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# On the Distance Paired-Domination of Circulant Graphs

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**Abstract.** Let $G = (V, E)$ be a graph without isolated vertices. A set $D \subseteq V$ is a *$d$-distance paired-dominating set* of $G$ if $D$ is a $d$-distance dominating set of $G$ and the induced subgraph $\langle D \rangle$ has a perfect matching. The minimum cardinality of a $d$-distance paired-dominating set for graph $G$ is the *$d$-distance paired-domination number*, denoted by $\gamma_p^d(G)$. In this paper, we study the *$d$-distance paired-domination number* of circulant graphs $C(n; \{1, k\})$ for $2 \le k \le 4$. We prove that for $k = 2$, $n \ge 5$ and $d \ge 1$,

$$\gamma_p^d(C(n; \{1, k\})) = 2 \left\lceil \frac{n}{2kd + 3} \right\rceil,$$

for $k = 3$, $n \ge 7$ and $d \ge 1$,

$$\gamma_p^d(C(n; \{1, k\})) = 2 \left\lceil \frac{n}{2kd + 2} \right\rceil,$$

and for $k = 4$ and $n \ge 9$,

(i) if $d = 1$, then

$$\gamma_p(C(n; \{1, k\})) = \begin{cases} 2\lceil \frac{3n}{23} \rceil + 2, & \text{if } n \equiv 15, 22 \pmod{23}; \\ 2\lceil \frac{3n}{23} \rceil, & \text{otherwise} \end{cases}$$

(ii) if $d \ge 2$, then

$$\gamma_p^d(C(n; \{1, k\})) = \begin{cases} 2\lceil \frac{2n}{4kd+1} \rceil + 2, & \text{if } n \equiv 2kd, 4kd - 1, 4kd \pmod{4kd + 1} \\ 2\lceil \frac{2n}{4kd+1} \rceil, & \text{otherwise.} \end{cases}$$

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## 1. Introduction

All graphs considered in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph without isolated vertices. The open neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $D \subseteq V(G)$, $N(D) = \bigcup_{v\in D} N(v)$ and $N[D] = \bigcup_{v\in D} N[v]$. For $D \subseteq V(G)$, let $\langle D\rangle$ be the subgraph induced by $D$.

A set $D \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - D$ is adjacent to at least one vertex in $D$. A set $D \subseteq V(G)$ is a *paired-dominating set* of $G$ if it is dominating and the induced subgraph $\langle D\rangle$ has a perfect matching. The *paired-domination number* $\gamma_p(G)$ is the cardinality of a smallest paired-dominating set of $G$. This type of domination was introduced by Haynes and Slater in [9, 10] and is well studied, for example, in [1–7, 11–13, 15].

For two vertices $x$ and $y$, let $d(x, y)$ denote the distance between $x$ and $y$ in $G$. A set $D \subseteq V(G)$ is a *$d$-distance dominating set* of $G$ if every vertex in $V(G) - D$ is within distance $d$ of at least one vertex in $D$. The *$d$-distance domination number* $\gamma^d(G)$ of $G$ is the minimum cardinality among all $d$-distance dominating sets of $G$. For a more detailed treatment of domination-related parameters and for terminology not defined here, the reader is referred to [8].

The $d$-distance paired-domination was introduced by Joanna Raczek [14] as a generalization of paired-domination. For a positive integer $d$, a set $D \subseteq V(G)$ is a *$d$-distance paired-dominating set* if every vertex in $V(G) - D$ is within distance $d$ of a vertex in $D$ and the induced subgraph $\langle D\rangle$ has a perfect matching. The *$d$-distance paired-domination number*, denoted by $\gamma_p^d(G)$, is the minimum cardinality of a $d$-distance paired-dominating set.

In the same paper, Joanna Raczek investigated properties of the $d$-distance paired-domination number of a graph. He also gave an upper bound and a lower bound on the $d$-distance paired-domination number of a non-trivial tree $T$ in terms of the size of $T$ and the number of leaves in $T$ and characterized the extremal trees.

The *circulant graph* $C(n; S)$ is the graph with the vertex set $V(C(n; S)) = \{v_i | 0 \le i \le n-1\}$ and the edge set $E(C(n; S)) = \{v_iv_j | 0 \le i, j \le n-1, (i-j) \bmod n \in S\}$, $S \subseteq \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.

In this paper, we determine the exact $d$-distance paired-domination number of the circulant graphs $C(n; \{1, k\})$ for $2 \le k \le 4$ and $d \ge 1$. We prove that for $k = 2$, $n \ge 5$ and $d \ge 1$,

$$\gamma_p^d(C(n; \{1, k\})) = 2\left\lceil \frac{n}{2kd+3} \right\rceil,$$

for $k = 3$, $n \ge 7$ and $d \ge 1$,

$$\gamma_p^d(C(n; \{1, k\})) = 2\left\lceil \frac{n}{2kd+2} \right\rceil,$$

and for $k = 4$ and $n \ge 9$,

(i) if $d = 1$, then

$$\gamma_p(C(n; \{1, k\})) = \begin{cases} 2\lceil \frac{3n}{23} \rceil + 2, & \text{if } n \equiv 15, 22 \pmod{23}; \\ 2\lceil \frac{3n}{23} \rceil, & \text{otherwise} \end{cases}$$

(ii) if $d \geq 2$, then

$$\gamma_p^d(C(n;\{1,k\})) = \begin{cases} 2\lceil \frac{2n}{4kd+1} \rceil + 2, & \text{if } n \equiv 2kd, 4kd-1, 4kd \pmod{4kd+1} \\ 2\lceil \frac{2n}{4kd+1} \rceil, & \text{otherwise.} \end{cases}$$

In this paper, let $D = \{x_i, y_i : i = 1, 2, \ldots, q\}$ be an arbitrary $d$-distance paired-dominating set of $C(n;\{1,k\})$, where $\{x_i y_i : i = 1, 2, \ldots, q\}$ is a perfect matching of $\langle D \rangle$, and let

$$D_p = \{(x_i, y_i) : i = 1, 2, \ldots, q\}.$$

For each pair $(x_j, y_j) \in D_p$ with $j \in \{1, 2, \ldots, q\}$, for convenience, we denote $x_j = v_{i_j}$, and $y_j = v_{i_j+1}$ or $y_j = v_{i_j+k}$, i.e., $(v_{i_j}, v_{i_j+1}) \in D_p$ or $(v_{i_j}, v_{i_j+k}) \in D_p$, where $0 = i_1 \leq i_2 \leq \cdots \leq i_q < n$.

We also denote

$$\delta_j = (i_{j+1} - i_j) \mod n$$

for $j = 1, 2, \ldots, q$, where the subscripts are modulo $q$.

For example, we consider the case for $C(12;\{1,4\})$. Let $d = 4$, $D = \{v_1, v_2, v_3, v_5, v_8, v_9\}$, and let $D_p = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ where $(x_1, y_1) = (v_1, v_5)$, $(x_2, y_2) = (v_2, v_3)$ and $(x_3, y_3) = (v_8, v_9)$. That is, $i_1 = 1, i_2 = 2, i_3 = 8$. We check that $\delta_1 = (2-1) \mod 12 = 1$, $\delta_2 = (8-2) \mod 12 = 6$ and $\delta_3 = (1-8) \mod 12 = 5$.

Clearly,

$$n = \delta_1 + \cdots + \delta_q.$$

Throughout the paper, the subscripts are taken modulo $n$ when it is unambiguous.

## 2. $d$-distance paired-domination number of $C(n;\{1,2\})$

In this section, we shall determine the exact $d$-distance paired-domination number of $C(n;\{1,k\})$ for $k = 2$ and $d \geq 1$.

For the circulant graphs $C(n;\{1,k\})$, if there exists $\ell \in \{1, 2, \ldots, q\}$ such that $\delta_\ell \geq (2d+1)k + 2$ for $k \geq 2$ and $d \geq 1$, then $v_{i_\ell+(d+1)k+1}$ would not be dominated by $D$. Hence, we have:

**Observation 2.1.** Suppose $k \geq 2$ and $d \geq 1$. Then $1 \leq \delta_j \leq (2d+1)k + 1$ for every $j \in \{1, 2, \ldots, q\}$.

**Theorem 2.1.** *For $k \geq 2$, $n \geq 2k+1$ and $d \geq 1$, $\gamma_p^d(C(n;\{1,k\})) \geq 2\lceil \frac{n}{(2d+1)k+1} \rceil$.*

*Proof.* By Observation 2.1, we have $n = \delta_1 + \cdots + \delta_q \leq q \times ((2d+1)k+1)$, and thus, $q \geq \lceil \frac{n}{(2d+1)k+1} \rceil$, which implies $\gamma_p^d(C(n;\{1,k\})) \geq 2\lceil \frac{n}{(2d+1)k+1} \rceil$. ∎

**Theorem 2.2.** *For $k = 2$, $n \geq 2k+1$ and $d \geq 1$, $\gamma_p^d(C(n;\{1,k\})) = 2\lceil \frac{n}{2kd+3} \rceil$.*

*Proof.* Let $D$ be a $d$-distance paired-dominating set of $C(n;\{1,k\})$ for $k = 2$. Let $m = \lfloor \frac{n}{2kd+3} \rfloor$, $t = n \bmod (2kd+3)$ and

$$D = \begin{cases} \{v_{(2kd+3)i}, v_{(2kd+3)i+2} : 0 \leq i \leq m-1\}, \text{ if } t = 0; \\ \{v_{(2kd+3)i}, v_{(2kd+3)i+2} : 0 \leq i \leq m-1\} \cup \{v_{(2kd+3)m-1}, v_{(2kd+3)m}\}, \\ \quad \text{if } t = 1; \\ \{v_{(2kd+3)i}, v_{(2kd+3)i+2} : 0 \leq i \leq m-1\} \cup \{v_{(2kd+3)m}, v_{(2kd+3)m+1}\}, \\ \quad \text{if } t = 2; \\ \{v_{(2kd+3)i}, v_{(2kd+3)i+2} : 0 \leq i \leq m\}, \text{ otherwise.} \end{cases}$$

It is not hard to verify that $D$ is a $d$-distance paired dominating set of $C(n;\{1,k\})$ for $k=2$ with $|D|=2\lceil\frac{n}{2kd+3}\rceil$. Hence, $\gamma_p^d(C(n;\{1,k\}))\leq 2\lceil\frac{n}{2kd+3}\rceil$ for $k=2$ and $d\geq 1$. On the other hand, by Theorem 2.2, we have that $\gamma_p^d(C(n;\{1,k\}))\geq 2\lceil\frac{n}{2kd+3}\rceil$ for $k=2$ and $d\geq 1$. The result immediately holds. ∎

In Figure 1, we show the $d$-distance paired-dominating sets of $C(n;\{1,2\})$ for $d=1$ and $7\leq n\leq 14$, and for $d=2$ and $11\leq n\leq 22$, where the vertices of $d$-distance paired dominating sets are in dark.

