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Martingale Convergence Theorems in JW-Algebras

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Abstract. In this paper certain properties of conditional expectations of reversible JW-algebra A with a faithful semifinite normal trace are studied. As an application of these results, we prove convergence theorems for supermartingales and martingales defined on such algebras.

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1. Introduction

It is known that in most mathematical formulations of the foundations of quantum mechanics, the bounded observables of a physical system are identified with a real linear space, L, of bounded self-adjoint operators on a Hilbert space H. Those bounded observables which correspond to the projections in L form a complete orthomodular lattice, P, otherwise known as the lattice of the quantum logic of the physical system. For the self-adjoint operators x and y on H their Jordan product is defined by $x \circ y = (xy + yx)/2 = (x + y)^2 - x^2 - y^2$. So, it is reasonable to assume that L is a Jordan algebra of self-adjoint operators on H which is closed in the weak operator topology. Hence L is a JW-algebra. It is known that the JW-algebra is a real non-associative analog of a von Neumann algebra that was first studied by Topping [19]. He had extended many results from the theory of von Neumann algebras on JW-algebras. To ergodic type theorems for Jordan algebras were devoted a lot of papers (see for example, [1]– [4], [15] etc.). The motivation of these investigations arose in quantum statistical mechanics and quantum field theory (see [9]). Asymptotic behavior of positive contractions of Jordan algebras has been studied in [16,17]. We refer the reader to the books [5,6,12] for the theory of JW-algebras.

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The main purposes of this paper is to study certain properties of conditional expectations of reversible JW-algebra A with a faithful normal trace, as an application of these results we prove convergence theorems for supermartingales and martingales defined on such algebras. Note that supermartingales and martingales in a von Neumann algebra setting were investigated in [7, 10, 13, 18].

2. Preliminaries

Throughout the paper H denotes a complex Hilbert space, $B(H)$ denotes the algebra of all bounded linear operators on H.

Recall that a *JW-algebra* is a real linear space of self-adjoint operators from $B(H)$ which is closed under the Jordan product $a \circ b = (ab + ba)/2$ and also closed in the weak operator topology. Here the sign ab denotes the usual operator multiplication of operators a and b taken from $B(H)$. A JW-algebra A is said to be *reversible* if $a_1 a_2 \cdots a_n + a_n a_{n-1} \cdots a_1 \in A$, whenever $a_1, a_2, \ldots, a_n \in A$. Examples of non-reversible JW-algebras are given by spin factors which are given in [5, 6].

Recall that a real $*$-algebra R in $B(H)$ is called a *real W^*-algebra* if it is closed in the weak operator topology and satisfies the conditions $R \cap iR = \{0\}$, $\mathbb{1} \in R$. It is obvious that if R is a real or complex W^*-algebra, then its selfadjoint part $R_{sa} = \{x \in R : x^* = x\}$ forms a reversible JW-algebra.

Given an arbitrary JW-algebra A, let $R(A)$ denote the weakly closed real $*$-algebra in $B(H)$ generated by A, and by $W(A)$ we denote the W^*-algebra (complex) generated by A.

Let A be a reversible JW-algebra and τ be a normal semifinite faithful (n.s.f.) trace on A. Then τ can be extended to a n.s.f. τ_1 on the W^*-algebra $W(A)$. Namely, for every $x \in W(A)^+$ (W^+ means the positive part of W), one has $x = a + ib$, where $a, b \in R(A)$, $a \in A^+$, $b^* = -b$ (skew-symmetric element). Then, one puts $\tau_1(x) = \tau(a)$ (see [6, Thm. 1.2.9], for more details).

Throughout the paper we always assume that a JW-algebra A is reversible. Therefore we do not stress on it, if it is not necessary.

Let A be a JW-algebra with a n.s.f. trace τ and τ_1 be its extension to $W(A)$. Set $\mathfrak{N}_\tau = \{x \in A : \tau(|x|) < \infty\}$, $\mathfrak{N}_{\tau_1} = \{x \in W(A) : \tau_1(|x|) < \infty\}$. Completion of $\mathfrak{N}_\tau$ (resp. $\mathfrak{N}_{\tau_1}$) w.r.t. the norm $\|x\|_1 = \tau(|x|)$ $x \in \mathfrak{N}_\tau$ (resp. $\|x\|_1 = \tau_1(|x|)$ $x \in \mathfrak{N}_{\tau_1}$) is denoted by $L_1(A, \tau)$ (resp. $L_1(W(A), \tau_1)$). It is obvious that $L_1(A, \tau) \subset L_1(W(A), \tau_1)$.

