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A Product Involving the β-Family in Stable Homotopy Theory

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Abstract. In the stable homotopy groups $\pi_{q(p^n+p^m+1)-3}(S)$ of the sphere spectrum S localized at the prime p greater than three, J. Lin constructed an essential family $\xi_{m,n}$ for $n \geqslant m+2 > 5$. In this paper, the authors show that the composite $\xi_{m,n}\beta_s \in \pi_{q(p^n+p^m+sp+s)-5}(S)$ for $2 \leqslant s < p$ is non-trivial, where $q = 2(p-1)$ and $\beta_s \in \pi_{q(sp+s-1)-2}(S)$ is the known β-family. We show our result by explicit combinatorial analysis of the (modified) May spectral sequence.

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1. Introduction and statement of results

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in homotopy theory.

So far, several methods have been found to determine the stable homotopy groups of spheres. For example we have the classical Adams spectral sequence (ASS) (cf. [1]) based on the Eilenberg-MacLane spectrum $K\mathbb{Z}_p$, whose E_2-term is $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)$ and the Adams differential is given by $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$. We also have the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum BP (cf. [3, 9, 10]).

Throughout this paper, we fix the prime $p \geqslant 5$ and set $q = 2(p-1)$. From [8], we know that $\mathrm{Ext}_A^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p,\mathbb{Z}_p)$ for all $i \geqslant 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of α_2, a_0^2, a_0h_i $(i > 0)$, g_i $(i \geqslant 0)$, k_i $(i \geqslant 0)$, b_i $(i \geqslant 0)$, and h_ih_j $(j \geqslant i+2, i \geqslant 0)$ whose

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internal degrees are $2q+1$, 2, p^iq+1, $q(p^{i+1}+2p^i)$, $q(2p^{i+1}+p^i)$, $p^{i+1}q$ and $q(p^i+p^j)$ respectively.

Let M denote the Moore spectrum modulo the prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{1.1}$$

Let $\alpha : \Sigma^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} V(1) \xrightarrow{j'} \Sigma^{q+1} M. \tag{1.2}$$

Let $\beta : \Sigma^{(p+1)q} V(1) \to V(1)$ be the v_2-map.

Definition 1.1. *We define, for $t \geqslant 1$, the β-family $\beta_t = jj'\beta^t i'i \in \pi_{q(tp+t-1)-2}(S)$. Here the maps i, j, i', j', and β are given as above.*

We have the following known result.

Theorem 1.1. [9, Theorem 2.12] *$\beta_t \neq 0 \in \pi_*(S)$ for $p \geqslant 5$ and $t \geqslant 1$.*

To determine the stable homotopy groups of spheres is very difficult. Thus not so many families of homotopy elements in the stable homotopy groups of spheres have been detected. See, for example, [2, 3, 8].

In [5], Liu obtained the following theorem, which is called the representative theorem.

Theorem 1.2. [5, Theorem 1.3] *For $p \geqslant 5$ and $2 \leqslant s < p$, there exists the second Greek letter element*

$$\widetilde{\beta}_s \in \mathrm{Ext}_A^{s,q(sp+s-1)+s-2}(\mathbb{Z}_p, \mathbb{Z}_p),$$

which converges to the β-family $\beta_s \in \pi_{q(sp+s-1)-2}(S)$ in the ASS. Moreover, $\widetilde{\beta}_s$ is represented by

$$a_2^{s-2} h_{2,0} h_{1,1} \in E_1^{s,q(sp+s-1)+s-2,*}$$

in the May spectral sequence (MSS).

In [4], Lin detected a new family of stable homotopy groups of spheres and showed the following theorem.

Theorem 1.3. [4] *For $p \geqslant 5$, $n \geqslant m+2 \geqslant 4$. Then*

$$h_0 h_n h_m \in \mathrm{Ext}_A^{3,q(p^n+p^m+1)}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and it converges to a family of homotopy elements of order p, denoted by $\xi_{m,n}$, in the stable homotopy groups of spheres $\pi_{q(p^n+p^m+1)-3}(S)$.

In this paper, we consider the non-triviality of the composite $\xi_{m,n}\beta_s$ and obtain the following theorem.

Theorem 1.4. *Let $p \geqslant 5$, $n \geqslant m+2 > 5$, $2 \leqslant s < p$. Then the product*

$$h_0 h_n h_m \widetilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+3,t(s)+s-2}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial family of homotopy elements $\xi_{m,n}\beta_s \in \pi_{t(s)+s-5}(S)$, where $t(s) = q(p^n + p^m + sp + s)$.

In this paper we make use of the ASS and the MSS to prove our theorem, especially the MSS. The method of the proof is very elementary. By this method, one can consider some similar problems, for example, the non-triviality of the composite $\xi_{m,n}\gamma_s$, where γ_s is the known γ-family (cf. [6]). The paper is arranged as follows: After giving some important lemmas on the MSS in Section 2, we will prove Theorem 1.4 in Section 3.