$G_{n,k}$ stands for $C(n;\{1,k\})$ in all figures of this paper.

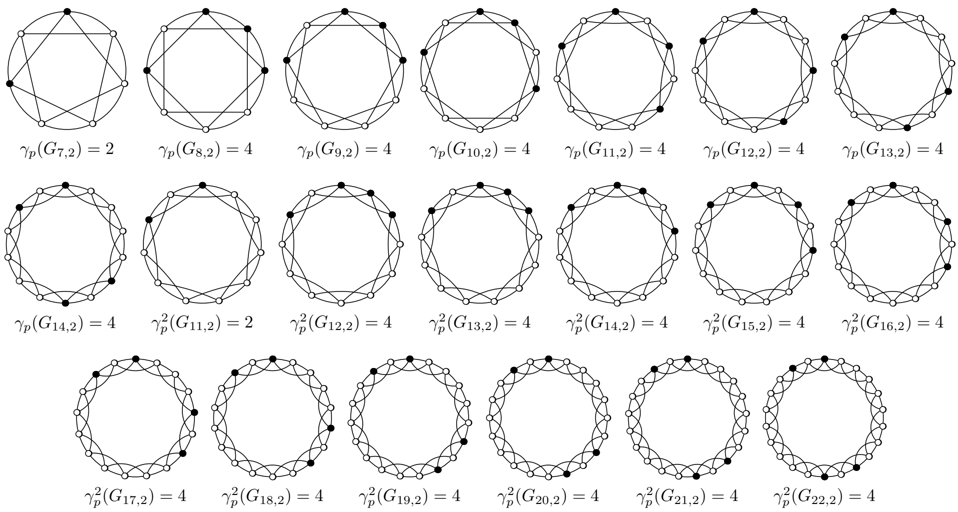



Figure 1. The $d$-distance paired dominating sets of $C(n;\{1,2\})$ for $d=1$ and $7\leq n\leq 14$, and for $d=2$ and $11\leq n\leq 22$.

## 3. $d$-distance paired-domination number of $C(n;\{1,3\})$

In this section, we shall determine the exact $d$-distance paired-domination number of $C(n;\{1,k\})$ for $k=3$ and $d\geq 1$.

**Lemma 3.1.** *For $k=3$, $n\geq 2k+1$ and $d\geq 1$, $\gamma_p^d(C(n;\{1,k\}))\leq 2\lceil\frac{n}{2kd+2}\rceil$.*

*Proof.* Let $D$ be a $d$-distance paired-dominating set of $C(n;\{1,k\})$ for $k=3$. Let $m=\lfloor\frac{n}{2kd+2}\rfloor$, $t=n \bmod (2kd+2)$ and

$$D=\begin{cases}\{v_{(2kd+2)i},v_{(2kd+2)i+1}:0\leq i\leq m-1\}, \text{ if } t=0;\\ \{v_{(2kd+2)i},v_{(2kd+2)i+1}:0\leq i\leq m-1\}\cup\{v_{(2kd+2)m-1},v_{(2kd+2)m}\}, \text{ if } t=1;\\ \{v_{(2kd+2)i},v_{(2kd+2)i+1}:0\leq i\leq m\}, \text{ otherwise}.\end{cases}$$

It is not hard to verify that $D$ is a $d$-distance paired dominating set of $C(n;\{1,k\})$ for $k=3$ with $|D|=2\lceil\frac{n}{2kd+2}\rceil$. Hence, $\gamma_p^d(C(n;\{1,k\}))\leq 2\lceil\frac{n}{2kd+2}\rceil$ for $k=3$ and $d\geq 1$. ∎

In Figure 2, we show the $d$-distance paired-dominating sets of $C(n;\{1,3\})$ for $d=1$ and $8\leq n\leq 16$, and for $d=2$ and $14\leq n\leq 28$, where the vertices of $d$-distance paired dominating sets are in dark.

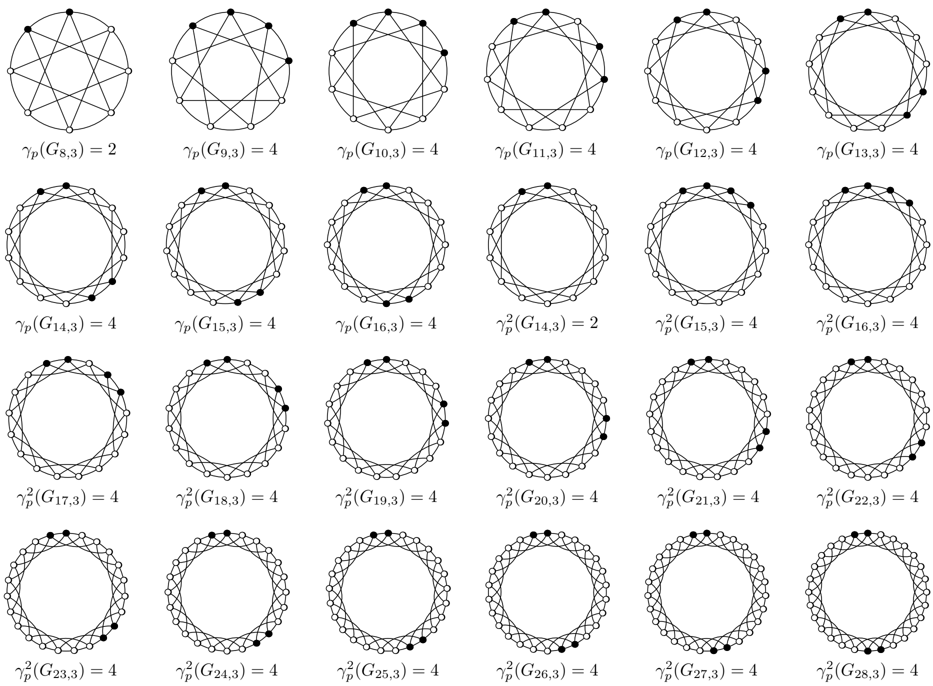



Figure 2. The $d$-distance paired dominating sets of $C(n; \{1, 3\})$ for $d = 1$ and $8 \le n \le 16$, and for $d = 2$ and $14 \le n \le 28$.

**Lemma 3.2.** *For $k = 3$, $n \ge 2k + 1$ and $d \ge 1$, $\gamma_p^d(C(n; \{1, k\})) \ge 2\lceil \frac{n}{2kd+2} \rceil$.*

*Proof.* Let $D = \{x_i, y_i : i = 1, 2, \ldots, q\}$ be a $d$-distance paired dominating set of $C(n; \{1, k\})$ for $k = 3$ with the minimum cardinality. By Observation 2.1, we have that

$$1 \le \delta_j \le 2kd + 4 \tag{3.1}$$

for every $j \in \{1, 2, \ldots, q\}$.

Suppose that there exists $\ell \in \{1, 2, \ldots, q\}$ such that $\delta_\ell \ge 2kd + 3$. Then $v_{i_\ell+kd+2}$ would not be dominated by $(x_\ell, y_\ell)$ and $(x_{\ell+1}, y_{\ell+1})$. To dominate $v_{i_\ell+kd+2}$, we have $v_{i_\ell+2} \in D$. It follows that $v_{i_\ell-1} \in D$, which implies $(x_{\ell-1}, y_{\ell-1}) = (v_{i_\ell-1}, v_{i_\ell+2})$, and thus

$$\delta_{\ell-1} = 1. \tag{3.2}$$

Let

$$S_1 = \{i : 1 \le i \le q, 2kd + 3 \le \delta_i \le 2kd + 4\},$$
$$S_2 = \{i : 1 \le i \le q, 2 \le \delta_i \le 2kd + 2\},$$
$$S_3 = \{i : 1 \le i \le q, \delta_i = 1\}.$$

By (3.1) and (3.2), we have that $\{1, 2, \ldots, q\} = S_1 \cup S_2 \cup S_3$, and there exists an injection $\phi : S_1 \to S_3$ defined by $\phi(i) = i - 1$ for any $i \in S_1$, i.e., $|S_1| \le |S_3|$. It follows that

$$\begin{aligned} n &= \delta_1 + \cdots + \delta_q \\ &= \sum_{i\in S_1} \delta_i + \sum_{i\in S_2} \delta_i + \sum_{i\in S_3} \delta_i \\ &\le (2kd+4)|S_1| + (2kd+2)|S_2| + |S_3| \\ &= (2kd+2)(|S_1|+|S_2|+|S_3|) + 2(|S_1|-|S_3|) - (2kd-1)|S_3| \\ &\le (2kd+2)q, \end{aligned}$$

which implies $q \ge \lceil \frac{n}{2kd+2} \rceil$, and thus $\gamma_p^d(C(n;\{1,k\})) \ge 2\lceil \frac{n}{2kd+2} \rceil$ for $k=3$ and $d \ge 1$. ∎

As an immediate consequence of Lemmas 3.1 and 3.2, we have the following:

**Theorem 3.1.** *For $k=3$, $n \ge 2k+1$ and $d \ge 1$, $\gamma_p^d(C(n;\{1,k\})) = 2\lceil \frac{n}{2kd+2} \rceil$.*

## 4. $d$-distance paired-domination number of $C(n;\{1,4\})$

In this section, we shall determine the $d$-distance paired domination number of $C(n;\{1,k\})$ for $k=4$ and $d \ge 1$.

We shall first consider the case for $d=1$. At this time, the $d$-distance paired-domination number $\gamma_p^d$ is just the paired-domination number $\gamma_p$.

**Lemma 4.1.** *For $n \ge 9$,*

$$\gamma_p(C(n;\{1,4\})) \le \begin{cases} 2\lceil \frac{3n}{23} \rceil + 2, & \text{if } n \equiv 15, 22 \pmod{23}; \\ 2\lceil \frac{3n}{23} \rceil, & \text{otherwise.} \end{cases}$$

*Proof.* It suffices to give a paired-dominating set $D$ of $C(n;\{1,4\})$ with the cardinality equaling to the exact values mentioned in this lemma.