3. Martingale convergence in JW-algebras

In this section we are going to study certain properties of conditional expectations of JW-algebra A.

Let A be a reversible JW-algebra with a n.s.f. trace τ. Let A_1 be its JW-subalgebra containing the identity operator $\mathbb{1}$.

Recall that a linear mapping $\phi : A \to A_1$ is called *the conditional expectation* with respect to a JW-subalgebra A_1 if the following conditions are satisfied:

(i) $\phi(\mathbb{1}) = \mathbb{1}$;
(ii) If $x \geq 0$, then $\phi(x) \geq 0$;
(iii) $\phi(xy) = \phi(x)y$ for $x \in A$, $y \in A_1$.

Let $\widetilde{\tau} := \tau \upharpoonright A_1$ be the restriction of the trace τ to A_1 such that $\widetilde{\tau}$ is also semifinite. Then the space $L_1(A_1, \widetilde{\tau})$ of integrable operators w.r.t. $(A_1, \widetilde{\tau})$, is a subspace of $L_1(A, \tau)$.

Theorem 3.1. [8] *Let A be a reversible JW-algebra with a n.s.f. trace τ and A_1 be its JW-subalgebra with $\mathbb{1}$. Let $\widetilde{\tau} = \tau \upharpoonright A_1$ be the restriction of the trace τ to A_1 such that $\widetilde{\tau}$ is also semifinite. Then there exists a positive linear mapping $E(\cdot/A_1) : A \to A_1$ satisfying the condition*

$$\tau(E(a/A_1)b) = \tau(ab)$$

for $a \in A$, $b \in L_1(A_1, \widetilde{\tau})$, and the mapping $E(\cdot/A_1)$ is a conditional expectation with respect to A_1.

Note that when trace is finite an analogous result has been proved in [2].

In what follows $E(\cdot/A_1)$ and its extension to $L_1(A, \tau)$ is called a *conditional expectation* w.r.t. A_1 [8].

We note that if $W(A)$ is the enveloping von Neumann algebra of a JW-algebra A, then the existence of conditional expectations from $W(A)$ onto A has been proved in [11].

Let A be a reversible JW-algebra with n.s.f. trace τ and τ_1 be its extension to $W(A)$. Further, we suppose that the restriction of τ to any considered subalgebras is semifinite. If B is a reversible JW-subalgebra of A, then we denote a conditional expectation to B by $E(\cdot/B)$. Similarly, a conditional expectation from $W(A)$ to a subalgebra $W(B)$ is denoted by $\widetilde{E}(\cdot/W(B))$.

In what follows we use the following lemma, whose proof immediately follows from the definition of τ_1.

Lemma 3.1. *Let $z \in \mathfrak{N}_{\tau_1} \cap R(A)$, and $z^* = -z$, then $\tau_1(z) = 0$.*

It is natural to ask: How the restriction of $\widetilde{E}(\cdot/W(B))$ to A is related to $E(\cdot/B)$? Our next result answers this question.

Theorem 3.2. *The restriction of $\widetilde{E}(\cdot/W(B))$ to A is equal to $E(\cdot/B)$.*

Proof. It is sufficient to prove that for every $a \in A$

$$\tau_1(ax) = \tau_1(E(a/B)x) \tag{3.1}$$

holds for all $x \in L_1(W(B), \tau_1)$, where $L_1(W(B), \tau_1) = L_1(R(B), \tau_1) + iL_1(R(B), \tau_1)$, and here $L_1(R(B), \tau_1)$ is the completion of $R(B) \cap \mathfrak{N}_{\tau_1}$ with respect to L_1-norm. We first prove (3.1) for $x \in L_1(R(B), \tau_1)$. A functional $\tau_1(h\cdot)$ is L_1-continuous on $L_1(W(B), \tau_1)$, for any $h \in W(B)$, therefore it is enough to prove (3.1) for any x taken from $R(B) \cap \mathfrak{N}_{\tau_1}$.