2. The ASS and some lemmas on the MSS

One of the main tools to determine the stable homotopy groups of spheres $\pi_*(S)$ is the ASS. In 1957, Adams constructed such a machinery in the form of a spectral sequence that making the doubly graded group $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ to the p-primary components of the stable homotopy groups of spheres by adapting the methods of homological algebra. From then on, the ASS has been a powerful tool in studying stable homotopy theory. Let X a spectrum of finite type and Y be a finite dimensional spectrum. Then there is a natural spectral sequence $\{E_r^{s,t}, d_r\}$ which is called ASS and

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X;\mathbb{Z}_p), H^*(Y;\mathbb{Z}_p)) \Rightarrow ([Y,X]_{t-s})_p, \tag{2.1}$$

where the differential is

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}. \tag{2.2}$$

If X and Y are sphere spectra S, then in the ASS

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p) \Rightarrow (\pi_{t-s}(S))_p, \tag{2.3}$$

the p-primary components of the group $\pi_{t-s}(S)$.

There are three problems in using the ASS: The calculation of the E_2-term, the computation of the differentials and the determination of the nontrivial extensions from E_∞ to $\pi_*(S)$. So, in order to compute the stable homotopy groups of spheres with the ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$. The most successful tool for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ is the MSS.

From [10], there is a MSS $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i}|m>0, i\geqslant 0)\otimes P(b_{m,i}|m>0, i\geqslant 0)\otimes P(a_n|n\geqslant 0), \tag{2.4}$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i}\in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i}\in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n\in E_1^{1,2p^n-1,2n+1}.$$

The r-th May differential is

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}, \tag{2.5}$$

and if $x\in E_r^{s,t,*}$ and $y\in E_r^{s',t',*}$, then $d_r(x\cdot y) = d_r(x)\cdot y + (-1)^s x\cdot d_r(y)$. From [7, Proposition 2.5], there exists a graded commutativity in the May E_1-term as follows:

$$\begin{cases} a_m h_{n,j} = h_{n,j} a_m, & h_{m,k} h_{n,j} = -h_{n,j} h_{m,k}, \\ a_m b_{n,j} = b_{n,j} a_m, & h_{m,k} b_{n,j} = b_{n,j} h_{m,k}, \\ a_m a_n = a_n a_m, & b_{m,n} b_{i,j} = b_{i,j} b_{m,n}. \end{cases} \tag{2.6}$$

The first May differential d_1 is given by

$$\left\{\begin{array}{l} d_1(h_{i,j})=\sum_{0<k<i} h_{i-k,k+j}h_{k,j},\\ d_1(a_i)=\sum_{0\leqslant k<i} h_{i-k,k}a_k,\\ d_1(b_{i,j})=0. \end{array}\right. \tag{2.7}$$

For each element $x \in E_1^{s,t,\mu}$, we define filt $x = s$, deg $x = t$, M $(x) = \mu$. Then we have

$$\left\{\begin{array}{l} \text{filt } h_{i,j} = \text{filt } a_i = 1,\ \text{filt } b_{i,j} = 2,\\ \deg h_{i,j} = 2(p^i-1)p^j = q(p^{i+j-1}+\cdots+p^j),\\ \deg b_{i,j} = 2(p^i-1)p^{j+1} = q(p^{i+j}+\cdots+p^{j+1}),\\ \deg a_i = 2p^i - 1 = q(p^{i-1}+\cdots+1)+1,\\ \deg a_0 = 1,\\ \text{M } (h_{i,j}) = \text{M } (a_{i-1}) = 2i-1,\\ \text{M } (b_{i,j}) = (2i-1)p, \end{array}\right. \tag{2.8}$$

where $i \geqslant 1$, $j \geqslant 0$.

In Section 3, we will need the following lemmas on the MSS.

By the knowledge on p-adic expression in number theory, we have that for each integer $t \geqslant 0$, it can be always expressed uniquely as

$$t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + c_{-1},$$

where $0 \leqslant c_i < p$ $(0 \leqslant i < n)$, $0 < c_n < p$, $0 \leqslant c_{-1} < q$.

