A Class of Integral Operators Preserving Subordination and Superordination

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Abstract. The purpose of the present paper is to obtain subordination- and superordination-preserving properties for a class of integral operators defined on the space of normalized analytic functions in the open unit disk. We also consider the sandwich-type theorems for these integral operators.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$

We denote by $\mathcal{A}$ the subclass of $\mathcal{H}[a, 1]$ with the normalization $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}^*$ and $\mathcal{K}$ be the subclasses of $\mathcal{A}$ consisting of all functions which are, respectively, starlike in $\mathbb{U}$ and convex in $\mathbb{U}$.

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Definition 1.1. [17] *Let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *and let* h *be univalent in* $\mathbb{U}$. *If* p *is analytic in* $\mathbb{U}$ *and satisfies the differential subordination*

(1.1)
$$\phi(p(z), zp'(z)) \prec h(z),$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

Recently, Miller and Mocanu [18] introduced the following differential superordinations, as the dual concept of differential subordinations.

Definition 1.2. [18] *Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination*

$$(1.2) \qquad\qquad h(z) \prec \varphi(p(z), zp'(z)),$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 1.3. [18] *We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial \mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \mathbb{U} \backslash E(f)$.

For a function $f \in \mathcal{A}$, we introduce the following integral operator I defined by

$$(1.3) \qquad\qquad I(f)(z) := \left(\frac{\beta + \gamma}{z^{\gamma}} \int_0^z t^{\delta - 1} f^{\alpha}(t) dt \right)^{1/\beta}$$

$$(f \in \mathcal{A}; \ \alpha, \gamma, \delta \in \mathbb{C}; \ \beta \in \mathbb{C} \backslash \{0\}; \ \alpha + \delta = \beta + \gamma; \ \mathrm{Re}\{\beta + \gamma\} > 0).$$

The integral operator defined by (1.3) have been extensively studied by many authors [7, 11, 12, 16, 19] with suitable restriction on the parameters α, β, γ and δ, and for f belonging to some favored classes of analytic functions. In particular, Bernard [7] showed that the integral operator $I(f)$ with $\alpha = \beta = 1$, $\delta = \gamma$ and $\mathrm{Re}\{\gamma\} > 0$ belongs to the classes $\mathcal{S}^*$ and $\mathcal{K}$, whenever f belongs to the classes $\mathcal{S}^*$ and $\mathcal{K}$, respectively, which include the results earlier by Libera [12]. Moreover, Miller *et al.* [19] proved that the integral operator I with some restrictions on the parameters α, β, γ and δ is preserved by the class $\mathcal{S}^*$.

Making use of the principle of subordination, Miller *et al.* [20] obtained some subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$. Moreover, Bulboacă [8, 9] investigated the subordination- and superordination-preserving properties of the integral operators defined by (1.3) with some restrictions on the parameters α, β, γ and δ. Some interesting developments involving subordination and superordination were considered by Ali *et al.* [1–6] and others [13, 22, 23]. In the present paper, we obtain the subordination- and superordination-preserving properties of the general integral operators I defined by (1.3) with the sandwich-type theorems.

The following lemmas will be required in our present investigation.

Lemma 1.1. [14] *Suppose that the function* $H : \mathbb{C}^2 \to \mathbb{C}$ *satisfies the condition*

$$\mathrm{Re}\{H(is, t)\} \leq 0,$$

for all real s *and* $t \leq -n(1 + s^2)/2$, *where* n *is a positive integer. If the function* $p(z) = 1 + p_n z^n + \cdots$ *is analytic in* $\mathbb{U}$ *and*

$$\mathrm{Re}\{H(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then $\mathrm{Re}\{p(z)\} > 0$ *in* $\mathbb{U}$.

Lemma 1.2. [15] *Let* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If* $\mathrm{Re}\{\beta h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, *then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \ (z \in \mathbb{U})$$

with $q(0) = c$ *is analytic in* $\mathbb{U}$ *and satisfies* $\mathrm{Re}\{\beta q(z) + \gamma\} > 0$ $(z \in \mathbb{U})$.

