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Linear Preservers of Regular Matrices over General Boolean Algebras

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Abstract. An $n \times n$ matrix A over a general Boolean algebra $\mathbb{B}_k$ is called regular if there exists an $n \times n$ matrix G over $\mathbb{B}_k$ such that $AGA = A$. We study some properties of regular matrices over general Boolean algebras. We also determine the linear operators that strongly preserve regular matrices over general Boolean algebras.

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1. Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. Although the linear operators concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings [1, 2, 4, 10].

In this paper, we study the problem of characterizing those operators T on the matrices over general Boolean algebra such that $T(X)$ is regular if and only if X is regular.

For a fixed positive integer k, let $\mathbb{B}_k$ be the *Boolean algebra* of subsets of a k-element set $\mathbb{S}_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of $\mathbb{S}_k$. Union is denoted by $+$ and intersection by juxtaposition; 0 denotes the null set and 1 the set $\mathbb{S}_k$. Under these two operations, $\mathbb{B}_k$ is a commutative, antinegative semiring (that is, only zero

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element has an additive inverse); all of its elements, except 0 and 1, are zero-divisors. In particular, if $k = 1$, $\mathbb{B}_1$ is called the *binary Boolean algebra*.

Let $\mathcal{M}_n(\mathbb{B}_k)$ denote the set of all $n \times n$ matrices with entries in $\mathbb{B}_k$. The usual definitions for addition and multiplication of matrices over fields are applied to matrices over $\mathbb{B}_k$ as well.

A matrix X in $\mathcal{M}_n(\mathbb{B}_k)$ is said to be *invertible* if there is a matrix Y in $\mathcal{M}_n(\mathbb{B}_k)$ such that $XY = YX = I_n$, where I_n is the $n \times n$ identity matrix.

In 1952, Luce [5] showed a matrix A in $\mathcal{M}_n(\mathbb{B}_1)$ possesses a two-sided inverse if and only if A is an orthogonal matrix in the sense that $AA^T = I_n$, and that, in this case, A^T is a two-sided inverse of A. In 1963, Rutherford [9] showed if a matrix A in $\mathcal{M}_n(\mathbb{B}_1)$ possesses a one-sided inverse, then the inverse is also a two-sided inverse. Furthermore such an inverse, if it exists, is unique and is A^T. Also, it is well known that the $n \times n$ permutation matrices are the only $n \times n$ invertible matrices over the binary Boolean algebra.

For any matrix $A = [a_{i,j}]$ in $\mathcal{M}_n(\mathbb{B}_k)$, the p^{th} *constituent*, A_p, of A is the matrix in $\mathcal{M}_n(\mathbb{B}_1)$ whose $(i,j)^{th}$ entry is 1 if and only if $a_{i,j} \supseteq \sigma_p$. Via the constituents, A can be written uniquely as $A = \sum_{p=1}^{k} \sigma_p A_p$ which is called the *canonical form* of A. It follows from the uniqueness of the decomposition and the fact that the singletons are mutually orthogonal idempotents that for all matrices $A, B, C \in \mathcal{M}_n(\mathbb{B}_k)$ and for all $\alpha \in \mathbb{B}_k$,

$$(A+B)_p = A_p + B_p, \quad (BC)_p = B_p C_p \quad \text{and} \quad (\alpha A)_p = \alpha_p A_p \tag{1.1}$$

for all $p = 1, \ldots, k$.

Lemma 1.1. [4] *For any matrix A in $\mathcal{M}_n(\mathbb{B}_k)$ with $k \geq 1$, A is invertible if and only if its all constituents are permutation matrices. In particular, if A is invertible, then $A^{-1} = A^T$.*

The notion of generalized inverse of an arbitrary matrix apparently originated in the work of Moore [6], and the generalized inverses have applications in network and switching theory and information theory [2].

Let A be a matrix in $\mathcal{M}_n(\mathbb{B}_k)$. Consider a matrix $X \in \mathcal{M}_n(\mathbb{B}_k)$ in the equation

$$AXA = A. \tag{1.2}$$

If (1.2) has a solution $X \in \mathcal{M}_n(\mathbb{B}_k)$, then X is called a *generalized inverse* of A. Furthermore A is called *regular* if there is a solution of (1.2).