Lemma 2.1. [6, Proposition 1.1] *Let t as above. Let s_1 be a positive integer with $0 < s_1 < p$. If there exists some $0 \leqslant j \leqslant n$ such that $c_j > s_1$, then in the MSS*

$$E_1^{s_1,t,*} = 0.$$

Lemma 2.2. *Let t as above. Let s_1 be a positive integer with $0 < s_1 < q$. If $c_{-1} > s_1$, then in the MSS,*

$$E_1^{s_1,t,*} = 0.$$

Proof. The proof is similar to that of [6, Proposition 1.1] and is omitted here. ∎

Let t as above and s a given positive integer. Suppose that in the MSS, a generator $\omega \in E_1^{s,t,*}$ is of the form $w = x_1x_2\cdots x_m$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $1 \leqslant i \leqslant m$, $0 \leqslant k \leqslant n+1$, $0 < u+z \leqslant n$, $0 < l+j \leqslant n+1$, $l > 0$, $j \geqslant 0$, $u > 0$, $z \geqslant 0$. By (2.8), we can assume that for any $1 \leqslant i \leqslant m$, $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,1}p + c_{i,0}) + c_{i,-1}$, where $c_{i,j} = 0$ or 1 for $0 \leqslant j \leqslant n$, $c_{i,-1} = 1$ if $x_i = a_{k_i}$, or $c_{i,-1} = 0$. It follows that

$$\deg \omega = \sum_{i=1}^m \deg x_i = q\left[\left(\sum_{i=1}^m c_{i,n}\right)p^n + \cdots + \left(\sum_{i=1}^m c_{i,1}\right)p^1 + \sum_{i=1}^m c_{i,0}\right] + \sum_{i=1}^m c_{i,-1}.$$

For convenience, we denote $\sum_{i=1}^m c_{i,j}$ by $\bar{c}_j$ for $j \geqslant -1$.

Lemma 2.3. *With notation as above. If there exist three integers $-1 \leqslant i_1 < i_2 < i_3 \leqslant n$ such that $\bar{c}_{i_1} + \bar{c}_{i_3} - m > \bar{c}_{i_2}$, then w is impossible to exist.*

Proof. By (2.8) and (2.4), one easily gets the lemma. ∎

Lemma 2.4. *With notation as above. Suppose that $m = s$, and there exist three integers i_1, i_2 and i_3 satisfying the following conditions that*

(i) $-1 \leqslant i_1 < i_2 < i_3 \leqslant n$;

(ii) $\overline{c}_{i_1} + \overline{c}_{i_3} - m \leqslant \overline{c}_{i_2}$;

(iii) $\overline{c}_j = \begin{cases} 0 & -1 \leqslant j < i_1 \\ 0 & i_3 < j \leqslant n. \end{cases}$

Then we have the following consequences:

(1) *When $i_1 > -1$, there are $(\overline{c}_{i_1} + \overline{c}_{i_3} - m)$ $h_{\overline{c}_{i_3}-\overline{c}_{i_1}+1,\overline{c}_{i_1}}$'s among ω. Furthermore, if $\overline{c}_{i_1} + \overline{c}_{i_3} - m > 1$, then $w = 0$.*

(2) *When $i_1 = -1$, there are $(\overline{c}_{i_1} + \overline{c}_{i_3} - m)$ a_{i_3+1}'s among ω.*

Proof. By (2.8) and (2.4), the desired results easily follow. ∎

3. Proof of Theorem 1.4

In this section, we will determine two Ext groups which will be used in the proof of Theorem 1.4. In order to do it, we first consider some May E_1-terms $E_1^{u,v,*}$ with two given integers u and v, and show the following lemma.

Lemma 3.1. *Let $p \geqslant 5$, $n \geqslant m+2 > 5$, $2 \leqslant s < p$ and $1 \leqslant r \leqslant s+3$. Then in the MSS, we have*

$$E_1^{s+3-r,t(s)+s-r-1,*} = \begin{cases} \mathbb{Z}_p\{\mathbf{g}_1, \cdots, \mathbf{g}_7\} & r = 1 \text{ and } s = p-1, \\ 0 & \text{other}. \end{cases} \tag{3.1}$$

Here, $t(s) = q(p^n+p^m+sp+s)$, and $\mathbf{g}_1, \cdots, \mathbf{g}_7$ equal elements $a_n^{p-3}h_{3,0}h_{1,m}h_{n-2,2}h_{n,0}$, $a_n^{p-3}h_{1,2}h_{m+1,0}h_{n-m,m}h_{n,0}$, $a_{m+1}a_n^{p-4}h_{3,0}h_{n-m,m}h_{n-2,2}h_{n,0}$, $a_n^{p-3}h_{3,0}h_{m-1,2}h_{n-m,m}h_{n,0}$, $a_n^{p-3}h_{3,0}h_{m+1,0}h_{n-m,m}h_{n-2,2}$, $a_3a_n^{p-4}h_{m+1,0}h_{n-m,m}h_{n-2,2}h_{n,0}$ and $a_m^{p-3}h_{3,0}h_{m,0}h_{m-2,2}h_{1,n}$, respectively.

Proof. We divide the proof into the following two cases.