Lemma 1.3. [17] *Let* $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let* $q(z) = a + a_n z^n + \cdots$ *be analytic in* $\mathbb{U}$ *with* $q(z) \not\equiv a$ *and* $n \geq 1$. *If* q *is not subordinate to* p, *then there exist points* $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U}\backslash E(f)$, *for which* $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,

$$q(z_0) = p(\zeta_0) \quad and \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Let $c \in \mathbb{C}$ with $\mathrm{Re}\{c\} > 0$ and let

$$N := N(c) = \frac{|c|\sqrt{1 + 2\mathrm{Re}\{c\}} + \mathrm{Im}\{c\}}{\mathrm{Re}\{c\}}.$$

If $R(z)$ is the univalent function defined in $\mathbb{U}$ by $R(z) = 2Nz/(1-z^2)$, then the open door function defined by

$$(1.4) \qquad R_c(z) := R\left(\frac{z+b}{1+\bar{b}z}\right) \quad (z \in \mathbb{U}),$$

where $b = R^{-1}(c)$ [17].

Remark 1.1. The function R_c defined by (1.4) is univalent in $\mathbb{U}$, $R_c(0) = c$ and $R_c(\mathbb{U}) = R(\mathbb{U})$ is the complex plane with slits along the half-lines $\mathrm{Re}\{w\} = 0$ and $\mathrm{Im}\{|w|\} \geq N$.

Lemma 1.4. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ *and* $\mathrm{Re}\{\alpha + \delta\} > 0$. *If* $f \in \mathcal{A}_{\alpha,\delta}$, *where*

$$\mathcal{A}_{\alpha,\delta} := \left\{f \in \mathcal{A} : \ \alpha\frac{zf'(z)}{f(z)} + \delta \prec R_{\alpha+\delta}(z)\right\}$$

and $R_{\alpha+\delta}(z)$ *is defined by (1.4) with* $c = \alpha + \delta$, *then* $I \in \mathcal{A}$, $I(f)(z)/z \neq 0$ *and*

$$\mathrm{Re}\left\{\alpha\frac{z(I(f)(z))'}{I(f)(z)} + \delta\right\} > 0 \quad (z \in \mathbb{U}),$$

where I *is the integral operator defined by (1.3).*

A function $L(z, t)$ defined on $\mathbb{U} \times [0, \infty)$ is the subordination chain (or Löwner chain) if $L(\cdot, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0, \infty)$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$ and $L(z, s) \prec L(z, t)$ for $0 \leq s < t$.

Lemma 1.5. [18] *Let $q \in \mathcal{H}[a, 1]$, let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and set $\varphi(q(z), zq'(z)) \equiv h(z)$. If $L(z, t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a, 1] \cap \mathcal{Q}$, then*

$$h(z) \prec \varphi(p(z), zp'(z))$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zp'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

Lemma 1.6. [21] *The function $L(z, t) = a_1(t)z + \cdots$, with $a_1(t) \neq 0$ and $\lim_{t \to \infty} |a_1(t)| = \infty$, is a subordination chain if and only if*

$$\mathrm{Re}\left\{ \frac{z \partial L(z, t)/\partial z}{\partial L(z, t)/\partial t} \right\} > 0 \quad (z \in \mathbb{U}; \ 0 \leq t < \infty).$$

2. Main results

Subordination theorem involving the integral operator I defined by (1.3) is contained in Theorem 2.1 below.