The equation (1.2) have been studied by several authors [3, 6, 7, 8]. Rao and Rao [8] characterized all regular matrices in $\mathcal{M}_n(\mathbb{B}_1)$. Also Plemmons [7] published algorithms for computing generalized inverses of regular matrices in $\mathcal{M}_n(\mathbb{B}_1)$ under certain conditions.

In this paper, we study some properties of regular matrices over general Boolean algebras $\mathbb{B}_k$. We also determine the linear operators on $\mathcal{M}_n(\mathbb{B}_k)$ that strongly preserve regular matrices.

2. Preliminaries and some results

The $n \times n$ identity matrix, I_n, and the $n \times n$ zero matrix, O_n, are defined as if $\mathbb{B}_k$ were a field. We denote the $n \times n$ matrix all of whose entries are 1 by J_n. We will suppress the subscripts on these matrices when the orders are evident from the context. For any matrix A in $\mathcal{M}_n(\mathbb{B}_k)$, A^T denotes the transpose of A. The $n \times n$ matrix all of whose entries are zero except its $(i,j)^{th}$, which is 1, is denoted by $E_{i,j}$. We call $E_{i,j}$ a *cell*.

Matrices J and O in $\mathcal{M}_n(\mathbb{B}_k)$ are regular because $JGJ = J$ and $OGO = O$ for all cells G in $\mathcal{M}_n(\mathbb{B}_k)$. Therefore in general, a solution of (2.1), although it exists, is not necessarily unique. Furthermore each cell E in $\mathcal{M}_n(\mathbb{B}_k)$ is regular because $EE^TE = E$.

Proposition 2.1. *Let A be a matrix in $\mathcal{M}_n(\mathbb{B}_k)$. If U and V are invertible matrices in $\mathcal{M}_n(\mathbb{B}_k)$, then the following are equivalent:*

(i) *A is regular;*
(ii) *UAV is regular;*
(iii) *A^T is regular.*

Proof. The proof is an easy exercise. ∎

Also we can easily show that

$$(2.1) \qquad A \quad \text{is regular if and only if} \quad \begin{bmatrix} A & O \\ O & B \end{bmatrix} \quad \text{is regular,}$$

for all matrices $A \in \mathcal{M}_n(\mathbb{B}_k)$ and for all regular matrices $B \in \mathcal{M}_m(\mathbb{B}_k)$. In particular, all idempotent matrices in $\mathcal{M}_n(\mathbb{B}_k)$ are regular.

For any zero-one matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathcal{M}_n(\mathbb{B}_k)$, we define $A \setminus B$ to be the zero-one matrix $C = [c_{i,j}]$ such that $c_{i,j} = 1$ if and only if $a_{i,j} = 1$ and $b_{i,j} = 0$ for all i and j.

Define an upper triangular matrix Λ_n in $\mathcal{M}_n(\mathbb{B}_k)$ by

$$\Lambda_n = [\lambda_{i,j}] \equiv \left(\sum_{i \le j}^{n} E_{i,j} \right) \setminus E_{1,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

Then the following lemma shows that Λ_n is not regular for $n \ge 3$.

Lemma 2.1. *Λ_n is regular in $\mathcal{M}_n(\mathbb{B}_k)$ if and only if $n \le 2$.*

Proof. For $n \le 2$, clearly Λ_n is regular because $\Lambda_n I_n \Lambda_n = \Lambda_n$.

