BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33**(3) (2010), 421–428

Rational Recursive Equations Characterizing Cotangent-tangent and Hyperbolic Cotangent-tangent Functions

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Abstract. Using a technique of Rhouma in 2005, closed form solutions of certain rational recursive equations characterizing the cotangent-tangent and the hyperbolic cotangent-tangent function solutions are derived.

2010 Mathematics Subject Classification: 39B32

Key words and phrases: Rational recursive equations, cotangent-tangent functions, hyperbolic cotangent-tangent functions.

1. Introduction

In 2005, Rhouma, [3], gave a closed form solution to the recursive difference equation

$$y_{n+2} = \frac{y_n y_{n+1} - 1}{y_n + y_{n+1}}, \tag{1.1}$$

which was originated from an open problem in the book [1] (see also [2]) as follows:

Theorem 1.1. *Let y_0 and y_1 be arbitrary real numbers such that y_n exists for all $n \in \mathbb{N} \cup \{0\}$ and $\{F_n\}$ is the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ $(n \geq 0)$. The solution to equation* (1.1) *exists for all $n \in \mathbb{N} \cup \{0\}$ if and only if*

$$F_{n-2}\theta_0 + F_{n-1}\theta_1 \neq 0 \pmod{2\pi},$$

where $\theta_0 = -2\text{arccot}\ y_0$ and $\theta_1 = -2\text{arccot}\ y_1$.

Communicated by Mohammad Sal Moslehian.
Received: March 25, 2009; *Revised:* June 11, 2009.

When it exists, the solution to (1.1) *is given by*

$$y_n = -\cot\left(\frac{F_{n-2}\theta_0 + F_{n-1}\theta_1}{2}\right) = \cot(F_{n-2}\ \mathrm{arccot}\ y_0 + F_{n-1}\ \mathrm{arccot}\ y_1). \tag{1.2}$$

Moreover,

(1) *if θ_0 and θ_1 are both rational multiples of π, then either $\{y_n\}$ diverges in finitely many steps or $\{y_n\}$ is periodic;*
(2) *if θ_0 is a rational multiple of π and θ_1 is not (or vice versa), then $\{y_n\}$ is aperiodic and does exist for all n.*

It is easily checked that the cotangent function in (1.2) satisfies (1.1) showing that the rational recursive equation (1.1) does indeed characterize the cotangent function. Rhouma's technique is first to transform (1.1) to an equivalent form of

$$y_{n+2} = i\frac{(y_{n+1}+i)(y_n+i) + (y_{n+1}-i)(y_n-i)}{(y_{n+1}+i)(y_n+i) - (y_{n+1}-i)(y_n-i)}, \tag{1.3}$$

or

$$\frac{y_{n+2}-i}{y_{n+2}+i} = \frac{y_{n+1}-i}{y_{n+1}+i}\cdot\frac{y_n-i}{y_n+i},$$

which is a difference equation of the shape

$$x_{n+2} = \alpha x_{n+1}x_n. \tag{1.4}$$

A closed form solution of this last equation is derived without difficulty.

The difference equation (1.1) is interesting at least in two respects. First, it resembles the well-known identity of the cotangent function. Second, putting $y_n = \cot z_n$, with the values of z_n restricted to the open interval $(0,\pi)$, the difference equation leads to the Fibonacci recurrence modulo π of the form $z_{n+2} = z_n + z_{n+1} \bmod \pi$. Motivated by the above result of Rhouma, we start by finding a closed form solution for any rational recursive equation extending (1.3), of the form

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\dots(y_n+i)^{A_\ell} + (y_{n+\ell-1}-i)^{A_1}\dots(y_n-i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\dots(y_n+i)^{A_\ell} - (y_{n+\ell-1}-i)^{A_1}\dots(y_n-i)^{A_\ell}},$$

and determine its asymptotic behavior. As the equation (1.1) characterizes the cotangent function, it is natural to consider its counterpart

$$y_{n+2} = \frac{y_n + y_{n+1}}{1 - y_n y_{n+1}}, \tag{1.5}$$

or equivalently,

$$y_{n+2} = i\frac{(y_{n+1}+i)(y_n+i) - (-y_{n+1}+i)(-y_n+i)}{(y_{n+1}+i)(y_n+i) + (-y_{n+1}+i)(-y_n+i)}, \tag{1.6}$$

which clearly has the tangent function as a solution. Our second objective is to find a closed form solution for any rational recursive equation extending (1.6) of the form

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} - (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} + (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}},$$

and determine its asymptotic behavior. Finally, we solve rational recursive equations which can be used to characterize the hyperbolic tangent and cotangent functions.