Theorem 2.1. *Let $f, g \in \mathcal{A}_{\alpha, \delta}$. Suppose also that*

$$(2.1) \qquad \mathrm{Re}\left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\rho \quad \left(z \in \mathbb{U}; \ \phi(z) := z\left(\frac{g(z)}{z}\right)^{\alpha} \right),$$

where

$$(2.2) \qquad \rho = \frac{1 + |\beta + \gamma - 1|^2 - |1 - (\beta + \gamma - 1)^2|}{4\mathrm{Re}\{\beta + \gamma - 1\}} \quad (\mathrm{Re}\ \{\beta + \gamma - 1\} > 0).$$

If f and g satisfy the following subordination condition

$$(2.3) \qquad z\left(\frac{f(z)}{z}\right)^{\alpha} \prec z\left(\frac{g(z)}{z}\right)^{\alpha},$$

then

$$z\left(\frac{I(f)(z)}{z}\right)^{\beta} \prec z\left(\frac{I(g)(z)}{z}\right)^{\beta},$$

where I is the integral operator defined by (1.3). Moreover, the function $z(I(g)(z)/z)^{\beta}$ is the best dominant.

Proof. Let us define the functions F and G by

$$(2.4) \qquad F(z) := z\left(\frac{I(f)(z)}{z}\right)^{\beta} \quad \text{and} \quad G(z) := z\left(\frac{I(g)(z)}{z}\right)^{\beta},$$

respectively. We note that F and G are well defined by Lemma 1.4.

We first show that, if the function q is defined by

$$(2.5) \qquad q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}),$$

then

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

From the definition of (1.3), we obtain

$$(2.6) \qquad \left(\frac{Ig(z)}{z}\right)^{\beta}\left(\beta\frac{z(I(g)(z))'}{I(g)(z)}+\gamma\right)\frac{1}{\beta+\gamma}=\left(\frac{g(z)}{z}\right)^{\alpha}.$$

We also have

$$(2.7) \qquad \beta\frac{z(I(g)(z))'}{I(g)(z)}=\beta-1+\frac{zG'(z)}{G(z)}.$$

It follows from (2.6) and (2.7) that

$$(2.8) \qquad 1+\frac{z\phi''(z)}{\phi'(z)}=q(z)+\frac{zq'(z)}{q(z)+\beta+\gamma-1}\equiv h(z).$$

From (2.1), we have

$$\operatorname{Re}\{h(z)+\beta+\gamma-1\}>0 \ \ (z\in\mathbb{U}),$$

and by using Lemma 1.2, we note that the differential equation (2.8) has a solution $q\in\mathcal{H}(\mathbb{U})$ with $q(0)=h(0)=1$. Let us put

$$(2.9) \qquad H(u,v)=u+\frac{v}{u+\beta+\gamma-1}+\rho,$$

where ρ is given by (2.2). From (2.1), (2.8) and (2.9), we obtain

$$\operatorname{Re}\{H(q(z),zq'(z))\}>0 \ \ (z\in\mathbb{U}).$$

Now we proceed to show that $\operatorname{Re}\{H(is,t)\}\leq 0$ for all real s and $t\leq-(1+s^2)/2$. From (2.9), we have

$$\operatorname{Re}\{H(is,t)\}=\operatorname{Re}\left\{is+\frac{t}{is+\beta+\gamma-1}+\rho\right\}$$

$$(2.10) \qquad\qquad \leq-\frac{E_{\rho}(s)}{2|\beta+\gamma-1+is|^2},$$

where

$$E_{\rho}(s):=(\operatorname{Re}\{\beta+\gamma-1\}-2\rho)s^2-4\rho(\operatorname{Im}\{\beta+\gamma-1\})s$$

$$(2.11) \qquad\qquad -2\rho|\beta+\gamma-1|^2+\operatorname{Re}\{\beta+\gamma-1\}.$$