Let $m_1 = \lfloor \frac{n}{23} \rfloor$ and $t = n \bmod 23$. Then $n = 23m_1 + t$.

For $2k+1 \le n \le 22$, let

$$D = \begin{cases} \{v_0, v_1, v_7, v_8\}, & \text{if } 9 \le n \le 14 \text{ and } n \ne 12; \\ \{v_0, v_1, v_2, v_3\}, & \text{if } n = 12; \\ \{v_0, v_1, v_7, v_8, v_{13}, v_{14}\}, & \text{if } n = 15; \\ \{v_0, v_1, v_7, v_8, v_{14}, v_{15}\}, & \text{if } 16 \le n \le 21 \text{ and } n \ne 19; \\ \{v_0, v_1, v_7, v_{11}, v_{13}, v_{17}\}, & \text{if } n = 19; \\ \{v_0, v_1, v_7, v_8, v_{14}, v_{15}, v_{20}, v_{21}\}, & \text{if } n = 22. \end{cases}$$

For $n \ge 23$ and $t \ne 5$, let $m_2 = \lfloor \frac{t}{7} \rfloor$,

$$\begin{aligned} D_{01} &= \{v_{23i}, v_{23i+1}, v_{23i+7}, v_{23i+11}, v_{23i+13}, v_{23i+17} : 0 \le i \le m_1 - 1\}, \\ D_{02} &= \{v_{23m_1+7i}, v_{23m_1+7i+1} : 0 \le i \le m_2 - 1\} \end{aligned}$$

and

$$D = \begin{cases} D_{01}, & \text{if } t = 0; \\ D_{01} \cup \{v_{23m_1-1}, v_{23m_1}\}, & \text{if } t = 1; \\ D_{01} \cup \{v_{23m_1}, v_{23m_1+1}\}, & \text{if } 2 \le t \le 7 \text{ and } t \ne 5; \\ D_{01} \cup D_{02} \cup \{v_{23m_1+7m_2-1}, v_{23m_1+7m_2}\}, & \text{if } t = 8, 15, 22; \\ D_{01} \cup D_{02} \cup \{v_{23m_1+7m_2}, v_{23m_1+7m_2+1}\}, & \text{if } 9 \le t \le 21 \text{ and } t \ne 12, 15, 19; \\ D_{01} \cup D_{02} \cup \{v_{23m_1+7m_2}, v_{23m_1+7m_2+4}\}, & \text{if } t = 12, 19. \end{cases}$$

For $t = 5$, let $m_3 = \frac{n-51}{23}$ where $n > 51$,

$$D_{03} = \{v_{23i}, v_{23i+4}, v_{23i+10}, v_{23i+11}, v_{23i+17}, v_{23i+21} : 0 \le i \le m_3 - 1\},$$
$$D_{04} = \{v_{23m_3+10+7i}, v_{23m_3+11+7i} : 0 \le i \le 4\}$$

and

$$D = \begin{cases} \{v_{7i}, v_{7i+1} : 0 \le i \le 3\}, & \text{if } n = 28; \\ \{v_{7i}, v_{7i+1} : 0 \le i \le 4\} \cup \{v_{35}, v_{39}, v_{41}, v_{45}\}, & \text{if } n = 51; \\ D_{03} \cup D_{04} \cup \{v_{23m_3}, v_{23m_3+4}, v_{n-6}, v_{n-2}\}, & \text{if } n > 51. \end{cases}$$

It is not hard to verify that $D$ is a paired-dominating set of $C(n; \{1, 4\})$ with the cardinality equaling to the exact values mentioned in this lemma. ∎

In Figure 3 and Figure 4, we show the paired-dominating sets of $C(n; \{1, 4\})$ for $9 \le n \le 22$ and $23 \le n \le 46$, respectively, where the vertices of paired-dominating sets are in dark.

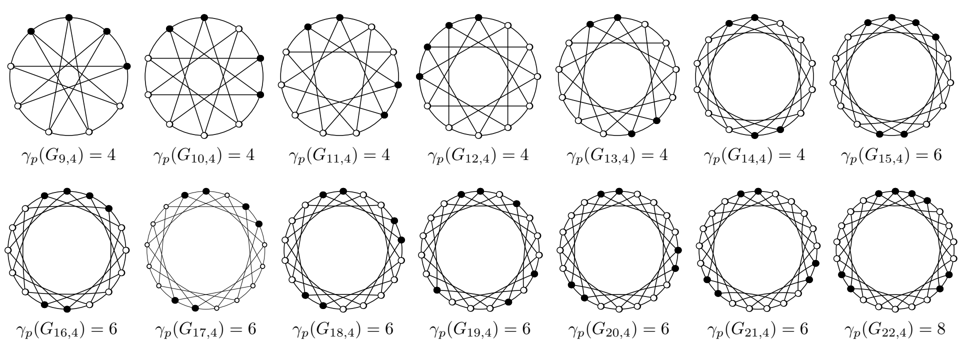



Figure 3. The paired-dominating sets of $C(n; \{1, 4\})$ for $9 \le n \le 22$.

For convenience, let

$$V'(i, t) = \{v_{i+j} \in V(C(n; \{1, 4\})) : 0 \le j \le t - 1\},$$

where $i \in \{0, 1, \ldots, n-1\}$ and $t \in \{1, 2, \ldots, n\}$.

For each vertex $v \in V(G)$, we define a function rdd counting the times that $v$ is re-dominated by vertex pairs $\{x_i, y_i\}$ in $D$ as follows:

$$\mathrm{rdd}(v) = |\{i : 1 \le i \le q, v \in N[\{x_i, y_i\}]\}| - 1.$$

For a vertex set $S \subseteq V(G)$, let

$$\mathrm{rdd}(S) = \sum_{v \in S} \mathrm{rdd}(v).$$

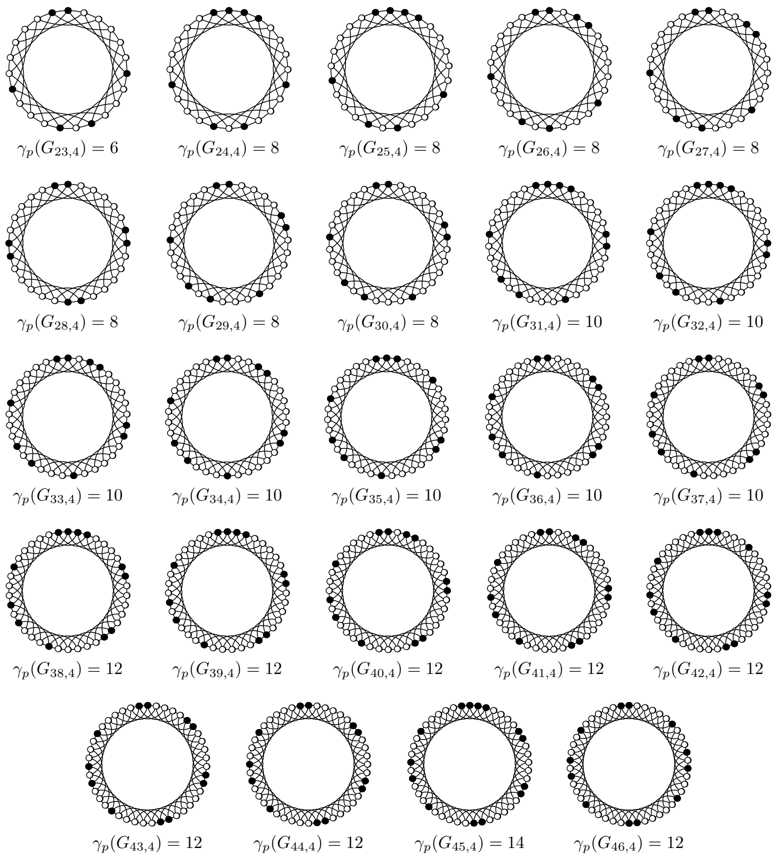



Figure 4. The paired-dominating sets of $C(n;\{1,4\})$ for $23 \leq n \leq 46$.

Since $x$ is not adjacent to $y$ for any two vertices $x, y \in N(v)$ where $v \in V(C(n;\{1,4\}))$, by the definition of rdd, we have:

**Observation 4.1.** $\text{rdd}(v) = |N(v) \cap D| - 1$ for every vertex $v \in V(C(n;\{1,4\}))$.

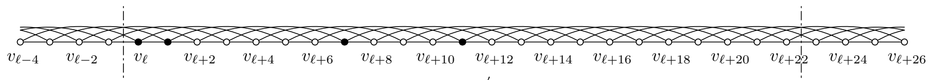
**Lemma 4.2.** *Suppose* $n \geq 23$. *Then* $\text{rdd}(V'(i, 23)) \geq 1$ *for every* $i \in \{0, 1, \ldots, n-1\}$.

*Proof.* Suppose to the contrary that there exists $\ell \in \{0, 1, \ldots, n-1\}$ such that

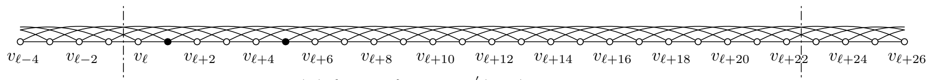
$$\text{rdd}(V'(\ell, 23)) = 0. \tag{4.1}$$

Suppose that there exists $s \in \{\ell, \ell+1, \ldots, \ell+21\}$ such that $(v_s, v_{s+1}) \in D_p$. For $s \in \{\ell, \ell+1, \ldots, \ell+10\}$, by (4.1), we have $v_{s-1}, v_{s+2}, v_{s+3}, v_{s+4}, v_{s+5}, v_{s+6}, v_{s+8}, v_{s+9} \notin D$. To dominate $v_{s+3}$, we have $v_{s+7} \in D$. It follows that $v_{s+10} \notin D$. Since $\langle D \rangle$ contains a perfect matching, we have $v_{s+11} \in D$. It follows that $v_{s+13} \notin D$ (see Figure 5(I) for $s = \ell$). Thus, $v_{s+9}$ would not be dominated by $D$, a contradiction. For $s \in \{\ell+11, \ell+12, \ldots, \ell+21\}$, by symmetry, we derive a contradiction. Hence, there does not exist $s \in \{\ell, \ell+1, \ldots, \ell+21\}$ such that $(v_s, v_{s+1}) \in D_p$.