Case 1. $s-r-1 < 0$. By the knowledge on p-adic expression in number theory and $1 \leqslant r \leqslant s+3$, we would have $s+3-r < s-r-1+q < q$. In this case

$$E_1^{s+3-r,t(s)+s-r-1,*} = 0$$

by Lemma 2.2.

Case 2. $s-r-1 \geqslant 0$. Thus $1 \leqslant r \leqslant s-1$. If $r \geqslant 4$, then $s+3-r < s$, which implies that in this case $E_1^{s+3-r,t(s)+s-r-1,*} = 0$ by Lemma 2.1. Consequently, in the rest of the proof, we always assume $r \leqslant 3$.

Consider $\omega = x_1x_2\cdots x_{m'} \in E_1^{s+3-r,t(s)+s-r-1,*}$ in the MSS, where x_i is one of a_k, $h_{l,j}$, $b_{u,z}$, $1 \leqslant i \leqslant m'$, $0 \leqslant k \leqslant n+1$, $0 < l+j \leqslant n+1$, $0 < u+z \leqslant n$, $l > 0$, $j \geqslant 0$, $u > 0$, $z \geqslant 0$. By (2.8), we can assume that deg $x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,1}p + c_{i,0}) + c_{i,-1}$, where $c_{i,j} = 0$ or 1 for $0 \leqslant j \leqslant n$, $c_{i,-1} = 1$ if $x_i = a_{k_i}$, or

$c_{i,-1} = 0$. It follows that

$$
(3.2)\qquad \begin{cases} \text{filt } \omega = \sum_{i=1}^{m'} \text{filt } x_i = s + 3 - r, \\ \deg \omega = \sum_{i=1}^{m'} \deg x_i = q\big[(\sum_{i=1}^{m'} c_{i,n})p^n + (\sum_{i=1}^{m'} c_{i,n-1})p^{n-1} + \cdots \\ \qquad + (\sum_{i=1}^{m'} c_{i,m})p^m + (\sum_{i=1}^{m'} c_{i,m-1})p^{m-1} + \cdots \\ \qquad + (\sum_{i=1}^{m'} c_{i,1})p + (\sum_{i=1}^{m'} c_{i,0})\big] \\ \qquad + (\sum_{i=1}^{m'} c_{i,-1}) = t(s) + s - r - 1. \end{cases}
$$

Note that filt $x_i = 1$ or 2 and $2 \leqslant s < p$. From $\sum_{i=1}^{m'}$ filt $x_i = s + 3 - r$, it follows that

$$
m' \leqslant s + 2 < p + 2.
$$

Using $0 \leqslant s,\ s - r - 1 < p$ and the knowledge on the p-adic expression in number theory, we have the following equations from (3.2).

$$
(3.3)\qquad \begin{cases} \sum_{i=1}^{m'} c_{i,-1} = s - r - 1 + \lambda_{-1} q, & \lambda_{-1} \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,0} + \lambda_{-1} = s + \lambda_0 p, & \lambda_0 \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,1} + \lambda_0 = s + \lambda_1 p, & \lambda_1 \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,2} + \lambda_1 = 0 + \lambda_2 p, & \lambda_2 \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,3} + \lambda_2 = 0 + \lambda_3 p, & \lambda_3 \geqslant 0, \\ \cdots & \cdots \\ \sum_{i=1}^{m'} c_{i,m-1} + \lambda_{m-2} = 0 + \lambda_{m-1} p, & \lambda_{m-1} \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,m} + \lambda_{m-1} = 1 + \lambda_m p, & \lambda_m \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,m+1} + \lambda_m = 0 + \lambda_{m+1} p, & \lambda_{m+1} \geqslant 0, \\ \cdots & \cdots \\ \sum_{i=1}^{m'} c_{i,n-2} + \lambda_{n-3} = 0 + \lambda_{n-2} p, & \lambda_{n-2} \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,n-1} + \lambda_{n-2} = 0 + \lambda_{n-1} p, & \lambda_{n-1} \geqslant 0, \\ \sum_{i=1}^{m'} c_{i,n} + \lambda_{n-1} = 1. \end{cases}
$$

By the knowledge on the p-adic expression and $m' < p + 2$, we have $\lambda_{-1} = \lambda_0 = \lambda_1 = 0$. For convenience, in the rest of the proof we will use $\bar{c}_j$ to denote $\sum_{i=1}^{m'} c_{i,j}$ for $-1 \leqslant j \leqslant n$. From the fourth equation of (3.3) $\bar{c}_2 = \lambda_2 p$, λ_2 may equal 0 or 1.

Subcase 2.1. $\lambda_2 = 0$.

Assertion 3.1 If $\lambda_2 = 0$, then $\lambda_3 = \cdots = \lambda_{m-1} = 0$.