Let $x \in R(B) \cap \mathfrak{N}_{\tau_1}$. Since $x = y + z$, $y, z \in R(B) \cap \mathfrak{N}_{\tau_1}$ such that $y^* = y$, $z^* = -z$, then

$$\tau_1(ax) = \tau_1(ay) + \tau_1(az).$$

Thus, we need to prove that $\tau_1(E(a/B)x) = \tau_1(E(a/B)y) + \tau_1(E(a/B)z)$. By Lemma 3.1, one finds $\tau_1(z) = 0$, and due to $(az + za)^* = -(az + za)$, again using Lemma 3.1 we get $\tau_1(az + za) = 0$, which means $\tau_1(az) + \tau_1(za) = 0$, so $2\tau_1(az) = 0$, i.e. $\tau_1(az) = 0$. Consequently, $\tau_1(E(a/B)z) = 0$. Since $y \in B \cap \mathfrak{N}_{\tau_1}$, then by Theorem

3.1 one has $\tau_1(ay) = \tau_1(E(a/B)y)$, i.e. $\tau_1(ax) = \tau_1(E(a/B)x)$ for any $x \in R(B)$. Using the linearity of τ_1 and by the standard argument one can prove (3.1) for an arbitrary element taken from $L_1(W(B), \tau_1)$. ∎

Suppose that $\{A_\alpha\}_{\alpha\in\mathbb{R}_+}$ is a family of reversible JW-subalgebras of A containing the identity operator $\mathbb{1}$ such that the set $\bigcup\limits_\alpha A_\alpha$ is weakly dense in A.

Definition 3.1. *A family $\{x_\alpha\}_{\alpha\in\mathbb{R}_+} \subset L_1(A, \tau)$ is called a supermartingale if for every $\alpha \in \mathbb{R}_+$*

(1) $x_\alpha \in L_1(A_\alpha, \tau)$

(2) *If $\alpha_1 \leq \alpha_2$, then $E(x_{\alpha_2}/A_{\alpha_1}) \leq x_{\alpha_1}$.*

Definition 3.2. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of reversible JW-subalgebras of A containing the identity operator $\mathbb{1}$, and the set $\bigcup_n A_n$ be weakly dense in A. A sequence $\{x_n\}$ of elements of A is called martingale adapted to the sequence $\{A_n\}$ if the following conditions are satisfied:*

(1) $x_n \in A_n$ $n = 1, 2, \ldots$
(2) $E(x_{n+1}/A_n) = x_n$ $n = 1, 2, \ldots$
(3) $\sup\limits_n \|x_n\| < \infty$.

Note that if in Definitions 3.1 and 3.2, as a JW-algebra A, we take self-adjoint part of a von Neumman algebra W, then we get usual definitions of supermartingale and martingale, respectively, in a von Neumann algebra setting.

In [7], the following result was proved:

Theorem 3.3. [7] *Let W be a von Neumann algebra with a f.s.n. trace τ, and suppose that $\{x_\alpha\}$ is a supermartingale in $L_1(W, \tau)$. If $\{x_\alpha\}$ is weakly relatively compact, then there is an $x \in L_1(W, \tau)$ such that $x_\alpha \to x$ in $L_1(W, \tau)$.*

Further, we shall prove an analog of such a theorem in a JW-algebra setting. To do it we shall need an auxiliary result.

It is known [5] that $(L_1(W, \tau_1))^* = W$, $(L_1(A, \tau))^* = A$, and therefore by $\sigma_W = \sigma_W(L_1(W, \tau_1), W)$, $\sigma_A = \sigma_A(L_1(A, \tau), A)$ we denote weakly topologies on $L_1(W, \tau_1)$, $L_1(A, \tau)$, respectively.

Lemma 3.2. *Let A be a reversible JW-algebra with a n.s.f. trace τ. Then one has $\sigma_{W(A)} |_{L_1(A,\tau)} = \sigma_A$.*

Proof. Let $\{x_\alpha\}$, $x \in L_1(A, \tau)$ and $x_\alpha \xrightarrow{\sigma_{W(A)}} x$, then $\tau_1(ax_\alpha) \to \tau_1(ax)$ for every $a \in W(A)_+$. If, in particular, $a \in A_+ \subset W(A)_+$ then

$$\tau(a \circ x_\alpha) = \frac{1}{2}(\tau(ax_\alpha) + \tau(x_\alpha a)) = \tau(ax_\alpha) \to \tau(ax) = \tau(a \circ x).$$

Hence $x_\alpha \xrightarrow{\sigma_A} x$.