To dominate $v_{\ell+9}$, we have that there exists $s \in \{\ell+1, \ldots, \ell+13\}$ such that $(v_s, v_{s+4}) \in D_p$. By (4.1), we have $v_{s-2}, v_{s+1}, v_{s+2}, v_{s+3}, v_{s+6} \notin D$ (see Figure 5(II) for $s = \ell+1$). It follows that $v_{s+2}$ would not be dominated by $D$, a contradiction. The lemma follows. ∎



(I) $\{v_s, v_{s+1}\} \subset D \cup V'(\ell, 23)$ for $s = \ell$



(II) $\{v_s, v_{s+4}\} \subset D \cup V'(\ell, 23)$ for $s = \ell+1$

Figure 5. The graphs for the proof of Lemma 4.2.

**Lemma 4.3.** *$\gamma_p(C(n; \{1,4\})) \geq 2\lceil \frac{3n}{23} \rceil$ for $n \geq 9$.*

*Proof.* Let $D = \{x_i, y_i : i = 1, 2, \ldots, q\}$ be a minimum paired-dominating set of $C(n; \{1,4\})$ where $\{x_i y_i : i = 1, 2, \ldots, q\}$ is a perfect matching of $\langle D \rangle$. Since each pair $\{x_i, y_i\}$ dominates exactly 8 vertices, we have $8q - n \geq 0$. It follows that $q \geq \lceil \frac{n}{8} \rceil$.

For $9 \leq n \leq 22$ and $n \neq 16$, since $\lceil \frac{n}{8} \rceil = \lceil \frac{3n}{23} \rceil$, we have $\gamma_p(C(n; \{1,4\})) \geq 2\lceil \frac{3n}{23} \rceil$.

For $n = 16$, it is easy to verify that two pairs of vertices would not dominate all vertices in $C(n; \{1,4\})$. Hence, $q \geq 3 = \lceil \frac{3n}{23} \rceil$, which implies $\gamma_p(C(n; \{1,4\})) \geq 2\lceil \frac{3n}{23} \rceil$.

For $n \geq 23$, by Lemma 4.2, we have $8q \geq n + \lceil \frac{n}{23} \rceil = \lceil \frac{24n}{23} \rceil$. It follows that $q \geq \lceil \frac{1}{8} \times \lceil \frac{24n}{23} \rceil \rceil \geq \lceil \frac{1}{8} \times \frac{24n}{23} \rceil = \lceil \frac{3n}{23} \rceil$, which implies $\gamma_p(C(n; \{1,4\})) \geq 2\lceil \frac{3n}{23} \rceil$. ∎

For convenience, we define

$$\Re = \sum_{i=0}^{n-1} (\mathrm{rdd}(V'(i, 23)) - 1).$$

**Lemma 4.4.** *If there exists $\ell \in \{0, 1, \ldots, n-1\}$ such that $\mathrm{rdd}(v_\ell) \geq 2$, then $\Re > 24$.*

*Proof.* By Observation 4.1, we have that $|N(v_\ell) \cap D| = \mathrm{rdd}(v_\ell) + 1 \geq 3$. Since $|N(v_\ell) \cap D| \leq |N(v_\ell)| = 4$, we have $\{v_{\ell+1}, v_{\ell+4}\} \subset D$ or $\{v_{\ell-1}, v_{\ell-4}\} \subset D$, say $\{v_{\ell+1}, v_{\ell+4}\} \subset D$. It follows that $\mathrm{rdd}(v_{\ell+5}) \geq 1$, and thus $\Re \geq \sum_{\ell-17 \leq i \leq \ell} (\mathrm{rdd}(V'(i, 23)) - 1) \geq 18 \times (\mathrm{rdd}(v_\ell) + \mathrm{rdd}(v_{\ell+5}) - 1) \geq 18 \times (2 + 1 - 1) > 24$. The lemma follows. ∎

In what follows, we admit that $\mathrm{rdd}(v_i) \in \{0, 1\}$ for every $i \in \{0, 1, \ldots, n-1\}$. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ be all the vertices re-dominated once, where $t = \mathrm{rdd}(V(C(n; \{1,4\})))$ and $0 \leq i_1 < i_2 < \cdots < i_t \leq n-1$. We define

$$\Theta_j = i_{j+1} - i_j$$

for $j = 1, 2, \ldots, t$, where the subscripts are modulo $t$. Obviously, $\Theta_1 + \cdots + \Theta_t = n$.

**Lemma 4.5.** *If $\Re \leq 24$, then $\Theta_j + \Theta_{j+1} \geq 22$ for every $j \in \{1, 2, \ldots, t\}$ where $t = \mathrm{rdd}(V(C(n; \{1,4\})))$.*

*Proof.* Choose arbitrary $\ell \in \{1, 2, \dots, t\}$. By the definition of $\Re$, we have $\Re = \sum_{i=1}^{t}(23 - \Theta_i) \geq (23 - \Theta_\ell) + (23 - \Theta_{\ell+1}) = 46 - (\Theta_\ell + \Theta_{\ell+1})$. Since $\Re \leq 24$, we have $46 - (\Theta_\ell + \Theta_{\ell+1}) \leq 24$. It follows that $\Theta_\ell + \Theta_{\ell+1} \geq 22$. The lemma follows. ∎

**Lemma 4.6.** *For $n > 23$, if there exists $\ell \in \{0, 1, \dots, n-1\}$ such that $v_\ell \in D$ and* $\mathrm{rdd}(v_\ell) = 1$*, then $\Re > 24$.*

*Proof.* Assume to the contrary that $\Re \leq 24$. By Lemma 4.4, we have that $\mathrm{rdd}(v_i) \in \{0, 1\}$ for every $i \in \{0, 1, \dots, n-1\}$. By Observation 4.1, we have $|N(v_\ell) \cap D| = \mathrm{rdd}(v_\ell) + 1 = 2$. Let $N(v_\ell) \cap D = \{w_1, w_2\}$. By symmetry, we have $\{w_1, w_2\} \in \{\{v_{\ell-1}, v_{\ell+1}\}, \{v_{\ell+1}, v_{\ell+4}\}, \{v_{\ell+1}, v_{\ell-4}\}, \{v_{\ell-4}, v_{\ell+4}\}\}$. Since $D$ contains a perfect matching, we infer that

$$\mathrm{rdd}(w_1) = 1 \text{ or } \mathrm{rdd}(w_2) = 1.$$

That is, there exists $j \in \{1, 2, \dots, t\}$ such that $\Theta_j \leq 4$. By Lemma 4.5, we have that

$$\Theta_{j-1} \geq 18 \text{ and } \Theta_{j+1} \geq 18. \tag{4.2}$$

From (4.2), we have $\{w_1, w_2\} \notin \{\{v_{\ell+1}, v_{\ell+4}\}, \{v_{\ell+1}, v_{\ell-4}\}\}$. If $\{w_1, w_2\} = \{v_{\ell-1}, v_{\ell+1}\}$, by (4.2), we have $V'(\ell - 5, 11) \cap D = \{v_{\ell-1}, v_\ell, v_{\ell+1}\}$ (see Figure 6(I)), which is contradicted with the fact that $D$ contains a perfect matching. If $\{w_1, w_2\} = \{v_{\ell-4}, v_{\ell+4}\}$, by (4.2), we have $v_{\ell-2}, v_{\ell+2}, v_{\ell+3}, v_{\ell+6} \notin D$. Since $v_{\ell+1} \notin D$, we have that $v_{\ell+2}$ would not be dominated by $D$ (see Figure 6(II)), a contradiction. ∎

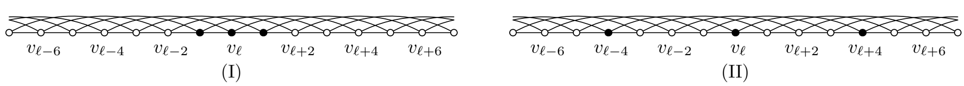



Figure 6. The graphs for the proof of Lemma 4.6.

As an immediate consequence of Lemmas 4.4 and 4.6, we have the following:

**Corollary 4.1.** *Suppose $(x, y) \in D_p$ and $\Re \leq 24$. Then $N(x) \cap D = \{y\}$.*

**Lemma 4.7.** *Suppose $n > 23$ and $\Re \leq 24$. If there exists $\ell \in \{0, 1, \dots, n-1\}$ such that $v_\ell \notin D$ and* $\mathrm{rdd}(v_\ell) = 1$*, then one of the following conditions holds.*

(a) $V'(\ell - 5, 11) \cap D = \{v_{\ell-5}, v_{\ell-1}, v_{\ell+1}, v_{\ell+5}\}$;
(b) $V'(\ell - 4, 9) \cap D = \{v_{\ell-4}, v_{\ell-3}, v_{\ell+3}, v_{\ell+4}\}$.

*Proof.* By Lemma 4.4, we have that $\mathrm{rdd}(v_i) \in \{0, 1\}$ for every $i \in \{0, 1, \dots, n-1\}$. By Observation 4.1, we have $|N(v_\ell) \cap D| = \mathrm{rdd}(v_\ell) + 1 = 2$. By symmetry, we distinguish four cases.

**Case 1.** $N(v_\ell) \cap D = \{v_{\ell-1}, v_{\ell+1}\}$.

By Lemma 4.6, we have $|\{v_{\ell-5}, v_{\ell-2}, v_{\ell+3}\} \cap D| = |\{v_{\ell-3}, v_{\ell+2}, v_{\ell+5}\} \cap D| = 1$. If $v_{\ell-2} \in D$, then $\mathrm{rdd}(v_{\ell-3}) = \mathrm{rdd}(v_{\ell+2}) = 1$ (see Figure 7(I) where the vertices that re-dominated once are in gray). By Lemma 4.5, we derive a contradiction. Hence $v_{\ell-2} \notin D$. By symmetry, we have $v_{\ell+2} \notin D$. If $v_{\ell+3} \in D$, then $\mathrm{rdd}(v_{\ell+2}) = 1$. Let

$i_j = \ell$. By Lemma 4.5, we have that $\Theta_j = 2$, $\Theta_{j-1} \geq 20$ and $\Theta_{j+1} \geq 20$. It follows that $v_{\ell-3}, v_{\ell+5} \notin D$ (see Figure 7(II)). Since $v_\ell, v_{\ell+2} \notin D$, we have that $D$ does not contain a perfect matching, a contradiction. Hence $v_{\ell+3} \notin D$. By symmetry, we have $v_{\ell-3} \notin D$. Therefore, we conclude that $v_{\ell-5}, v_{\ell+5} \in D$ (see Figure 7(III)). Since $v_{\ell-4}, v_{\ell+4} \notin D$, we have $V^{'}(\ell-5, 11) \cap D = \{v_{\ell-5}, v_{\ell-1}, v_{\ell+1}, v_{\ell+5}\}$.