Suppose $\lambda_3 = 1$. Then from the fifth equation of (3.3) we would have $\bar{c}_3 = p$, which implies that m' can only equal p or $p+1$. Note that $2 \leqslant s < p$. From $\bar{c}_3 = p$, $\bar{c}_2 = 0$ and $\bar{c}_1 = s$, one would have $\bar{c}_3 + \bar{c}_1 - m' = p + s - m' \geqslant 1 > 0 = \bar{c}_2$. Thus by Lemma 2.3, ω is impossible to exist. Thus, $\lambda_3 = 0$. Similarly, one can show that $\lambda_4 = \cdots = \lambda_{m-1} = 0$. Assertion 3.1 is proved.

From the $(m+2)$-th equation of (3.3) $\bar{c}_m = 1 + \lambda_m p$, λ_m may equal 0 or 1.

Subcase 2.1.1. $\lambda_m = 0$. An argument similar to that used in Assertion 3.1 shows that $\lambda_{m+1} = \cdots = \lambda_{n-1} = 0$. Thus we have

$\overline{c}_n$	$\overline{c}_{n-1}$	$\cdots$	$\overline{c}_{m+1}$	$\overline{c}_m$	$\overline{c}_{m-1}$	$\cdots$	$\overline{c}_3$	$\overline{c}_2$	$\overline{c}_1$	$\overline{c}_0$	$\overline{c}_{-1}$
1	0	$\cdots$	0	1	0	$\cdots$	0	0	s	s	$s-r-1$

If ω has $h_{1,n}h_{1,m}$ as factors, one can let $\omega = h_{1,n}h_{1,m}\omega_1$ by (2.6). Then filt $\omega_1 = s+1-r$ and deg $\omega_1 = spq + sq + (s-r-1)$. When $r > 1$, ω_1 is impossible to exist by Lemma 2.1. So ω is impossible to exist either. When $r = 1$, ω_1 has $(s-2)$ a_2's among ω if ω exists by Lemma 2.4. Then up to sign $\omega_1 = a_2^{s-2}\omega_2$ with $\omega_2 \in E_1^{2,2pq+2q,*} = 0$, which means $\omega = 0$.

Similarly, ω cannot have $h_{1,n}b_{1,m-1}$, $b_{1,n-1}h_{1,m}$, $b_{1,n-1}b_{1,m-1}$ as factors either.

Subcase 2.1.2. $\lambda_m = 1$. In this case $\lambda_{m+1} = \cdots = \lambda_{n-1} = 1$. An argument similar to that used in Assertion 3.1 can show that in this case ω is impossible to exist.

Subcase 2.2. $\lambda_2 = 1$. In this case, $(\lambda_3, \cdots, \lambda_{m-1})$ must equal $(1, \cdots, 1)$. From the $(m+2)$-th equation of (3.3) $\overline{c}_m = \lambda_m p$, λ_m may equal 0 or 1.

Subcase 2.2.1. $\lambda_m = 1$. In this case, $(\lambda_{m+1}, \cdots, \lambda_{n-1})$ must equal $(1, \cdots, 1)$. Thus we have

$\overline{c}_n$	$\overline{c}_{n-1}$	$\cdots$	$\overline{c}_{m+1}$	$\overline{c}_m$	$\overline{c}_{m-1}$	$\cdots$	$\overline{c}_3$	$\overline{c}_2$	$\overline{c}_1$	$\overline{c}_0$	$\overline{c}_{-1}$
0	$p-1$	$\cdots$	$p-1$	p	$p-1$	$\cdots$	$p-1$	p	s	s	$s-r-1$

If $r = 2$ or 3, then by $p \geqslant 5$, $2 \leqslant s < p$ and $m' \leqslant s+3-r$ one can have

$$\overline{c}_2 + \overline{c}_m - m' = p + p - m' \geqslant p + p - (s+1) \geqslant p > p - 1 = \overline{c}_3,$$

which implies that ω is impossible to exist by Lemma 2.3.

If $r = 1$, then one has filt $\omega = s+2$. From $\overline{c}_m = p$, one has $m' \geqslant p$ by $c_{i,m} = 0$ or 1. Thus m' may equal p or $p+1$. If $m' = p$, then $\overline{c}_2 + \overline{c}_m - m' = p > p - 1 = \overline{c}_3$, which implies that ω is impossible to exist by Lemma 2.3. Thus, in the rest of Subcase 2.2.1 we always assume that $r = 1$ and $m' = p+1$. Thus we have $s = p-1$, filt $\omega = p+1$ and $\omega = x_1 \ \cdots \ x_{p+1} \in E(h_{i,j}|i > 0, j \geqslant 0) \otimes P(a_n|n \geqslant 0)$. The table above becomes