Conversely, let $x_\alpha \xrightarrow{\sigma_A} x$ and take $b \in W(A)_+$ with $b = c + id$, then $0 \leq c \in A$, $d^* = -d$ (see [6]). So, $\tau(cx_\alpha) \to \tau(cx)$ since $\frac{dx_\alpha + x_\alpha d}{2}$ is a skew-symmetric element in $R(A)$ then by Lemma 3.1 one gets $\tau_1(dx_\alpha) = \tau_1(\frac{dx_\alpha + x_\alpha d}{2}) = 0$. Hence $\tau_1(bx_\alpha) = \tau(cx_\alpha) \to \tau(cx) = \tau_1(bx)$. ∎

Theorem 3.4. *Let A be a reversible JW-algebra with a n.s.f. trace τ. Suppose that $\{x_\alpha\}$ is a supermartingale in $L_1(A,\tau)$. If the set $\{x_\alpha\}$ is weakly relatively compact in $L_1(A,\tau)$, then there is $x \in L_1(A,\tau)$ such that $x_\alpha \to x$ in L_1-norm.*

Proof. Let $W(A_\alpha)$ be an enveloping von Neumann algebra of A_α, then

$$x_\alpha \in L_1(A_\alpha, \tau) \subset L_1(W(A_\alpha), \tau_1).$$

Moreover, since $\widetilde{E}(\cdot/W(A_\alpha))$ is a conditional expectation on $W(A)$, then by Theorem 3.2, for $\alpha_1 \le \alpha_2$ one has $\widetilde{E}(x_{\alpha_2}/W(A_{\alpha_1})) = E(x_{\alpha_2}/A_{\alpha_1}) \le x_{\alpha_1}$, i.e. $\{x_\alpha\}$ is a supermartingale in $L_1(W(A),\tau_1)$. So Lemma 3.2 implies that $\{x_\alpha\}$ is weakly relatively compact in $L_1(W(A),\tau_1)$. Then, Theorem 3.3 yields the existence of $x \in L_1(W(A),\tau_1)$ such that $x_\alpha \to x$ in L_1-norm. Completeness of $L_1(A,\tau)$ w.r.t. L_1-norm implies that x belongs to $L_1(A,\tau)$. This completes the proof. ∎

Remark 3.1. Note that when the family $\{x_\alpha\}$ is a martingale, a similar result was proved in [2,8], therefore, our result extends the mentioned one. When trace is finite, a similar result was studied in [2,4].

Recall that a sequence $\{x_n\} \subset A$ is said to be *almost uniformly convergent* to $x \in A$ ($x_n \xrightarrow{a.u.} x$) if, for each $\varepsilon > 0$, there exists $p \in \mathrm{Proj}\ A$ with $\tau(p^\perp) < \varepsilon$ such that $\|(x_n - x)p\| \to 0$ as $n \to \infty$. Here Proj A stands for the set of all projections in A.

Theorem 3.5. *Let A be a reversible $JW-$ algebra with a faithful normal finite trace τ. Let $\{x_n\}$ be a martingale adapted to an increasing sequence $\{A_n\}$ of JW-subalgebras of A with conditional expectations. Then there exists a unique $x_\infty \in A_\infty = A(A_n, n \ge 1)$ such that $\{x_n\}$ strongly and almost uniformly converges to x_∞. Moreover, $x_n = E(x_\infty/A_n)$, for every $n \in \mathbb{N}$.*

Proof. Let $W(A_n)$ be the enveloping von Neumann algebra of A_n then $x_n \in A_n \subset W(A_n)$. By Theorem 3.2 one has $\widetilde{E}(x_{n+1}/W(A_n)) = E(x_{n+1}/A_n) = x_n$ i.e. $\{x_n\}$ is a martingale on A. Then [18, Theorem 9] implies the existence of $x_\infty \in W(A)$ such that $\{x_n\}$ strongly and almost uniformly converges to x_∞ and $x_n = \widetilde{E}(x_\infty/W(A_n))$.

Obviously, the strong topologies on A and $W(A)$ coincide, and from the weak closeness of A, we obtain $x_n \to x_\infty$ (strongly) and $x_\infty \in A$. Also, from a result of [14] we have $x_n \to x_\infty$ almost uniformly in A. ∎

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