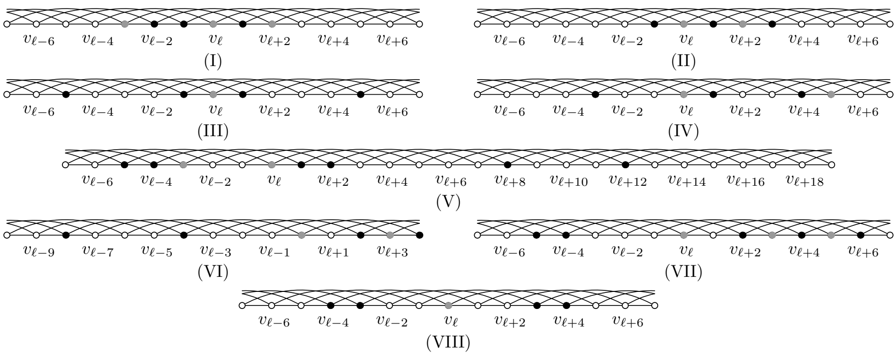



Figure 7. The graphs for proof of Lemma 4.7.

**Case 2.** $N(v_\ell) \cap D = \{v_{\ell+1}, v_{\ell+4}\}$.

Then $\text{rdd}(v_{\ell+5}) = 1$. Let $i_j = \ell$. By Lemma 4.5, we have that $\Theta_j = 5$, $\Theta_{j-1} \geq 17$ and $\Theta_{j+1} \geq 17$. It follows that $v_{\ell-2}, v_{\ell+2}, v_{\ell+3}, v_{\ell+5} \notin D$. Since $D$ contains a perfect matching, we have $v_{\ell-3} \in D$. It follows that $v_{\ell-5} \notin D$ (see Figure 7(IV)). Thus, $v_{\ell-1}$ would not be dominated by $D$, a contradiction.

**Case 3.** $N(v_\ell) \cap D = \{v_{\ell+1}, v_{\ell-4}\}$.

Then $\text{rdd}(v_{\ell-3}) = 1$. Let $i_j = \ell - 3$. By Lemma 4.5, we have that $\Theta_j = 3$, $\Theta_{j-1} \geq 19$ and $\Theta_{j+1} \geq 19$. It follows that $v_{\ell-6}, v_{\ell-3}, v_{\ell-2}, v_{\ell+3} \notin D$. To dominate $\{v_{\ell-2}, v_{\ell-1}\}$, we have $v_{\ell+2}, v_{\ell-5} \in D$. It follows that $v_{\ell+4}, v_{\ell+5}, v_{\ell+6}, v_{\ell+7} \notin D$. To dominate $v_{\ell+4}$, we have $v_{\ell+8} \in D$. It follows that $v_{\ell+9}, v_{\ell+10}, v_{\ell+11} \notin D$. Since $D$ contains a perfect matching, we have $v_{\ell+12} \in D$. It follows that $v_{\ell+14} \notin D$ (see Figure 7(V)). Thus, $v_{\ell+10}$ would not be dominated by $D$, a contradiction.

**Case 4.** $N(v_\ell) \cap D = \{v_{\ell-4}, v_{\ell+4}\}$.

By Lemma 4.6, we have $|\{v_{\ell-8}, v_{\ell-5}, v_{\ell-3}\} \cap D| = |\{v_{\ell+3}, v_{\ell+5}, v_{\ell+8}\} \cap D| = 1$.

Suppose $v_{\ell-8} \in D$. By Lemma 4.5, we have $v_{\ell-6} \notin D$. By Corollary 4.1, we have $v_{\ell-7}, v_{\ell-5}, v_{\ell-3} \notin D$. If $v_{\ell+2} \notin D$, then either $v_{\ell-2}$ would not be dominated by $D$ or $D$ would not contain a perfect matching. Hence $v_{\ell+2} \in D$. It follows that $\text{rdd}(v_{\ell+3}) = 1$. Let $i_j = \ell$. By Lemma 4.5, we have that $\Theta_j = 3$, $\Theta_{j-1} \geq 19$ and $\Theta_{j+1} \geq 19$. It follows that $v_{\ell-10}, v_{\ell-2} \notin D$ (see Figure 7(VI)), and thus $v_{\ell-6}$ would not be dominated by $D$, a contradiction. Hence $v_{\ell-8} \notin D$. By symmetry, we have $v_{\ell+8} \notin D$.

Suppose $v_{\ell-5} \in D$. By Corollary 4.1, we have $v_{\ell-6}, v_{\ell-3} \notin D$. By Lemma 4.5, we have $v_{\ell-2} \notin D$. Since $v_{\ell-1} \notin D$, to dominate $v_{\ell-2}$, we have $v_{\ell+2} \in D$. It follows that $\text{rdd}(v_{\ell+3}) = 1$. Let $i_j = \ell$. By Lemma 4.5, we have that $\Theta_j = 3$,

$\Theta_{j-1} \geq 19$ and $\Theta_{j+1} \geq 19$. It follows that $v_{\ell+3}, v_{\ell+6} \notin D$ (see Figure 7(VII)). Since $v_{\ell+1}, v_{\ell-2} \notin D$, we have that $D$ does not contain a perfect matching, a contradiction. Hence $v_{\ell-5} \notin D$. By symmetry, we have $v_{\ell+5} \notin D$.

Therefore, we conclude that $v_{\ell-3}, v_{\ell+3} \in D$ (see Figure 7(VIII)). By Corollary 4.1, we have $v_{\ell-2}$, $v_{\ell+2} \notin D$, i.e., $V^{'}(\ell-4,9) \cap D = \{v_{\ell-4}, v_{\ell-3}, v_{\ell+3}, v_{\ell+4}\}$.

This completes the proof of Lemma 4.7. ∎

**Lemma 4.8.** *Let* $t = \mathrm{rdd}(V(C(n;\{1,4\})))$. *If* $\Re \leq 24$, *then the following conditions hold.*

(a) $\Theta_i \in \{7, 15, 23\}$ *for every* $i \in \{1, 2, \ldots, t\}$;
(b) $|\{1 \leq i \leq t : \Theta_i = 15\}|$ *is even.*

*Proof.* (a) Let $A_1 = \{0 \leq i \leq n-1 : \mathrm{rdd}(v_i) = 1, V^{'}(i-5, 11) \cap D = \{v_{i-5}, v_{i-1}, v_{i+1}, v_{i+5}\}\}$ and $A_2 = \{0 \leq i \leq n-1 : \mathrm{rdd}(v_i) = 1, V^{'}(i-4, 9) \cap D = \{v_{i-4}, v_{i-3}, v_{i+3}, v_{i+4}\}\}$. By Lemma 4.7, we have $A_1 \cap A_2 = \emptyset$ and

$$A_1 \cup A_2 = \{0 \leq i \leq n-1 : \mathrm{rdd}(v_i) = 1\}. \tag{4.3}$$

By Lemma 4.2, we have $\Theta_i \leq 23$ for every $i \in \{1, 2, \ldots, t\}$. Let $\Theta$ be an arbitrary integer of $\{\Theta_1, \ldots, \Theta_t\}$. That is, there exists $\ell \in \{0, 1, \ldots, n-1\}$ such that $\mathrm{rdd}(v_\ell) = \mathrm{rdd}(v_{\ell+\Theta}) = 1$ and $\mathrm{rdd}(v_{\ell+j}) = 0$ for every $j \in \{1, 2, \ldots, \Theta - 1\}$. To prove (a), it suffices to show $\Theta \in \{7, 15, 23\}$.

**Case 1.** $\ell \in A_1$.

By Corollary 4.1, we have $v_{\ell+6}, v_{\ell+9} \notin D$. By Lemma 4.5, we have $v_{\ell+7}, v_{\ell+8}, v_{\ell+10} \notin D$. To dominate $\{v_{\ell+7}, v_{\ell+8}\}$, we have $v_{\ell+11}, v_{\ell+12} \in D$. It follows from Corollary 4.1 that $v_{\ell+13}, v_{\ell+15}, v_{\ell+16} \notin D$. By Lemma 4.5, we have $v_{\ell+14}, v_{\ell+17} \notin D$. To dominate $v_{\ell+14}$, we have $v_{\ell+18} \in D$. Since $D$ contains a perfect matching, it follows from Corollary 4.1 that $|\{v_{\ell+19}, v_{\ell+22}\} \cap D| = 1$.

If $v_{\ell+19} \in D$, then $\mathrm{rdd}(v_{\ell+15}) = 1$ and $\ell + 15 \in A_2$ (see Figure 8(I) where the vertices that re-dominated once are in gray). Thus, $\Theta = 15$. If $v_{\ell+22} \in D$, by (4.3), we have $v_{\ell+24}, v_{\ell+28} \in D$ and $\mathrm{rdd}(v_{\ell+23}) = 1$, i.e., $\ell + 23 \in A_1$ (see Figure 8(II)). Thus, $\Theta = 23$.

**Case 2.** $\ell \in A_2$.

By Corollary 4.1, we have $v_{\ell+5}, v_{\ell+7}, v_{\ell+8} \notin D$. By Lemma 4.5, we have $v_{\ell+6}, v_{\ell+9} \notin D$. To dominate $v_{\ell+6}$, we have $v_{\ell+10} \in D$. Since $D$ contains a perfect matching, it follows from Corollary 4.1 that $|\{v_{\ell+11}, v_{\ell+14}\} \cap D| = 1$.

If $v_{\ell+11} \in D$, then $\mathrm{rdd}(v_{\ell+7}) = 1$ and $\ell + 7 \in A_2$ (see Figure 8(III)). Thus, $\Theta = 7$. If $v_{\ell+14} \in D$, by (4.3), we have $v_{\ell+16}, v_{\ell+20} \in D$ and $\mathrm{rdd}(v_{\ell+15}) = 1$, i.e., $\ell + 15 \in A_1$ (see Figure 8(IV)). Thus, $\Theta = 15$.