$\overline{c}_n$	$\overline{c}_{n-1}$	$\cdots$	$\overline{c}_{m+1}$	$\overline{c}_m$	$\overline{c}_{m-1}$	$\cdots$	$\overline{c}_3$	$\overline{c}_2$	$\overline{c}_1$	$\overline{c}_0$	$\overline{c}_{-1}$
0	$p-1$	$\cdots$	$p-1$	p	$p-1$	$\cdots$	$p-1$	p	$p-1$	$p-1$	$p-3$

Assertion 3.2. ω has $p-1$ factors whose degrees are q(higher terms on $p + p^m + \cdots + p^2 +$ lower terms on $p) + \epsilon$, where $\epsilon = 0$ or 1, and two factors whose degree are q(higher terms on $p + p^m$) and $q(p^2 +$ lower terms on $p) + \epsilon$, respectively.

This assertion can be easily verified by (2.8) and (3.2).

Assertion 3.3. ω cannot have $h_{2,1}$ or $h_{j,m}$ $(2 \leqslant j < n-m)$ as a factor.

Otherwise, we can let $\omega = \omega_1 h_{2,1}$ by (2.6). Then filt $\omega_1 = p$, deg $\omega_1 = q[(p-1)p^{n-1} + \cdots + (p-1)p^{m+1} + pp^m + (p-1)p^{m-1} + \cdots + (p-1)p^3 + (p-1)p^2 + (p-2)p + (p-1)] + p - 3$. In this case ω_1 is impossible to exist by Lemma 2.3. Thus ω cannot have $h_{2,1}$ as a factor. Similarly, ω cannot have $h_{j,m}$ $(2 \leqslant j < n-m)$ as a factor.

From Assertions 3.2 and 3.3, there must be one of $h_{1,2}h_{1,m}$, $h_{3,0}h_{1,m}$, $a_3h_{1,m}$, $h_{1,2}h_{n-m,m}$, $h_{3,0}h_{n-m,m}$ or $a_3h_{n-m,m}$ among ω if ω exists. By (2.6), we let $\omega = \omega_1\omega_2$, where ω_2 is one of the six factors above. Then filt $\omega_1 = p-1$.

(i) If $\omega_2 = h_{1,2}h_{1,m}$, then deg $\omega_1 = q[(p-1)p^{n-1}+\cdots+(p-1)p^m+\cdots+(p-1)p+(p-1)]+p-3$. So there must be two $h_{n,0}$'s in ω by Lemma 2.4 (1), which implies that $\omega_1 = 0$. Then $\omega = 0$.

Similarly, one can show that $\omega = 0$ if $\omega_2 = a_3h_{1,m}$.

(ii) If $\omega_2 = h_{3,0}h_{1,m}$, then deg $\omega_1 = q[(p-1)p^{n-1}+\cdots+(p-1)p^m+\cdots+(p-1)p^2+(p-2)p+(p-2)]+p-3$. By Lemma 2.4, ω_1 must equal $a_n^{p-3}h_{n,0}h_{n-2,2}$ up to sign . Thus up to sign $\omega = a_n^{p-3}h_{3,0}h_{1,m}h_{n-2,2}h_{n,0}$, denoted by $\mathbf{g}_1$.

(iii) If $\omega_2 = h_{1,2}h_{n-m,m}$, then deg $\omega_1 = q[(p-2)p^{n-1}+\cdots+(p-2)p^{m+1}+(p-1)p^m+(p-1)p^{m-1}+\cdots+(p-1)p^2+(p-1)p+(p-1)]+p-3$, so ω_1 has at least $p-4$ a_n's by Lemma 2.4. We let $\omega_1 = \omega_3a_n^{p-4}$ by (2.6). Thus filt $\omega_3 = 3$, deg $\omega_3 = q(2p^{n-1}+\cdots+2p^{m+1}+3p^m+3p^{m-1}+\cdots+3p^3+3p^2+3p+3)+1$. Then $\omega_3 \in E_1^{3,\deg\,\omega_3,*} = \mathbb{Z}_p\{a_nh_{n,0}h_{m+1,0}\}$. Thus up to sign $\omega = a_n^{p-3}h_{1,2}h_{m+1,0}h_{n-m,m}h_{n,0}$, denoted by $\mathbf{g}_2$.

(iv) If $\omega_2 = h_{3,0}h_{n-m,m}$, an argument similar to that used in (iii) shows $\omega_1 = a_n^{p-4}\omega_3$ with $\omega_3 \in E_1^{3,t,*} = \mathbb{Z}_p\{a_{m+1}h_{n-2,2}h_{n,0}, a_nh_{m-1,2}h_{n,0}, a_nh_{m+1,0}h_{n-2,2}\}$, where $t = q(2p^{n-1}+\cdots+2p^{m+1}+3p^m+\cdots+3p^2+2p+2)+1$. Thus up to sign $\omega = a_{m+1}a_n^{p-4}h_{3,0}h_{n-m,m}h_{n-2,2}h_{n,0}$, $a_n^{p-3}h_{3,0}h_{m-1,2}h_{n-m,m}h_{n,0}$ or $a_n^{p-3}h_{3,0}h_{m+1,0}h_{n-m,m}h_{n-2,2}$, denoted by $\mathbf{g}_3$, $\mathbf{g}_4$, $\mathbf{g}_5$ respectively.