The method adopted here is an extension of the original Rhouma's technique. Indeed, in the last section of his paper, Rhouma [3] indicated how to find closed form solutions of other difference equations generalizing (1.4) with some simple examples and our results here may be considered as more elaborate examples illustrating his technique in the direction of characterizing the cotangent-tangent and hyperbolic tangent-cotangent functions.

2. Results

Our first lemma follows easily from a simple calculation whose trivial proof is omitted.

Lemma 2.1. *Let $\ell \in \mathbb{N}, \ell \geq 2$; $b, x_1, \ldots, x_\ell, z$ be complex numbers and let $A_1, \ldots, A_\ell$ be nonzero integers such that*

$b(x_1+b)^{A_1} \ldots (x_\ell+b)^{A_\ell} \left\{(x_1+b)^{A_1} \ldots (x_\ell+b)^{A_\ell} - (x_1-b)^{A_1} \ldots (x_\ell-b)^{A_\ell}\right\} \neq 0.$

Then

$$\frac{z-b}{z+b} = \left(\frac{x_1-b}{x_1+b}\right)^{A_1} \cdots \left(\frac{x_\ell-b}{x_\ell+b}\right)^{A_\ell}$$

if and only if

$$z = b\,\frac{(x_1+b)^{A_1} \cdots (x_\ell+b)^{A_\ell} + (x_1-b)^{A_1} \cdots (x_\ell-b)^{A_\ell}}{(x_1+b)^{A_1} \cdots (x_\ell+b)^{A_\ell} - (x_1-b)^{A_1} \cdots (x_\ell-b)^{A_\ell}}.$$

The next lemma relates generalized Fibonacci numbers with elements in rational recursive sequences.

Lemma 2.2. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$ and let $\{F_n\}$ be the (generalized Fibonacci) sequence satisfying a linear recurrence relation of the form*

$$F_{n+\ell} = A_1 F_{n+\ell-1} + A_2 F_{n+\ell-2} + \cdots + A_\ell F_n, \tag{2.1}$$

where $A_1, \ldots, A_\ell$ are nonzero integers such that $A_1 + \ldots + A_\ell \neq 0$, with initial values

$$F_0 = A_\ell,\ F_1 = A_{\ell-1}, \ldots,\ F_{\ell-1} = A_1.$$

If

$$x_{n+\ell} = \alpha^{A_1^2+\cdots+A_\ell^2 - \frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}} x_{n+\ell-1}^{A_1} x_{n+\ell-2}^{A_2} \cdots x_n^{A_\ell} \quad (n \geq 0), \tag{2.2}$$

then

$$x_n = \alpha^{F_n - \frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}} x_0^{F_{n-\ell}} x_1^{F_{n-\ell+1}} \cdots x_{\ell-1}^{F_{n-1}} \tag{2.3}$$

for all $n \geq \ell$.

Proof. For the starting case, the condition (2.2) and the recurrence (2.1) yield

$$x_\ell = \alpha^{A_1^2+\cdots+A_\ell^2 - \frac{A_1^2+\ldots+A_\ell^2}{A_1+\cdots+A_\ell}} x_{\ell-1}^{A_1} x_{\ell-2}^{A_2} \cdots x_0^{A_\ell} = \alpha^{F_\ell - \frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}} x_{\ell-1}^{A_1} x_{\ell-2}^{A_2} \cdots x_0^{A_\ell},$$

which agrees with (2.3) when $n = \ell$. Next, suppose that (2.3) is true for all $\ell \leq n \leq k$. From (2.2), using the induction hypothesis and the recurrence (2.1), we get

$$x_{k+1} = \alpha^{A_1^2+\cdots+A_\ell^2 - \frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}} x_k^{A_1} \cdots x_{k-\ell+1}^{A_\ell}$$