From the above discuss, we see that $\Theta_i \in \{7, 15, 23\}$ for every $i \in \{1, 2, \ldots, t\}$ if $\Re \leq 24$.

(b) Let $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ be all the vertices that re-dominated once, where $0 \leq i_1 < i_2 < \cdots < i_t \leq n-1$. Then $\Theta_j = i_{j+1} - i_j$ for $j = 1, 2, \ldots, t$. By the arguments of (a), we conclude that $\Theta_j = 15$ if and only if either $i_j \in A_1$ and $i_{j+1} \in A_2$, or $i_j \in A_2$ and $i_{j+1} \in A_1$. Note that $i_{t+1} = i_1$. We infer that $|\{1 \leq i \leq t : \Theta_i = 15\}|$ is even. ∎

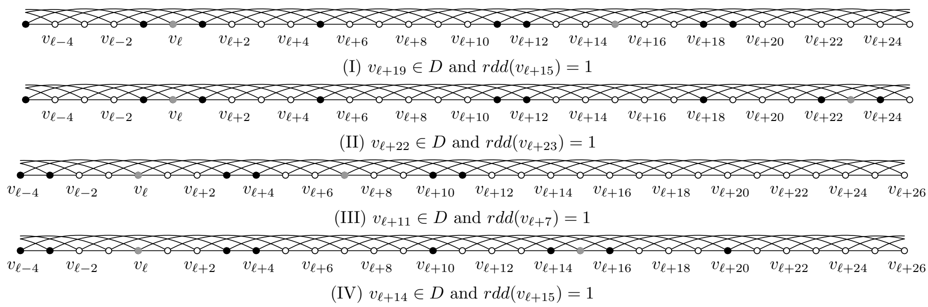



Figure 8. The graphs for proof of Lemma 4.8.

**Lemma 4.9.** $\gamma_p(C(n;\{1,4\})) \geq 2\lceil \frac{3n}{23} \rceil + 2$ *for* $n \equiv 15, 22 \pmod{23}$.

*Proof.* Suppose to the contrary that $\gamma_p(C(n;\{1,4\})) < 2\lceil \frac{3n}{23} \rceil + 2$, i.e., there exists a paired dominating set $D = \{x_i, y_i : i = 1, 2, \ldots, q\}$ such that

$$q = \left\lceil \frac{3n}{23} \right\rceil. \tag{4.4}$$

For $n = 15$ (22), it is not hard to verify that two (three) pairs of vertices would not dominate all vertices in $C(n;\{1,4\})$. Hence, we need only consider the case for $n > 23$.

Since each pair $\{x_i, y_i\}$ in $C(n;\{1,4\})$ dominates exactly 8 vertices, we have $8q - n = \text{rdd}(V(C(n;\{1,4\})))$. By the definition of $\Re$, we have that $23 \times (8q - n) = 23 \times \text{rdd}(V(C(n;\{1,4\}))) = 23 \times \sum_{v \in V(C(n;\{1,4\}))} \text{rdd}(v) = \sum_{0 \leq i \leq n-1} \text{rdd}(V'(i, 23)) = n + \Re$, and thus $q = \frac{3n + \Re/8}{23}$. By (4.4), we conclude that $\Re = 8$ for $n \equiv 15 \pmod{23}$ and $\Re = 24$ for $n \equiv 22 \pmod{23}$.

By Lemma 4.4, we have that $\text{rdd}(v_i) \in \{0, 1\}$ for every $i \in \{0, 1, \ldots, n-1\}$. Let $t = \text{rdd}(V(C(n;\{1,k\})))$. By Lemma 4.8, we have that $\Theta_i \in \{7, 15, 23\}$ for every $i \in \{1, 2, \ldots, t\}$ if $\Re \leq 24$. Let $N_7 = |\{1 \leq i \leq t : \Theta_i = 7\}|$ and $N_{15} = |\{1 \leq i \leq t : \Theta_i = 15\}|$. Then $\Re = (23 - 23) \times (t - N_7 - N_{15}) + (23 - 7) \times N_7 + (23 - 15) \times N_{15} = 16N_7 + 8N_{15}$.

For $\Re = 8$, we have $(N_7, N_{15}) = (0, 1)$. For $\Re = 24$, we have $(N_7, N_{15}) = \{(1, 1), (0, 3)\}$. In either case, we have that $N_{15}$ is odd, which is contradicted with Lemma 4.8 (b). ∎

From Lemmas 4.1, 4.3 and 4.9, we have the following:

**Theorem 4.1.** *For* $n \geq 9$,

$$\gamma_p(C(n;\{1,4\})) = \begin{cases} 2\lceil \frac{3n}{23} \rceil + 2, & \textit{if } n \equiv 15, 22 \pmod{23}; \\ 2\lceil \frac{3n}{23} \rceil, & \textit{otherwise.} \end{cases}$$

In the rest of this section, we shall consider the case for $d \geq 2$.

For the readers' convenience, we shall show the cases for the vertices dominated by a specific vertex pair $(x, y) \in D_p$ in Figure 9, where the vertex pair $(x, y)$ are in dark and the vertices dominated by the vertex pair $(x, y)$ are in gray.

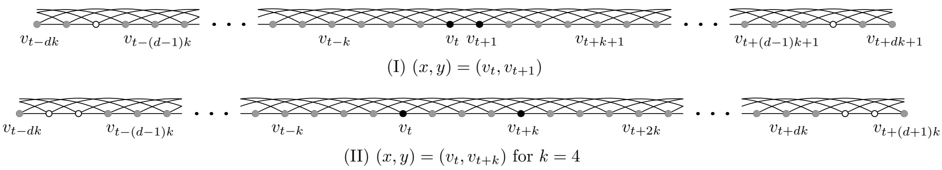



Figure 9. The cases for the vertices dominated by a specific vertex pair.

**Lemma 4.10.** *For $k=4$, $n \geq 2k+1$ and $d \geq 2$,*

$$\gamma_p^d(C(n;\{1,k\})) \leq \begin{cases} 2\lceil \frac{2n}{4kd+1} \rceil + 2, & \textit{if } n \equiv 2kd, 4kd-1, 4kd \pmod{4kd+1} \\ 2\lceil \frac{2n}{4kd+1} \rceil, & \textit{otherwise.} \end{cases}$$

*Proof.* It suffices to give a $d$-distance paired-dominating set $D$ of $C(n;\{1,k\})$ for $k=4$ and $d \geq 2$ with the cardinality equaling to the exact values mentioned in this lemma.

For $9 \leq n \leq 4kd$, let

$$D = \begin{cases} \{v_0, v_4\}, & \text{if } 9 \leq n \leq 2kd-1; \\ \{v_0, v_1, v_{2kd-2}, v_{2kd-1}\}, & \text{if } n = 2kd; \\ \{v_0, v_1, v_{2kd-1}, v_{2kd}\}, & \text{if } 2kd+1 \leq n \leq 2kd+3; \\ \{v_0, v_1, v_{2kd-1}, v_{2kd+3}\}, & \text{if } 2kd+4 \leq n \leq 4kd-2; \\ \{v_0, v_1, v_{2kd-1}, v_{2kd+3}, v_{n-2}, v_{n-1}\}, & \text{if } n = 4kd-1, 4kd. \end{cases}$$

For $n \geq 4kd+1$, let $\alpha = 4kd+1$, $\beta = 2kd-1$, $m_1 = \lfloor \frac{n}{\alpha} \rfloor$ and $t = n \bmod \alpha$. Let

$$D_{01} = \{v_{\alpha i}, v_{\alpha i+1}, v_{\alpha i+\beta}, v_{\alpha i+\beta+4} : 0 \leq i \leq m_1 - 1\},$$
$$D_{02} = \{v_{\alpha m_1}, v_{\alpha m_1+1}, v_{\alpha m_1+\beta}, v_{\alpha m_1+\beta+4}\},$$

and

$$D = \begin{cases} D_{01}, & \text{if } t = 0; \\ D_{01} \cup \{v_{\alpha m_1-1}, v_{\alpha m_1}\}, & \text{if } t = 1; \\ D_{01} \cup \{v_{\alpha m_1}, v_{\alpha m_1+1}\}, & \text{if } 2 \leq t \leq 2kd-1 \\ & \text{and } t \neq 2kd-3; \\ D_{01} \cup \{v_{\alpha m_1-5}, v_{\alpha m_1-1}\}, & \text{if } t = 2kd-3; \\ D_{01} \cup \{v_{\alpha m_1}, v_{\alpha m_1+1}, v_{\alpha m_1+\beta-1}, v_{\alpha m_1+\beta}\}, & \text{if } t = 2kd; \\ D_{01} \cup \{v_{\alpha m_1}, v_{\alpha m_1+1}, v_{\alpha m_1+\beta}, v_{\alpha m_1+\beta+1}\}, & \text{if } 2kd+1 \leq t \leq 2kd+3; \\ D_{01} \cup D_{02}, & \text{if } 2kd+4 \leq t \leq 4kd-2; \\ D_{01} \cup D_{02} \cup \{v_{n-2}, v_{n-1}\}, & \text{if } t = 4kd-1, 4kd. \end{cases}$$

It is not hard to verify that $D$ is a $d$-distance paired dominating set of $C(n;\{1,k\})$ for $k=4$ and $d \geq 2$ with the cardinality equaling to the exact values mentioned in this lemma. ∎

For convenience, we give a map $\varphi : \{1,2,\ldots,q\} \to \{1,4\}$ defined by $\varphi(s)=1$ for $(x_s,y_s)=(v_{i_s},v_{i_s+1})$ and $\varphi(s)=4$ for $(x_s,y_s)=(v_{i_s},v_{i_s+4})$.

**Lemma 4.11.** *Suppose $k=4$, $d \geq 2$ and $\ell \in \{1,2,\ldots,q\}$.*

(a) *If $\delta_{\ell-1} \geq 2kd+3$, then $\delta_\ell \leq 2$.*
(b) *If $\varphi(\ell)=1$, then either $\delta_{\ell-1} \leq 5$ or $\delta_\ell \leq 2kd-1$.*

(c) *If* $\varphi(\ell) = 4$*, then either* $\delta_{\ell-1} \leq 2$ *or* $\delta_\ell \leq 2kd + 2$*.*

(d) *If* $\varphi(\ell) = \varphi(\ell + 1) = 4$ *and* $2kd \leq \delta_\ell \leq 2kd + 2$*, then either* $\delta_{\ell-1} \leq 2$ *or* $\delta_{\ell+1} \leq 2$*.*

*Proof.* (a) Suppose $\delta_{\ell-1} \geq 2kd+3$. If $\delta_\ell \geq 3$, then $v_{i_\ell - kd+2}$ would not be dominated by $D$, a contradiction. Hence $\delta_\ell \leq 2$.