(v) If $\omega_2 = a_3h_{n-m,m}$, by an argument similar to that used in (iii) we have $\omega_1 = a_n^{p-5}\omega_3$ with $\omega_3 \in E_1^{4,t',*} = \mathbb{Z}_p\{a_nh_{m+1,0}h_{n,0}h_{n-2,2}\}$, where $t' = q(3p^{n-1}+\cdots+3p^{m+1}+4p^m+\cdots+4p^2+3p+3)+1$. Thus up to sign $\omega = a_3a_n^{p-4}h_{m+1,0}h_{n-m,m}h_{n-2,2}h_{n,0}$, denoted by $\mathbf{g}_6$.

Subcase 2.2.2 $\lambda_m = 0$. By an argument similar to that used in Assertion 3.1, we have $\lambda_{m+1} = \cdots = \lambda_{n-1} = 0$. Thus we have

$\overline{c}_n$	$\overline{c}_{n-1}$	$\cdots$	$\overline{c}_{m+1}$	$\overline{c}_m$	$\overline{c}_{m-1}$	$\cdots$	$\overline{c}_3$	$\overline{c}_2$	$\overline{c}_1$	$\overline{c}_0$	$\overline{c}_{-1}$
1	0	$\cdots$	0	0	$p-1$	$\cdots$	$p-1$	p	s	s	$s-r-1$

Obviously ω must have a factor $h_{1,n}$. One can let $\omega = \omega_1h_{1,n}$ by (2.6).

If $r = 2$ or 3, it is easy to show that ω_1 is impossible to exist by Lemma 2.1, which implies that ω is impossible to exist either.

If $r = 1$, by Lemma 2.1 it is easy to get that in this case s must equal $p-1$ and m' must equal $p+1$. By an argument similar to that used in Subcase 2.2.1, we get that up to sign $\omega = a_m^{p-3}h_{3,0}h_{m,0}h_{m-2,2}h_{1,n} \in E^{p+1,t(p-1)+p-3,(2m+1)p-2m-3}$, denoted by $\mathbf{g}_7$.

Combining Cases 1 and 2, we complete the proof of the lemma. ∎

By use of Lemma 3.1, we now show the non-triviality of $h_0h_nh_m\widetilde{\beta}_s$.

Theorem 3.1. *Let $p \geqslant 5$, $n \geqslant m+2 > 5$, $2 \leqslant s < p$. Then the product*

$$h_0h_nh_m\widetilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+3,t(s)+s-2}(\mathbb{Z}_p,\mathbb{Z}_p),$$

where $t(s) = q(p^n+p^m+sp+s)$.

Proof. Since $h_{1,n} (n \geqslant 0)$ and $a_2^{s-2}h_{2,0}h_{1,1}$ are permanent cycles in the MSS and converge nontrivially to h_n, $\widetilde{\beta}_s \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ respectively, $a_2^{s-2}h_{2,0}h_{1,1}h_{1,0}h_{1,n}h_{1,m} \in E_1^{s+3,t(s)+s-2,*}$ is a permanent cycle in the MSS and converges to $h_0h_nh_m\widetilde{\beta}_s \in \mathrm{Ext}_A^{s+3,t(s)+s-2}(\mathbb{Z}_p, \mathbb{Z}_p)$.

Case 1. $s = p-1$. By (2.7), one can have that up to sign

$$(3.4) \qquad \begin{cases} d_1(\mathbf{g}_1) = a_n^{p-3}h_{1,0}h_{3,0}h_{1,m}h_{n-2,2}h_{n-1,1} + \cdots & \neq 0; \\ d_1(\mathbf{g}_2) = a_n^{p-3}h_{1,0}h_{1,2}h_{m+1,1}h_{n-m,m}h_{n-1,1} + \cdots & \neq 0; \\ d_1(\mathbf{g}_3) = a_n^{p-4}a_{m+1}h_{1,0}h_{3,0}h_{n-m,m}h_{n-2,2}h_{n-1,1} + \cdots & \neq 0; \\ d_1(\mathbf{g}_4) = a_n^{p-3}h_{1,0}h_{3,0}h_{m-1,2}h_{n-m,m}h_{n-1,1} + \cdots & \neq 0; \\ d_1(\mathbf{g}_5) = a_n^{p-3}h_{1,0}h_{3,0}h_{m,1}h_{n-m,m}h_{n-2,2} + \cdots & \neq 0; \\ d_1(\mathbf{g}_6) = a_n^{p-4}a_3h_{1,0}h_{m+1,0}h_{n-m,m}h_{n-2,2}h_{n-1,1} + \cdots & \neq 0; \\ d_1(\mathbf{g}_7) = a_m^{p-3}h_{1,2}h_{3,0}h_{m-3,3}h_{1,n}h_{n,0} + \cdots & \neq 0. \end{cases}$$