$$= \alpha^{A_1^2+\cdots+A_\ell^2-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}} \left(\alpha^{-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}+F_k} x_0^{F_{k-\ell}} x_1^{F_{k-\ell+1}} \cdots x_{\ell-1}^{F_{k-1}}\right)^{A_1} \times \cdots$$

$$\times \left(\alpha^{-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}+F_{k-\ell+1}} x_0^{F_{k-\ell+1-\ell}} x_1^{F_{k-\ell+1-\ell+1}} \cdots x_{\ell-1}^{F_{k-\ell+1-1}}\right)^{A_\ell}$$

$$= \alpha^{-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}(-(A_1+\cdots+A_\ell)+1+A_1+\cdots+A_\ell)+A_1F_k+\cdots+A_\ell F_{k-\ell+1}}$$

$$\times x_0^{A_1F_{k-\ell}+\cdots+A_\ell F_{k-\ell-\ell+1}} \cdots x_{\ell-1}^{A_1F_{k-1}+\cdots+A_\ell F_{k-1-\ell+1}}$$

$$= \alpha^{-\frac{A_1^2+\cdots+A_\ell^2}{A_1+\cdots+A_\ell}+F_{k+1}} x_0^{F_{(k+1)-\ell}} \cdots x_{\ell-1}^{F_{(k+1)-1}}. \quad \blacksquare$$

Our main result reads:

Theorem 2.1. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $y_0, \ldots, y_{\ell-1}$ be real numbers such that those y_n which satisfy*

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\ldots(y_n+i)^{A_\ell} + (y_{n+\ell-1}-i)^{A_1}\ldots(y_n-i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\ldots(y_n+i)^{A_\ell} - (y_{n+\ell-1}-i)^{A_1}\ldots(y_n-i)^{A_\ell}}, \tag{2.4}$$

exist for all $n \in \mathbb{N}\cup\{0\}$. Let $\{F_n\}$ be the sequence satisfying a linear recurrence relation of the form

$$F_{n+\ell} = A_1F_{n+\ell-1} + A_2F_{n+\ell-2} + \ldots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots, F_{\ell-1} = A_1$. Then the solution to (2.4) exists if and only if

$$A_1F_{n-\ell}\theta_0 + \ldots + A_\ell F_{n-1}\theta_{\ell-1} \not\equiv 0 \pmod{2\pi} \quad (n \in \mathbb{N}\cup\{0\}),$$

where $\theta_j = \frac{-2}{A_{j+1}}$ arccot y_j for all $j \in \{0, \ldots, \ell-1\}$.

When it exists, the solution is given by

$$y_n = i\frac{(y_0+i)^{F_{n-\ell}}\cdots(y_{\ell-1}+i)^{F_{n-1}} + (y_0-i)^{F_{n-\ell}}\cdots(y_{\ell-1}-i)^{F_{n-1}}}{(y_0+i)^{F_{n-\ell}}\cdots(y_{\ell-1}+i)^{F_{n-1}} - (y_0-i)^{F_{n-\ell}}\cdots(y_{\ell-1}-i)^{F_{n-1}}}, \tag{2.5}$$

or

$$y_n = \cot\left(\frac{-A_1F_{n-\ell}\theta_0 - \cdots - A_\ell F_{n-1}\theta_{\ell-1}}{2}\right)$$
$$= \cot(F_{n-\ell}\ \text{arccot}\ y_0 + \cdots + F_{n-1}\ \text{arccot}\ y_{\ell-1}).$$

Moreover,

(1) *if all the θ_j are rational multiples of π, then either $\{y_n\}$ diverges in finitely many steps or $\{y_n\}$ is periodic;*
(2) *if $\theta_0, \theta_1, \ldots, \theta_{\ell-1}, \pi$ are linearly independent over $\mathbb{Q}$, and $A_1, \ldots, A_\ell$ are nonzero integers, then y_n exists for all n and the sequence $\{y_n\}$ is never periodic.*

