_n \subseteq A_{n+1} \leq A$ with all A_n/C nice in A/C and direct sums of cyclic groups. Therefore, $A = \cup_{n<\omega} A_n$ where $p^\omega A_n \subseteq C$. Hence $p^\omega A_n \subseteq C \cap p^\omega A = 0$ and we conclude that all members of the union are separable. On the other hand, as aforementioned, every A_n/C being a direct sum of cyclic groups yields by [5] (see [1] too) that A_n is a direct sum of cyclic groups. Finally, because of the niceness of C in A, [7, v. II, p. 92, Lemma 79.3] applies to get that every A_n is nice in A, as needed. ∎

Remark 2.2. Note that the restriction $C \cap p^\omega A = 0$ is essential and cannot be dropped off. Specifically, there is an Abelian p-group A which does not possess a nice basis with $p^\omega A$ countable inseparable. In fact, suppose G is any separable thick group and let A be any group such that $p^\omega A$ is reduced, countable and not a direct sum of cyclic groups with $A/p^\omega A \cong G$. Then [6] applies to show that A does not have a nice basis since $p^\omega A$ is not a direct sum of cyclic groups.

We terminate the work with four problems.

Problem 1: Suppose G is a group with G/G^1 a direct sum of cyclic groups. Does it follow that G has a (strong, weak) nice basis if and only if G^1 has a (strong, weak) nice basis?

Problem 2: Let A be an Abelian p-group with $p^\alpha A$ countable for some $\alpha > \omega$. If A has a nice basis, does it follow that $A/p^\alpha A$ also has a nice basis?

Problem 3: If C is a countable subgroup of an Abelian p-group A such that A/C is totally projective, does it follow that A has a nice basis?

Notice that if C is nice in A, then A is totally projective by [4] and, furthermore, [2] is applicable to derive that A has a nice basis, indeed.

Problem 4: Characterize those groups $G = A/C$ such that for all Abelian p-groups A and all countable subgroups C it follows that A has a nice basis.

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References

[1] B. Charles, Note sur la structure des groupes abeliens primaires, *Compt. Rend. Acad. Sci. Paris* **252** (1961), 1547–1548.
[2] P. V. Danchev, A nice basis for $S(FG)/G_p$, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* **53** (2005), no. 1, 3–11.
[3] P. V. Danchev, Nice bases for primary abelian groups, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **53** (2007), no. 1, 39–50.
[4] P. V. Danchev, Abelian groups with a nice basis, *C. R. Acad. Bulgare Sci.* **60** (2007), no. 3, 219–224.
[5] P. V. Danchev, Generalized Dieudonné and Hill criteria, *Port. Math.* **65** (2008), no. 1, 121–142.
[6] P. V. Danchev and P. W. Keef, Nice bases and thickness in primary abelian groups, *Rocky Mount. J. Math.*, to appear.
[7] L. Fuchs, *Infinite abelian groups. Vol. 2* (Russian), Translated from the English by A. A. Manovcev and A. P. Misina, Translation edited by L. Ja. Kulikov, Izdat. "Mir", Moscow, 1977.
[8] P. Hill, Primary groups whose subgroups of smaller cardinality are direct sums of cyclic groups, *Pacific J. Math.* **42** (1972), 63–67.
[9] P. Hill and C. Megibben, Mixed groups, *Trans. Amer. Math. Soc.* **334** (1992), no. 1, 121–142.
[10] R. Hunter and F. Richman, Global Warfield groups, *Trans. Amer. Math. Soc.* **266** (1981), no. 2, 555–572.
[11] C. K. Megibben, Separable mixed groups, *Comment. Math. Univ. Carolin.* **21** (1980), no. 4, 755–768.