For ρ given by (2.2), the coefficient of s^2 in the quadratic expression $E_{\rho}(s)$ given by (2.11) is positive or equal to zero. Moreover, the quadratic expression $E_{\rho}(s)$ by s in (2.11) is a perfect square for the assumed value of ρ given by (2.2). Hence from (2.10), we see that $\operatorname{Re}\{H(is,t)\}\leq 0$ for all real s and $t\leq-(1+s^2)/2$. Thus, by using Lemma 1.1, we conclude that $\operatorname{Re}\{q(z)\}>0$ for all $z\in\mathbb{U}$. That is, G defined by (2.4) is convex (univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (2.3) implies that

$$(2.12) \qquad F(z)\prec G(z)$$

for the functions F and G defined by (2.5). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and $g'(\zeta)\neq 0$ for $|\zeta|=1$. Now we consider the function $L(z,t)$ given by

$$L(z,t):=\frac{\beta+\gamma-1}{\beta+\gamma}G(z)+\frac{1+t}{\beta+\gamma}zG'(z) \ \ (z\in\mathbb{U};\ 0\leq t<\infty).$$

Since G is convex and $\mathrm{Re}\{\beta + \gamma - 1\} > 0$, we note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{\beta + \gamma + t}{\beta + \gamma}\right) \neq 0 \quad (0 \leq t < \infty)$$

and

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\left\{\beta + \gamma - 1 + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right)\right\} > 0.$$

Therefore, by virtue of Lemma 1.6, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$L(\zeta, t) \notin L(\mathbb{U}, 0) = \phi(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty).$$

Now suppose that F is not subordinate to G, then by Lemma 1.3, there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).$$

Hence we have

$$\begin{aligned}
L(\zeta_0, t) &= \frac{\beta + \gamma - 1}{\beta + \gamma} G(\zeta_0) + \frac{1+t}{\beta + \gamma}\zeta_0 G'(\zeta_0) \\
&= \frac{\beta + \gamma - 1}{\beta + \gamma} F(z_0) + \frac{1}{\beta + \gamma} z_0 F'(z_0) \\
&= z_0 \left(\frac{f(z_0)}{z_0}\right)^\alpha \in \phi(\mathbb{U}),
\end{aligned}$$

by virtue of the subordination condition (2.3). This contradicts the above observation that $L(\zeta_0, t) \notin \phi(\mathbb{U})$. Therefore, the subordination condition (2.3) must imply the subordination given by (2.12). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 2.1. ∎

Remark 2.1. We note that ρ given by (2.2) in Theorem 2.1 satisfies the inequality $0 < \rho \leq 1/2$.

We next prove a dual problem of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.2. *Let $f, g \in \mathcal{A}_{\alpha,\delta}$. Suppose also that*

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\rho \quad \left(z \in \mathbb{U};\ \phi(z) := z\left(\frac{g(z)}{z}\right)^\alpha\right),$$

where ρ is given by (2.2). If $z\left(f(z)/z\right)^\alpha$ is univalent in $\mathbb{U}$ and $z\left(I(f)(z)/z\right)^\beta \in \mathcal{Q}$, where I is the integral operator defined by (1.3), then

$$(2.13) \qquad z\left(\frac{g(z)}{z}\right)^\alpha \prec z\left(\frac{f(z)}{z}\right)^\alpha$$

implies that

$$(2.14) \qquad z\left(\frac{I(g)(z)}{z}\right)^\beta \prec z\left(\frac{I(f)(z)}{z}\right)^\beta.$$

Moreover, the function $z\left(I(g)(z)/z\right)^\beta$ is the best subordinant.

Proof. The first part of the proof is similar to that of Theorem 2.1 and so we will use the same notation as in the proof of Theorem 2.1.

Now let us define the functions F and G, respectively, by (2.4). We first note that from (2.6) and (2.7), we obtain

$$\phi(z) = \frac{\beta + \gamma - 1}{\beta + \gamma} G(z) + \frac{1}{\beta + \gamma} z G'(z)$$

(2.15)
$$=: \varphi(G(z), zG'(z)).$$

Then by using the same method as in the proof of Theorem 2.1, we can prove that $\mathrm{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$, where the function q is defined by (2.5). That is, G defined by (2.4) is convex (univalent) in $\mathbb{U}$.