Proof. Taking $z = y_{n+\ell}$, $x_1 = y_{n+\ell-1}, \ldots, x_\ell = y_n$, $b = i$ in Lemma 2.1, the rational recursive equation (2.4) is equivalent to

$$\frac{y_{n+\ell}-i}{y_{n+\ell}+i} = \left(\frac{y_{n+\ell-1}-i}{y_{n+\ell-1}+i}\right)^{A_1} \cdots \left(\frac{y_n-i}{y_n+i}\right)^{A_\ell}. \tag{2.6}$$

Putting $U_n = \frac{y_n - i}{y_n + i}$, the relation (2.6) becomes

$$U_{n+\ell} = U_{n+\ell-1}^{A_1} U_{n+\ell-2}^{A_2} \cdots U_n^{A_\ell},$$

whose solution is, by virtue of Lemma 2.2,

$$U_n = U_0^{F_{n-\ell}} U_1^{F_{n-\ell+1}} \cdots U_{\ell-1}^{F_{n-1}} \ (n \geq \ell),$$

and so

$$\text{(2.7)} \qquad \frac{y_n - i}{y_n + i} = \left(\frac{y_0 - i}{y_0 + i}\right)^{F_{n-\ell}} \left(\frac{y_1 - i}{y_1 + i}\right)^{F_{n-\ell+1}} \cdots \left(\frac{y_{\ell-1} - i}{y_{\ell-1} + i}\right)^{F_{n-1}},$$

which, by Lemma 2.1, becomes

$$\text{(2.8)} \qquad y_n = i \frac{(y_0 + i)^{F_{n-\ell}} \cdots (y_{\ell-1} + i)^{F_{n-1}} + (y_0 - i)^{F_{n-\ell}} \cdots (y_{\ell-1} - i)^{F_{n-1}}}{(y_0 + i)^{F_{n-\ell}} \cdots (y_{\ell-1} + i)^{F_{n-1}} - (y_0 - i)^{F_{n-\ell}} \cdots (y_{\ell-1} - i)^{F_{n-1}}}.$$

Next, setting $e^{i\theta_0 A_1} = \frac{y_0 - i}{y_0 + i}, \ \dots \ , e^{i\theta_{\ell-1} A_\ell} = \frac{y_{\ell-1} - i}{y_{\ell-1} + i}$, we have

$$\frac{y_n - i}{y_n + i} = e^{i(A_1 F_{n-\ell}\theta_0 + \cdots + A_\ell F_{n-1}\theta_{\ell-1})},$$

i.e.,

$$\begin{aligned} y_n &= i \frac{1 + e^{i(A_1 F_{n-\ell}\theta_0 + \cdots + A_\ell F_{n-1}\theta_{\ell-1})}}{1 - e^{i(A_1 F_{n-\ell}\theta_0 + \cdots + A_\ell F_{n-1}\theta_{\ell-1})}} \\ &= \cot\left(\frac{-A_1 F_{n-\ell}\theta_0 - \cdots - A_\ell F_{n-1}\theta_{\ell-1}}{2}\right) \\ &= \cot(F_{n-\ell} \operatorname{arccot} y_0 + \cdots + F_{n-1} \operatorname{arccot} y_{\ell-1}), \end{aligned}$$

provided $A_1 F_{n-\ell}\theta_0 + \cdots + A_\ell F_{n-1}\theta_{\ell-1} \neq 0 \pmod{2\pi}$.

If all the $\theta_j \ \ (j = 0, 1, \dots, \ell - 1)$ are rational multiples of π, say,

$$\theta_j = m_j \pi / t_j \text{ with } m_j, t_j \ (> 0) \in \mathbb{Z}, \ \gcd(m_j, t_j) = 1,$$

then it is easily checked that $\sum_{k=1}^{\ell} A_k F_{n-\ell+k-1} \theta_{k-1} \bmod 2\pi$ is equivalent to

$$G_n = \sum_{k=1}^{\ell} A_k F_{n-\ell+k-1} m_{k-1} \prod_{\substack{j=0 \\ j \neq k}}^{\ell-1} t_j \mod \prod_{j=0}^{\ell-1} 2t_j.$$