(b) Suppose $\varphi(\ell) = 1$. If $\delta_{\ell-1} \geq 6$ and $\delta_\ell \geq 2kd$, then $v_{i_\ell + kd-1}$ would not be dominated by $D$, a contradiction. Hence either $\delta_{\ell-1} \leq 5$ or $\delta_\ell \leq 2kd - 1$.

(c) Suppose $\varphi(\ell) = 4$. If $\delta_{\ell-1} \geq 3$ and $\delta_\ell \geq 2kd + 3$, then $v_{i_\ell + kd+2}$ would not be dominated by $D$, a contradiction. Hence either $\delta_{\ell-1} \leq 2$ or $\delta_\ell \leq 2kd + 2$.

(d) Suppose $\varphi(\ell) = \varphi(\ell + 1) = 4$ and $2kd \leq \delta_\ell \leq 2kd + 2$. If $\delta_{\ell-1} \geq 3$ and $\delta_{\ell+1} \geq 3$, then at least one of $\{v_{i_\ell+kd+2}, v_{i_\ell+kd+3}\}$ would not be dominated by $D$, a contradiction. Hence either $\delta_{\ell-1} \leq 2$ or $\delta_{\ell+1} \leq 2$. ∎

We denote $\Omega_i = \delta_i + \delta_{i+1}$ for $i = 1, 2, \ldots, q$, where the subscripts are taken modulo $q$.

**Lemma 4.12.** *Suppose* $k = 4$ *and* $d \geq 2$*. Let* $\ell \in \{1, 2, \ldots, q\}$*. Then either* $\Omega_\ell \leq 4kd + 1$*, or* $\frac{\Omega_{\ell-1}+\Omega_\ell}{2} < 4kd + 1$ *and* $\delta_{\ell-1} \leq 5$*.*

*Proof.* Suppose

$$\Omega_\ell \geq 4kd + 2. \tag{4.5}$$

By Observation 2.1, we have that $\delta_i \leq 2kd + 5$ for every $i \in \{1, 2, \ldots, q\}$. If $\delta_\ell \leq 2kd-4$ or $\delta_{\ell+1} \leq 2kd-4$, then $\Omega_\ell = \delta_\ell + \delta_{\ell+1} \leq (2kd+5)+(2kd-4) = 4kd+1$, a contradiction with (4.5). Therefore,

$$\delta_\ell \geq 2kd - 3 \geq 13 \tag{4.6}$$

and

$$\delta_{\ell+1} \geq 2kd - 3 \geq 13. \tag{4.7}$$

It follows from (4.7) and Lemma 4.11 (a) that

$$\delta_\ell \leq 2kd + 2.$$

**Case 1.** $\varphi(\ell + 1) = 1$.

By (4.6) and Lemma 4.11 (b), we have $\delta_{\ell+1} \leq 2kd - 1$. It follows that $\Omega_\ell = \delta_\ell + \delta_{\ell+1} \leq (2kd + 2) + (2kd - 1) = 4kd + 1$, a contradiction with (4.5).

**Case 2.** $\varphi(\ell + 1) = 4$.

By (4.6) and Lemma 4.11 (c), we have $\delta_{\ell+1} \leq 2kd + 2$.

Suppose $\varphi(\ell) = 1$. By Lemma 4.11 (b), we have that either $\delta_{\ell-1} \leq 5$ or $\delta_\ell \leq 2kd - 1$. If $\delta_\ell \leq 2kd - 1$, then $\Omega_\ell = \delta_\ell + \delta_{\ell+1} \leq (2kd - 1) + (2kd + 2) = 4kd + 1$, a contradiction with (4.5). Hence $\delta_\ell > 2kd - 1$, i.e.,

$$\delta_{\ell-1} \leq 5.$$

It follows that

$$\begin{aligned}\frac{\Omega_{\ell-1} + \Omega_\ell}{2} &= \frac{(\delta_{\ell-1} + \delta_\ell) + (\delta_\ell + \delta_{\ell+1})}{2} \\ &\leq \frac{5 + (2kd + 2) + (2kd + 2) + (2kd + 2)}{2} < 4kd + 1.\end{aligned}$$

Suppose $\varphi(\ell) = 4$. If $\delta_\ell \leq 2kd-1$ or $\delta_{\ell+1} \leq 2kd-1$, then $\Omega_\ell = \delta_\ell + \delta_{\ell+1} \leq (2kd-1)+(2kd+2) = 4kd+1$, a contradiction with (4.5). Hence $\delta_\ell \geq 2kd$ and $\delta_{\ell+1} \geq 2kd$. By Lemma 4.11 (d), we have that

$$\delta_{\ell-1} \leq 2,$$

and thus

$$\begin{aligned}\frac{\Omega_{\ell-1}+\Omega_\ell}{2} &= \frac{(\delta_{\ell-1}+\delta_\ell)+(\delta_\ell+\delta_{\ell+1})}{2}\\ &\leq \frac{2+(2kd+2)+(2kd+2)+(2kd+2)}{2} < 4kd+1.\end{aligned}$$

This completes the proof of Lemma 4.12. ∎

**Lemma 4.13.** *For $k=4$, $n \geq 2k+1$ and $d \geq 2$, $\gamma_p^d(C(n;\{1,k\})) \geq 2\lceil \frac{2n}{4kd+1} \rceil$.*

*Proof.* Let $S_1 = \{1 \leq i \leq q : \Omega_i \leq 4kd+1\}$ and $S_2 = \{1 \leq i \leq q : \Omega_i \geq 4kd+2\}$. Then $S_1 \cup S_2 = \{1,2,\dots,q\}$. By Lemma 4.12, there exists an injection $\phi : S_2 \to S_1$ defined by $\phi(i) = i-1$, where $i \in S_2$. Then $\Omega_i + \Omega_{\phi(i)} < 2(4kd+1)$ for any $i \in S_2$. It follows that

$$\begin{aligned}2n &= \sum_{i=1}^{q} \Omega_i\\ &= \sum_{i\in S_1}\Omega_i + \sum_{i\in S_2}\Omega_i\\ &= \sum_{i\in S_1\setminus\phi(S_2)}\Omega_i + \sum_{i\in S_2}\Omega_i + \sum_{i\in\phi(S_2)}\Omega_i\\ &= \sum_{i\in S_1\setminus\phi(S_2)}\Omega_i + \sum_{i\in S_2}(\Omega_i+\Omega_{\phi(i)})\\ &\leq (|S_1|-|S_2|)\times(4kd+1) + |S_2|\times 2(4kd+1)\\ &= (|S_1|+|S_2|)\times(4kd+1)\\ &= q\times(4kd+1),\end{aligned}$$

which implies $q \geq \lceil \frac{2n}{4kd+1} \rceil$, and thus $\gamma_p^d(C(n;\{1,k\})) \geq 2\lceil \frac{2n}{4kd+1} \rceil$ for $k=4$, $n \geq 2k+1$ and $d \geq 2$. ∎

**Lemma 4.14.** *For $k=4$, $n \geq 2k+1$ and $d \geq 2$, suppose $\delta_i \geq 6$ for every $i \in \{1,2,\dots,q\}$. Let $s \in \{1,2,\dots,q\}$.*

(a) *If $(\varphi(s),\varphi(s+1)) = (1,1)$, then $\delta_s \leq 2kd-1$ and $\delta_s \neq 2kd-3$.*
(b) *If $(\varphi(s),\varphi(s+1)) = (1,4)$, then $\delta_s \leq 2kd-1$ and $\delta_s \notin \{2kd-3, 2kd-2\}$.*
(c) *If $(\varphi(s),\varphi(s+1)) = (4,1)$, then $\delta_s \leq 2kd+2$ and $\delta_s \notin \{2kd, 2kd+1\}$.*
(d) *If $(\varphi(s),\varphi(s+1)) = (4,4)$, then $\delta_s \leq 2kd-1$.*

*Proof.* (a) Suppose $(\varphi(s),\varphi(s+1)) = (1,1)$. If $\delta_s \geq 2kd$ or $\delta_s = 2kd-3$, then $v_{i_s+kd-1}$ would not be dominated by $D$, a contradiction. Hence $\delta_s \leq 2kd-1$ and $\delta_s \neq 2kd-3$.

(b) Suppose $(\varphi(s),\varphi(s+1)) = (1,4)$. If $\delta_s \geq 2kd$ or $\delta_s \in \{2kd-3, 2kd-2\}$, then $v_{i_s+kd-1}$ would not be dominated by $D$, a contradiction. Hence $\delta_s \leq 2kd-1$ and $\delta_s \notin \{2kd-3, 2kd-2\}$.

(c) Suppose $(\varphi(s), \varphi(s+1)) = (4, 1)$. If $\delta_s \geq 2kd+3$ or $\delta_s = 2kd$, then $v_{i_s+kd+2}$ would not be dominated by $D$, a contradiction. If $\delta_s = 2kd+1$, then $v_{i_s+kd+3}$ would not be dominated by $D$, a contradiction. Hence $\delta_s \leq 2kd+2$ and $\delta_s \notin \{2kd, 2kd+1\}$.

(d) Suppose $(\varphi(s), \varphi(s+1)) = (4, 4)$. If $\delta_s \geq 2kd$, then at least one of $\{v_{i_s+kd+2}, v_{i_s+kd+3}\}$ would not be dominated by $D$, a contradiction. Hence $\delta_s \leq 2kd-1$. ∎

From Lemma 4.14, we can easily derive the following result.