Conversely, assume that Λ_n is regular for some $n \ge 3$. Then there is a nonzero matrix $B = [b_{i,j}]$ in $\mathcal{M}_n(\mathbb{B}_k)$ such that $\Lambda_n = \Lambda_n B \Lambda_n$. From $0 = \lambda_{1,n} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} b_{i,j}$, we obtain all entries of the second column of B are zero except for the entry $b_{n,2}$. From $0 = \lambda_{2,1} = \sum_{i=2}^{n} b_{i,1}$, we have all entries of the first column of B are zero except for $b_{1,1}$. Also, from $0 = \lambda_{3,2} = \sum_{i=3}^{n} \sum_{j=1}^{2} b_{i,j}$, we obtain $b_{n,2} = 0$. If we combine these three results, we conclude all entries of the first two columns are zero except for $b_{1,1}$. But we have $1 = \lambda_{2,2} = \sum_{i=2}^{n} \sum_{j=1}^{2} b_{i,j} = 0$, a contradiction. Hence Λ_n is not regular for all $n \ge 3$. ∎

In particular, $\Lambda_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not regular in $\mathcal{M}_3(\mathbb{B}_k)$. Let

$$\Phi_n = \begin{bmatrix} \Lambda_3 & O \\ O & O \end{bmatrix} \tag{2.2}$$

for all $n \geq 3$. Then Φ_n is not regular in $\mathcal{M}_n(\mathbb{B}_k)$ by (2.1).

Note that for a matrix $A = [a_{i,j}]$ in $\mathcal{M}_n(\mathbb{B}_k)$, the p^{th} constituent, A_p, of A is the matrix in $\mathcal{M}_n(\mathbb{B}_1)$ whose $(i,j)^{th}$ entry is 1 if and only if $a_{i,j} \supseteq \sigma_p$.

Example 2.1. Let $k \geq 2$. Consider the matrix

$$A = \begin{bmatrix} 1 & \sigma_1 & 0 \\ 0 & \sigma_1 & \sigma_1 \\ 0 & 0 & \sigma_1 \end{bmatrix} \in \mathcal{M}_3(\mathbb{B}_k).$$

Then we have $A_1 = \Lambda_3$ is not regular in $\mathcal{M}_3(\mathbb{B}_1)$, while $A_p = E_{1,1}$ is regular in $\mathcal{M}_3(\mathbb{B}_1)$ for all $p = 2, 3, \ldots, k$. The theorem below shows that A is not regular in $\mathcal{M}_3(\mathbb{B}_k)$.

Theorem 2.1. *Let A be a matrix in $\mathcal{M}_n(\mathbb{B}_k)$. Then A is regular in $\mathcal{M}_n(\mathbb{B}_k)$ if and only if its all constituents are regular in $\mathcal{M}_n(\mathbb{B}_1)$.*

Proof. If A is regular in $\mathcal{M}_n(\mathbb{B}_k)$, then all constituents of A are regular in $\mathcal{M}_n(\mathbb{B}_1)$ by (1.1).

Conversely, assume that each constituent A_p of A is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$. Then there are matrices $G_1, \ldots, G_k$ in $\mathcal{M}_n(\mathbb{B}_1)$ such that $A_p G_p A_p = A_p$ for all $p = 1, \ldots, k$. If $G = \sum_{p=1}^{k} \sigma_p G_p$, then we can easily show that $AGA = A$ and hence A is regular in $\mathcal{M}_n(\mathbb{B}_k)$. ∎

Theorem 2.1 shows that the regularity of a matrix A in $\mathcal{M}_n(\mathbb{B}_k)$ depends only on the regularities of its all constituents in $\mathcal{M}_n(\mathbb{B}_1)$. Henceforth we suffice to consider properties of regular matrices in $\mathcal{M}_n(\mathbb{B}_1)$.

The *factor rank* [1], $b(A)$, of a nonzero matrix $A \in \mathcal{M}_n(\mathbb{B}_k)$ is defined as the least integer r for which there are matrices B and C of orders $n \times r$ and $r \times n$, respectively such that $A = BC$. The rank of a zero matrix is zero. Also we can easily obtain that

$$0 \leq b(A) \leq n \quad \text{and} \quad b(AB) \leq \min\{b(A), b(B)\} \tag{2.3}$$

for all $A, B \in \mathcal{M}_n(\mathbb{B}_k)$.