Since each G_n takes at most $\prod_{j=0}^{\ell-1}(2t_j)$ distinct values, each ℓ-tuple $(G_t, \dots, G_{t+\ell-1})$ takes at most $\prod_{j=0}^{\ell-1}(2t_j)^\ell$ distinct values. Since the sequence $\{(G_t, \dots, G_{t+\ell-1})\}_{t \geq 0}$ is infinite, there are integers $N_1 \neq N_2$ such that

$$(G_{N_1}, \dots, G_{N_1+\ell-1}) = (G_{N_2}, \dots, G_{N_2+\ell-1}).$$

Since

$$G_{j+\ell} = G_j + \cdots + G_{j+\ell-1} \quad (j \in \mathbb{N}),$$

we deduce that $G_{N_1+k} \equiv G_{N_2+k}$ for all $k \in \mathbb{N}$, i.e., the sequence $\{G_n\}$ is periodic. If some G_n is zero, then clearly the sequence $\{y_n\}$ diverges.

If $\theta_0, \theta_1, \dots, \theta_{\ell-1}, \pi$ are linearly independent over $\mathbb{Q}$, then

$$A_1 F_{n-\ell}\theta_0 + \ldots + A_\ell F_{n-1}\theta_{\ell-1} \neq 2k\pi \ (k \in \mathbb{Z}),$$

showing that y_n exists for each n and the sequence $\{y_n\}$ is never periodic. ∎

Theorem 2.1 enables us to solve certain other rational recursive equations characterizing related trigonometric and hyperbolic functions. We begin with the tangent function.

Corollary 2.1. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $y_0, \ldots, y_{\ell-1}$ be real numbers such that those y_n which satisfy the rational recursive equation*

$$y_{n+\ell} = i\frac{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} - (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}}{(y_{n+\ell-1}+i)^{A_1}\cdots(y_n+i)^{A_\ell} + (-y_{n+\ell-1}+i)^{A_1}\cdots(-y_n+i)^{A_\ell}}. \tag{2.9}$$

exist for all $n \in \mathbb{N} \cup \{0\}$. Let $\{F_n\}$ be a sequence satisfying a linear recurrence relation of the form

$$F_{n+\ell} = A_1F_{n+\ell-1} + A_2F_{n+\ell-2} + \ldots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots,$ $F_{\ell-1} = A_1$. Then the solution to the equation (2.9) *exists if and only if $A_1F_{n-\ell}\theta_0 + A_2F_{n-\ell+1}\theta_1 + \cdots + A_\ell F_{n-1}\theta_{\ell-1}$ is not an odd multiple of π, where $\theta_j = \frac{-2}{A_{j+1}}\arctan\ y_j$ $(j = 0, 1, \ldots, \ell-1)$.*

When the solution exists, it is given by

$$y_n = i\frac{(y_0+i)^{A_1}\cdots(y_{\ell-1}+i)^{A_\ell} - (-y_0+i)^{A_1}\cdots(-y_{\ell-1}+i)^{A_\ell}}{(y_0+i)^{A_1}\cdots(y_{\ell-1}+i)^{A_\ell} + (-y_0+i)^{A_1}\cdots(-y_{\ell-1}+i)^{A_\ell}}, \tag{2.10}$$

or

$$\begin{aligned} y_n &= -\tan\left(\frac{A_1F_{n-\ell}\theta_0 + \cdots + A_\ell F_{n-1}\theta_{\ell-1}}{2}\right) \\ &= \tan\left(F_{n-\ell}\arctan\ y_0 + \cdots + F_{n-1}\arctan\ y_{\ell-1}\right). \end{aligned}$$

Moreover,

1. *if all θ_j are rational multiples of π, then either the sequence $\{y_n\}$ diverges in finitely many steps or is periodic;*
2. *if $\theta_0, \theta_1, \ldots, \theta_{\ell-1}, \pi$ are linearly independent over $\mathbb{Q}$, and $A_1, \ldots, A_\ell$ are nonzero integers, then the value y_n exists for all n and the sequence $\{y_n\}$ is never periodic.*

Proof. Substituting y_n by $1/y_n$ turns the equation (2.9) into a rational recursive equation of the form (2.4) and so the corollary follows at once from Theorem 2.1. ∎

Remark 2.1. Although the substitution y_n by $1/y_n$ employed in Corollary 2.1 allows us to obtain a closed form solution of the equation (2.9), there remains a difficulty should there exist integer N such that $y_N = 0$. To overcome this shortcoming, we may either interpret the infinite value of the two expressions on both sides of the solution as equal or repeat the technique used in the proof of Theorem 2.1 to solve the equation (2.9) using auxiliary results analogous to Lemmas 2.1 and 2.2.