Now consider the function $L(z,t)$ defined by

$$L(z,t) := \frac{\beta + \gamma - 1}{\beta + \gamma} G(z) + \frac{t}{\beta + \gamma} z G'(z) \ (z \in \mathbb{U}; \ 0 \le t < \infty).$$

Since G is convex and $\mathrm{Re}\{\beta + \gamma - 1\} > 0$, we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore according to Lemma 1.5, we conclude that the superordination condition (2.13) must imply the superordination given by (2.14). Furthermore, since the differential equation (2.15) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.2. ∎

Remark 2.2. If we take the parameters α, β, γ and δ with the restrictions $\alpha = \beta$, $\delta = \gamma$ and $1 < \beta + \gamma \le 2$ in Theorem 2.2, then we have the result obtained by Bulboacă [9].

If we combine Theorem 2.1 and Theorem 2.2, then we obtain the following differential sandwich-type theorem.

Theorem 2.3. *Let $f, g_k \in \mathcal{A}_{\alpha,\delta}(k = 1, 2)$. Suppose also that*

(2.16) $\mathrm{Re}\left\{1 + \dfrac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\rho \quad \left(z \in \mathbb{U}; \ \phi_k(z) := z\left(\dfrac{g_k(z)}{z}\right)^\alpha; \ k = 1, 2\right),$

where ρ is given by (2.2). If $z\left(f(z)/z\right)^\alpha$ is univalent in $\mathbb{U}$ and $z\left(I(f)(z)/z\right)^\beta \in \mathcal{Q}$, where I is the integral operator defined by (1.3), then

$$z\left(\frac{g_1(z)}{z}\right)^\alpha \prec z\left(\frac{f(z)}{z}\right)^\alpha \prec z\left(\frac{g_2(z)}{z}\right)^\alpha$$

implies that

$$z\left(\frac{I(g_1)(z)}{z}\right)^\beta \prec z\left(\frac{I(f)(z)}{z}\right)^\beta \prec z\left(\frac{I(g_2)(z)}{z}\right)^\beta.$$

Moreover, the functions $z\left(I(g_1)(z)/z\right)^\beta$ and $z\left(I(g_2)(z)/z\right)^\beta$ are the best subordinant and the best dominant, respectively.

The assumption of Theorem 2.3 that the functions $z\left(f(z)/z\right)^\alpha$ and $z(I(f)(z)/z)^\alpha$ need to be univalent in $\mathbb{U}$ may be replaced by another conditions in the following result.

Corollary 2.1. *Let $f, g_k \in \mathcal{A}_{\alpha,\delta}(k = 1, 2)$. Suppose also that the condition (2.16) is satisfied and*

$$(2.17) \qquad \mathrm{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\rho \quad \left(z \in \mathbb{U}; \ \psi(z) := z\left(\frac{f(z)}{z}\right)^{\alpha}; \ f \in \mathcal{Q}\right),$$

where ρ is given by (2.2). Then

$$z\left(\frac{g_1(z)}{z}\right)^{\alpha} \prec z\left(\frac{f(z)}{z}\right)^{\alpha} \prec z\left(\frac{g_2(z)}{z}\right)^{\alpha}$$

implies that

$$z\left(\frac{I(g_1)(z)}{z}\right)^{\beta} \prec z\left(\frac{I(f)(z)}{z}\right)^{\beta} \prec z\left(\frac{I(g_2)(z)}{z}\right)^{\beta},$$

where I is the integral operator defined by (1.3). Moreover, the functions $z\left(I(g_1)(z)/z\right)^{\beta}$ and $z\left(I(g_2)(z)/z\right)^{\beta}$ are the best subordinant and the best dominant, respectively.

Proof. In order to prove Corollary 2.1, we have to show that the condition (2.17) implies the univalence of $\psi(z)$ and $F(z) := z(I(f)(z)/z)^{\beta}$. Since $0 < \rho \leq 1/2$ from Remark 2.1, the condition (2.17) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [10]) and hence ψ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 2.3, we can prove the convexity (univalence) of F and so the details may be omitted. Therefore, by applying Theorem 2.3, we obtain Corollary 2.1. ∎

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