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ be a matrix in $\mathcal{M}_n(\mathbb{B}_k)$, where $\mathbf{a}_j$ denotes the j^{th} column of A for all $j = 1, \ldots, n$. Then the *column space* of A is the set $\{\sum_{j=1}^{n} \alpha_j \mathbf{a}_j \mid \alpha_j \in \mathbb{B}_k\}$, and denoted by $< A >$; the *row space* of A is $< A^T >$.

For a matrix $A \in \mathcal{M}_n(\mathbb{B}_k)$ with $b(A) = r$, A is said to be *space decomposable* if there are matrices B and C of orders $n \times r$ and $r \times n$, respectively such that $A = BC$, $< A >=< B >$ and $< A^T >=< C^T >$.

Theorem 2.2. [8] *A is regular in $\mathcal{M}_n(\mathbb{B}_1)$ if and only if A is space decomposable.*

Let A be a matrix in $\mathcal{M}_n(\mathbb{B}_k)$. By Theorems 2.1 and 2.2, we have A is regular in $\mathcal{M}_n(\mathbb{B}_k)$ if and only if its all constituents are space decomposable in $\mathcal{M}_n(\mathbb{B}_1)$.

Lemma 2.2. *If A is a matrix in $\mathcal{M}_n(\mathbb{B}_1)$ with $b(A) \leq 2$, then A is regular.*

Proof. If $b(A) = 0$, then $A = O$ is clearly regular. If $b(A) = 1$, then there exist permutation matrices P and Q such that $PAQ = \begin{bmatrix} J & O \\ O & O \end{bmatrix}$, and hence PAQ is regular by (2.1). It follows from Proposition 2.1 that A is regular.

Suppose $b(A) = 2$. Then there are matrices $B = [\mathbf{b}_1 \quad \mathbf{b}_2]$ and $C = [\mathbf{c}_1 \quad \mathbf{c}_2]^T$ of orders $n \times 2$ and $2 \times n$, respectively such that $A = BC$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are distinct nonzero columns of B, and $\mathbf{c}_1$ and $\mathbf{c}_2$ are distinct nonzero columns of C^T. Then we can easily show that all columns of A are of the forms $\mathbf{0}, \mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_1 + \mathbf{b}_2$ so that $< A > = < B >$. Similarly, all columns of A^T are of the forms $\mathbf{0}, \mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_1 + \mathbf{c}_2$ so that $< A^T > = < C^T >$. Therefore A is space decomposable and hence A is regular by Theorem 2.2. ∎

For matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathcal{M}_n(\mathbb{B}_k)$, we say B *dominates* A (written $B \geq A$ or $A \leq B$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathcal{M}_n(\mathbb{B}_k)$.

The number of nonzero entries of a matrix A in $\mathcal{M}_n(\mathbb{B}_k)$ is denoted by $|A|$. The number of elements in a set $\mathbb{S}$ is also denoted by $|\mathbb{S}|$.

Corollary 2.1. *Let A be a nonzero matrix in $\mathcal{M}_n(\mathbb{B}_1)$, where $n \geq 3$.*

(i) *If $|A| \leq 4$, then A is regular;*
(ii) *If $|A| \leq 2$, there is a matrix B such that $|A + B| = 5$ and $A + B$ is not regular;*
(iii) *If $|A| = 3$ and $b(A) = 2$ or 3, there is a matrix C with $|C| = 2$ such that $A + C$ is not regular;*
(iv) *If $|A| = 5$ and A has a row or a column that has at least 3 nonzero entries, then A is regular.*

Proof. (i) By Lemma 2.2, we lose no generality in assuming that $b(A) \geq 3$ so that $b(A) = 3$ or 4. Consider the matrix $X = \begin{bmatrix} A & O \\ O & 0 \end{bmatrix}$ in $\mathcal{M}_{n+1}(\mathbb{B}_1)$. Since $|A| \leq 4$ and $b(A) = 3$ or 4, we can easily show that there are permutation matrices P and Q of orders $n + 1$ such that $PXQ = \begin{bmatrix} Y & O \\ O & O \end{bmatrix}$ for some idempotent matrix Y in $\mathcal{M}_4(\mathbb{B}_1)$ with $|Y| = 3$ or 4. By (2.1) and Proposition 2.1, X is regular and hence A is regular by (2.1).