Next, we deal with rational recursive equations characterizing the hyperbolic cotangent-tangent functions.

Corollary 2.2. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $y_0, \ldots, y_{\ell-1}$ be real numbers such that those y_n which satisfy*

$$y_{n+\ell} = \frac{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} + (y_{n+\ell-1}-1)^{A_1} \cdots (y_n-1)^{A_\ell}}{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} - (y_{n+\ell-1}-1)^{A_1} \cdots (y_n-1)^{A_\ell}}, \tag{2.11}$$

exist for all $n \in \mathbb{N} \cup \{0\}$. Let $\{F_n\}$ be a sequence satisfying a linear recurrence relation of the form

$$F_{n+\ell} = A_1 F_{n+\ell-1} + A_2 F_{n+\ell-2} + \ldots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots,$ $F_{\ell-1} = A_1$. Then the solution to equation (2.11) *exists and is given by*

$$y_n = \frac{(y_0+1)^{F_{n-\ell}} \cdots (y_{\ell-1}+1)^{F_{n-1}} + (y_0-1)^{F_{n-\ell}} \cdots (y_{\ell-1}-1)^{F_{n-1}}}{(y_0+1)^{F_{n-\ell}} \cdots (y_{\ell-1}+1)^{F_{n-1}} - (y_0-1)^{F_{n-\ell}} \cdots (y_{\ell-1}-1)^{F_{n-1}}}, \tag{2.12}$$

or

$$y_n = \coth\left(F_{n-\ell} \text{ arccoth } y_0 + \ldots + F_{n-1} \text{ arccoth } y_{\ell-1}\right).$$

Proof. Substituting y_n by iy_n in the equation (2.11) turns it into a rational recursive equation of the form (2.4) and so Theorem 2.1 yields the desired result. ∎

Corollary 2.3. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$; $A_1, \ldots, A_\ell$ be nonzero integers such that $A_1 + \cdots + A_\ell \neq 0$. Let $y_0, \ldots, y_{\ell-1}$ be real numbers such that those y_n which satisfy*

$$y_{n+\ell} = \frac{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} - (-y_{n+\ell-1}+1)^{A_1} \cdots (-y_n+1)^{A_\ell}}{(y_{n+\ell-1}+1)^{A_1} \cdots (y_n+1)^{A_\ell} + (-y_{n+\ell-1}+1)^{A_1} \cdots (-y_n+1)^{A_\ell}}, \tag{2.13}$$

exist for all $n \in \mathbb{N} \cup \{0\}$. Let $\{F_n\}$ be a sequence satisfying a linear recurrence relation of the form

$$F_{n+\ell} = A_1 F_{n+\ell-1} + A_2 F_{n+\ell-2} + \cdots + A_\ell F_n,$$

with initial values $F_0 = A_\ell$, $F_1 = A_{\ell-1}, \ldots,$ $F_{\ell-1} = A_1$. Then the solution to equation (2.13) *exists and is given by*

$$y_n = \frac{(y_0+1)^{F_{n-\ell}} \ldots (y_{\ell-1}+1)^{F_{n-1}} - (-y_0+1)^{F_{n-\ell}} \ldots (-y_{\ell-1}+1)^{F_{n-1}}}{(y_0+1)^{F_{n-\ell}} \ldots (y_{\ell-1}+1)^{F_{n-1}} + (-y_0+1)^{F_{n-\ell}} \ldots (-y_{\ell-1}+1)^{F_{n-1}}}, \tag{2.14}$$

or

$$y_n = \tanh\left(\text{arctanh } y_0 F_{n-\ell} + \cdots + \text{arctan}h \ y_{\ell-1} F_{n-1}\right).$$

Proof. Replacing y_n by iy_n in the equation (2.13), we get a rational recursive equation of the form (2.9) and Corollary 2.1 yields the required result. ∎

Acknowledgement. This research is supported by the Thailand Research Funds RTA 5180005, the Centre of Excellence in Mathematics, CHE, and the KU Center for Advanced Studies, Thailand.

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