Obviously the first May differential of each of the seven generators contains at least a term which is not in the first May differentials of the other generators, which implies that $d_1(\mathbf{g}_1), \cdots, d_1(\mathbf{g}_7)$ are linearly independent. Thus, $E_2^{p+1,t(p-1)+p-3,*} = 0$. It follows that

$$E_r^{p+1,t(p-1)+p-3,*} = 0 \text{ for } r \geqslant 2.$$

Meanwhile, by (2.8) we have that $M(\mathbf{g}_i) = (2n+1)p - 2n - 3$ $(1 \leqslant i \leqslant 6)$, $M(\mathbf{g}_7) = (2m+1)p - 2m - 3$ and $M(a_2^{p-3}h_{2,0}h_{1,1}h_{1,0}h_{1,n}h_{1,m}) = 5p - 8$. Then from (2.5) one has

$$a_2^{p-3}h_{2,0}h_{1,1}h_{1,0}h_{1,n}h_{1,m} \notin d_1(E_1^{p+1,t(p-1)+p-3,p(2n+1)-2n-3})$$

and

$$a_2^{p-3}h_{2,0}h_{1,1}h_{1,0}h_{1,n}h_{1,m} \notin d_1(E_1^{p+1,t(p-1)+p-3,p(2m+1)-2m-3}).$$

Thus we have the permanent cycle

$$a_2^{p-3}h_{2,0}h_{1,1}h_{1,0}h_{1,n}h_{1,m} \in E_r^{p+2,t(p-1)+p-3,*}$$

cannot be hit by any May differential. It follows that $h_0h_nh_m\widetilde{\beta}_{p-1} \neq 0$.

Case 2. $2 \leqslant s < p-1$. From Lemma 3.1 one has that in this case the May E_1-term

$$E_1^{s+2,t(s)+s-2,*} = 0.$$

Thus one has

$$E_r^{s+2,t(s)+s-2,*} = 0 \text{ for } r > 1.$$

Consequently, the permanent cycle

$$h_{1,0}h_{1,n}h_{1,m}a_2^{s-2}h_{2,0}h_{1,1} \in E_r^{s+3,t(s)+s-2,*}$$

cannot be hit by any differential in the MSS. Then

$$h_0h_nh_m\widetilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+3,t(s)+s-2}(\mathbb{Z}_p, \mathbb{Z}_p).$$

From Cases 1 and 2, we complete the proof of the theorem. ∎

Theorem 3.2. *Let $p \geqslant 5$, $n \geqslant m+2 > 5$, $2 \leqslant s < p$ and $2 \leqslant r \leqslant s+3$. Then*

$$\mathrm{Ext}_A^{s+3-r,t(s)+s-r-1,*}(\mathbb{Z}_p,\mathbb{Z}_p)=0,$$

where $t(s)=q(p^n+p^m+sp+s)$.

Proof. From the case $2 \leqslant r \leqslant s+3$ in Lemma 3.1, we have that in the MSS

$$E_1^{s+3-r,t(s)+s-r-1,*}=0.$$

The theorem follows easily by the MSS. ∎

Now we give the proof of Theorem 1.4 .

Proof of Theorem 1.4. From Theorem 1.2, we have that $\widetilde{\beta}_s$ converges to β-family $\beta_s \in \pi_{spq+(s-1)q+s-2}(S)$ in ASS. From Theorem 1.3, $h_0h_nh_m \in \mathrm{Ext}_A^{3,p^nq+p^mq+q}$ $(\mathbb{Z}_p,\mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial family of homotopy elements $\xi_{m,n} \in \pi_{p^nq+p^mq-3}(S)$. Hence, we have that the composite

$$\xi_{m,n}\beta_s$$

is represented up to a nonzero scalar by

$$h_0h_nh_m\widetilde{\beta}_s \neq 0 \in \mathrm{Ext}_A^{s+3,t(s)+s-2}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS (cf. Theorem 3.1).

Moreover, from Theorem 3.2, $h_0h_nh_m\widetilde{\beta}_s$ cannot be hit by any differential in the ASS. Consequently, the corresponding homotopy element $\xi_{m,n}\beta_s$ is nontrivial. This proves Theorem 1.4. ∎

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