(ii) If $|A| \leq 2$, we can easily show that there are permutation matrices P and Q such that $PAQ \leq \Phi_n$. Let $B' = \Phi_n \setminus PAQ$. Then we have $PAQ + B' = \Phi_n$ so that $A + P^T B' Q^T = P^T \Phi_n Q^T$ is not regular by Proposition 2.1. If we let $B = P^T B' Q^T$, then we have $|A + B| = 5$ and $A + B$ is not regular.

(iii) Similar to (ii).

(iv) If $|A| = 5$ and A has a row or a column that has at least 3 nonzero entries, then we can easily show that $b(A) \leq 3$. By Lemma 2.2, it suffices to consider $b(A) = 3$. Then A has either a row or a column that has just 3 nonzero entries. Suppose that a row of A has just 3 nonzero entries. Since $b(A) = 3$, there are permutation matrices P and Q such that

$$PAQ = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,i} + E_{3,j}$$

for some $i, j \in \{1, \ldots, n\}$ with $i < j$. If $j \geq 4$, then PAQ is regular by the above result (i) and (2.1), and hence A is regular by Proposition 2.1. If $1 \leq i < j \leq 3$, then there are permutation matrices P' and Q' such that $P'PAQQ' = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$, where $D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can easily show that D is idempotent in $\mathcal{M}_3(\mathbb{B}_1)$, and hence D is regular. It follows from (2.1) and Proposition 2.1 that A is regular.

If a column of A has just 3 nonzero entries, a parallel argument shows that A is regular. ∎

Linearity of operators on $\mathcal{M}_n(\mathbb{B}_k)$ is defined as for vector spaces over fields. A linear operator on $\mathcal{M}_n(\mathbb{B}_k)$ is completely determined by its behavior on the set of cells in $\mathcal{M}_n(\mathbb{B}_k)$.

An operator T on $\mathcal{M}_n(\mathbb{B}_k)$ is said to be

(1) *singular* if $T(X) = O$ for some nonzero matrix $X \in \mathcal{M}_n(\mathbb{B}_k)$; otherwise T is *nonsingular*;
(2) *preserve regularity* if $T(A)$ is regular whenever A is regular in $\mathcal{M}_n(\mathbb{B}_k)$;
(3) *strongly preserve regularity* if $T(A)$ is regular if and only if A is regular in $\mathcal{M}_n(\mathbb{B}_k)$.

Example 2.2. Let A be any regular matrix in $\mathcal{M}_n(\mathbb{B}_k)$, where at least one entry of A is 1. Define an operator T on $\mathcal{M}_n(\mathbb{B}_k)$ by

$$T(X) = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} \right) A$$

for all $X = [x_{i,j}] \in \mathcal{M}_n(\mathbb{B}_k)$. Then we can easily show that T is nonsingular and T is a linear operator that preserves regularity. But T does not preserve any matrix that is not regular in $\mathcal{M}_n(\mathbb{B}_k)$.

Thus, we are interested in linear operators on $\mathcal{M}_n(\mathbb{B}_k)$ that strongly preserve regularity.

Lemma 2.3. *Let $n \geq 3$. If T is a linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ that strongly preserves regularity, then T is nonsingular.*

Proof. If $T(X) = O$ for some nonzero matrix X in $\mathcal{M}_n(\mathbb{B}_1)$, then we have $T(E) = O$ for all cells $E \leq X$. By Corollary 2.1 (ii), there is a matrix B such that $|B| = 4$ and $E + B$ is not regular, while B is regular by Corollary 2.7(i). Nevertheless, $T(E + B) = T(B)$, a contradiction to the fact that T strongly preserves regularity. Hence $T(X) \neq O$ for all nonzero matrix X in $\mathcal{M}_n(\mathbb{B}_1)$. Therefore T is nonsingular. ∎

If $n \leq 2$, then all matrices in $\mathcal{M}_n(\mathbb{B}_1)$ are regular by (2.3) and Lemma 2.2. Therefore all matrices in $\mathcal{M}_n(\mathbb{B}_k)$ are also regular by Theorem 2.1. This proves:

Theorem 2.3. *If $n \leq 2$, then all operators on $\mathcal{M}_n(\mathbb{B}_k)$ strongly preserve regularity.*

3. The binary Boolean case

In this section we have characterizations of the linear operators that strongly preserve regular matrices over the binary Boolean algebra $\mathbb{B}_1$.