**Lemma 4.15.** *For $k = 4$, $n \geq 2k+1$ and $d \geq 2$, suppose $\delta_i \geq 6$ for every $i \in \{1, 2, \ldots, q\}$. Let $s \in \{1, 2, \ldots, q\}$.*

(a) *If $(\varphi(s), \varphi(s+1), \varphi(s+2)) \in \{(1, 1, 1), (1, 4, 4), (4, 4, 4)\}$, then $\Omega_s \leq 4kd-2$.*
(b) *If $(\varphi(s), \varphi(s+1), \varphi(s+2)) = (1, 1, 4)$, then $\Omega_s \leq 4kd-2$ and $\Omega_s \neq 4kd-4$.*
(c) *If $(\varphi(s), \varphi(s+1), \varphi(s+2)) \in \{(1, 4, 1), (4, 1, 4)\}$, then $\Omega_s \notin \{4kd, 4kd-1\}$.*
(d) *If $(\varphi(s), \varphi(s+1), \varphi(s+2)) = (4, 1, 1)$, then $\Omega_s \neq 4kd-1$.*

**Lemma 4.16.** *Suppose $k = 4$, $n \geq 2k+1$ and $d \geq 2$. Then $\gamma_p^d(C(n; \{1, k\})) \geq 2\lceil \frac{2n}{4kd+1} \rceil + 2$ for $n \equiv 2kd, 4kd-1, 4kd \pmod{4kd+1}$.*

*Proof.* Suppose to the contrary that $\gamma_p^d(C(n; \{1, k\})) < 2\lceil \frac{2n}{4kd+1} \rceil + 2$, i.e., there exists a $d$-distance paired dominating set $D = \{x_i, y_i : i = 1, 2, \ldots, q\}$ such that

$$q = \lceil \frac{2n}{4kd+1} \rceil. \tag{4.8}$$

Let $x \in \mathbb{Z}$ be such that

$$2n = \sum_{i=1}^{q} \Omega_i = q \times (4kd+1) - x. \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$\lceil \frac{2n}{4kd+1} \rceil = q = \frac{2n+x}{4kd+1}. \tag{4.10}$$

Since $2n \equiv 4kd, 4kd-1, 4kd-3 \pmod{4kd+1}$, by (4.10), we have

$$x = 1, 2, 4 \tag{4.11}$$

for $n \equiv 2kd, 4kd, 4kd-1 \pmod{4kd+1}$, respectively.

Let $S_1 = \{1 \leq i \leq q : \Omega_i \leq 4kd+1\}$ and $S_2 = \{1 \leq i \leq q : \Omega_i \geq 4kd+2\}$. Then $S_1 \cup S_2 = \{1, 2, \ldots, q\}$. By Lemma 4.12, there exists an injection $\phi : S_2 \to S_1$ defined by $\phi(i) = i-1$, where $i \in S_2$. Then $\Omega_i + \Omega_{\phi(i)} < 2(4kd+1)$ for any $i \in S_2$.

If there exists $\ell \in \{1, 2, \ldots, q\}$ such that $\Omega_\ell \geq 4kd+2$, by Lemma 4.12, we have $\delta_{\ell-1} \leq 5$. It follows from Observation 2.1 that $\Omega_{\ell-1} = \delta_{\ell-1} + \delta_\ell \leq 5 + (2kd+5) \leq (4kd+1) - 7$ and $\Omega_{\ell-2} = \delta_{\ell-2} + \delta_{\ell-1} \leq (2kd+5) + 5 \leq (4kd+1) - 7$, which implies $\ell - 2 \in S_1 \setminus \phi(S_2)$. It follows that

$$\begin{aligned}
\sum_{i=1}^{q} \Omega_i &= \sum_{i \in S_1} \Omega_i + \sum_{i \in S_2} \Omega_i \\
&= \sum_{i \in S_1 \setminus (\phi(S_2) \cup \{\ell-2\})} \Omega_i + \Omega_{\ell-2} + \sum_{i \in \phi(S_2)} \Omega_i + \sum_{i \in S_2} \Omega_i \\
&= \sum_{i \in S_1 \setminus (\phi(S_2) \cup \{\ell-2\})} \Omega_i + \Omega_{\ell-2} + \sum_{i \in S_2} (\Omega_i + \Omega_{\phi(i)})
\end{aligned}$$

$$\le (|S_1| - |S_2| - 1) \times (4kd + 1) + ((4kd + 1) - 7) + |S_2| \times 2(4kd + 1)$$
$$= (|S_1| + |S_2|) \times (4kd + 1) - 7 = q \times (4kd + 1) - 7.$$

By (4.9), we have $x \ge 7$, which is a contradiction with (4.11). Hence

$$\Omega_i \le 4kd + 1 \tag{4.12}$$

for every $i \in \{1, 2, \ldots, q\}$ when $n \equiv 2kd, 4kd, 4kd - 1 \pmod{4kd + 1}$.

For $n = 2kd$, i.e., $q = 1$, we may assume $(x_1, y_1) \in \{(v_0, v_1), (v_0, v_4)\}$. Then $v_{kd+2}$ would not be dominated by $D$, a contradiction.

For $n = 4kd - 1, 4kd$, i.e., $q = 2$, by Observation 2.1, we have $\delta_j \le 2kd + 5$ for $j = 1, 2$. It follows that $\delta_j \ge (4kd - 1) - (2kd + 5) = 2kd - 6 > 6$ for $j = 1, 2$. If $(\varphi(1), \varphi(2)) \in \{(1, 1), (4, 4)\}$, by Lemma 4.14 (a) and (d), we have $n = \delta_1 + \delta_2 \le (2kd - 1) + (2kd - 1) = 4kd - 2$, a contradiction. If $(\varphi(1), \varphi(2)) \in \{(1, 4), (4, 1)\}$, by Lemma 4.14 (b) and (c), we have $n = \delta_1 + \delta_2 \ne 4kd, 4kd - 1$, a contradiction. Therefore, it remains to consider the case for $n \notin \{2kd, 4kd - 1, 4kd\}$, i.e., $q \ge 3$.

**Case 1.** $n \equiv 2kd, 4kd \pmod{4kd + 1}$.

Then $x = 1, 2$. It follows from (4.9) and (4.12) that $4kd - 1 \le \Omega_i \le 4kd + 1$ for every $i \in \{1, 2, \ldots, q\}$, and there exists $\ell \in \{1, 2, \ldots, q\}$ such that $\Omega_\ell < 4kd + 1$. By Observation 2.1, we have that $\delta_i = \Omega_i - \delta_{i+1} \ge (4kd - 1) - (2kd + 5) = 2kd - 6 > 6$ for every $i \in \{1, 2, \ldots, q\}$. By Lemma 4.15 (a) and (b), we conclude that for any $i \in \{1, 2, \ldots, q\}$, $\varphi(i) \ne \varphi(i + 1)$. Since $q \ge 3$, by Lemma 4.15 (c), we derive a contradiction.

**Case 2.** $n \equiv 4kd - 1 \pmod{4kd + 1}$.

Then $x = 4$. It follows from (4.9) and (4.12) that $4kd - 3 \le \Omega_i \le 4kd + 1$ for every $i \in \{1, 2, \ldots, q\}$, and there exists $\ell \in \{1, 2, \ldots, q\}$ such that $\Omega_\ell < 4kd + 1$.

By Observation 2.1, we have that $\delta_i = \Omega_i - \delta_{i+1} \ge (4kd - 1) - (2kd + 5) = 2kd - 6 > 6$ for every $i \in \{1, 2, \ldots, q\}$. If $\Omega_i \ge 4kd - 1$ for every $i \in \{1, 2, \ldots, q\}$, by Lemma 4.15 (a) and (b), we conclude that for any $i \in \{1, 2, \ldots, q\}$, $\varphi(i) \ne \varphi(i + 1)$. Since $q \ge 3$, by Lemma 4.15 (c), we have that $\Omega_i = 4kd + 1$ for every $i \in \{1, 2, \ldots, q\}$, which is a contradiction. Hence, there exists $s \in \{1, 2, \ldots, q\}$ such that $\Omega_s \in \{4kd - 2, 4kd - 3\}$.

**Case 2.1** Suppose $\Omega_s = 4kd - 3$.

By (4.9) and (4.12), we have that $\Omega_s = 4kd + 1$ for every $i \in \{1, 2, \ldots, q\} \setminus \{s\}$. It follows that either $\delta_s \le 2kd - 2$ or $\delta_{s+1} \le 2kd - 2$. If $\delta_s \le 2kd - 2$, by Lemma 4.14, then $\Omega_{s-1} = \delta_{s-1} + \delta_s \le (2kd + 2) + (2kd - 2) = 4kd$, a contradiction. If $\delta_{s+1} \le 2kd - 2$, by Lemma 4.14, then $\Omega_{s+1} = \delta_{s+1} + \delta_{s+2} \le (2kd - 2) + (2kd + 2) = 4kd$, a contradiction.

**Case 2.2** Suppose $\Omega_s = 4kd - 2$.

By (4.9) and (4.12), there exists $t \in \{1, 2, \ldots, q\} \setminus \{s\}$ such that $\Omega_t = 4kd$ and $\Omega_i = 4kd + 1$ for every $i \in \{1, 2, \ldots, q\} \setminus \{s, t\}$. By Lemma 4.15, we conclude that $(\varphi(t), \varphi(t + 1), \varphi(t + 2)) \in \{(4, 1, 1), (4, 4, 1)\}$.

Suppose $(\varphi(t), \varphi(t + 1), \varphi(t + 2)) = (4, 1, 1)$. By Lemma 4.14 (a) and (c), we have that $\delta_t = 2kd + 2$ and $\delta_{t+1} = 2kd - 2$. By Lemma 4.14 (a) and (b), we have that $\Omega_{t+1} = \delta_{t+1} + \delta_{t+2} \le (2kd - 2) + (2kd - 1) = 4kd - 3$, a contradiction.

Suppose $(\varphi(t), \varphi(t+1), \varphi(t+2)) = (4,4,1)$. By Lemma 4.14 (a) and (c), we have that $\delta_{t+1} = 2kd+2$ and $\delta_t = 2kd-2$. By Lemma 4.14 (b) and (d), we have that $\Omega_{t-1} = \delta_{t-1} + \delta_t \leq (2kd-1) + (2kd-2) = 4kd-3$, a contradiction. ∎

From Lemmas 4.10, 4.13 and 4.16, we have the following

**Theorem 4.2.** *For $k=4$, $n \geq 2k+1$ and $d \geq 2$,*

$$\gamma_p^d(C(n;\{1,k\})) = \begin{cases} 2\lceil \frac{2n}{4kd+1} \rceil + 2, & \text{if } n \equiv 2kd, 4kd-1, 4kd \pmod{4kd+1} \\ 2\lceil \frac{2n}{4kd+1} \rceil, & \text{otherwise.} \end{cases}$$

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