As shown in Theorem 2.3, each operator T on $\mathcal{M}_n(\mathbb{B}_1)$ strongly preserve regularity if $n \leq 2$. Thus in the followings, unless otherwise stated, we assume that T is a linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ that strongly preserve regularity for $n \geq 3$.

The next lemmas and propositions are necessary to prove the main theorem.

Lemma 3.1. *Let A be a matrix in $\mathcal{M}_n(\mathbb{B}_1)$ with $|A| = k$ and $b(A) = k$. Then $J \setminus A$ is regular if and only if $k \leq 2$.*

Proof. If $k \leq 2$, then there are permutation matrices P and Q such that $P(J \setminus A)Q = J \setminus (aE_{1,1} + bE_{2,2})$, where $a, b \in \{0, 1\}$, and hence

$$P(J \setminus A)Q = \begin{bmatrix} a' & 1 \\ 1 & b' \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

so that $b(J \setminus A) = b(P(J \setminus A)Q) \leq 2$, where $a + a' = b + b' = 1$ with $a \neq a'$ and $b \neq b'$. Thus we have $J \setminus A$ is regular by Lemma 2.2. ∎

Conversely, assume that $J \setminus A$ is regular for some $k \geq 3$. It follows from $|A| = k$ and $b(A) = k$ that there are permutation matrices U and V such that

$$U(J \setminus A)V = J \setminus \sum_{t=1}^{k} E_{t,t}.$$

Let $J \setminus \left(\sum_{t=1}^{k} E_{t,t} \right) = X = [x_{i,j}]$. By Proposition 2.1, X is regular, and hence there is a nonzero matrix $B = [b_{i,j}] \in \mathcal{M}_n(\mathbb{B}_1)$ such that $X = XBX$. Then the $(t,t)^{th}$ entry of XBX becomes

$$\sum_{i \in I} \sum_{j \in J} b_{i,j} \tag{3.1}$$

for all $t = 1, \ldots, k$, where $I = J = \{1, \ldots, n\} \setminus \{t\}$. From $x_{1,1} = 0$ and (3.1), we have

$$b_{i,j} = 0 \quad \text{for all } \; i, j \in \{2, \ldots, n\}. \tag{3.2}$$

Consider the first row and the first column of B. It follows from $x_{2,2} = 0$ and (3.1) that

$$b_{i,1} = 0 = b_{1,j} \quad \text{for all } \; i, j \in \{1, 3, 4, \ldots, n\}. \tag{3.3}$$

Also, from $x_{3,3} = 0$, we obtain $b_{1,2} = b_{2,1} = 0$, and hence $B = O$ by (3.2) and (3.3). This contradiction shows that $k \leq 2$.

Proposition 3.1. *Let A and B be matrices in $\mathcal{M}_n(\mathbb{B}_1)$ such that $A \leq B$ and $|A| < |B|$. If $|B| \leq (n-2)n$, then we have $|T(A)| < |T(B)|$.*

Proof. Suppose that $|T(A)| = |T(B)|$ for some $A, B \in \mathcal{M}_n(\mathbb{B}_1)$ with $A \leq B$, $|A| < |B|$ and $|B| \leq (n-2)n$. Then $T(A) = T(B)$ and there is a cell E such that $E \leq B$ and $E \not\leq A$. Since $|A| < (n-2)n$, there must be two distinct cells F and G different from E such that $F \not\leq A$, $G \not\leq A$ and $b(E+F+G) = 3$. Let $C = J \setminus (E+F+G)$. Then

$$A + C = J \setminus (E+F+G) \quad \text{and} \quad B + C = J \setminus (F+G).$$

It follows from $T(A) = T(B)$ that $T(J \setminus (E+F+G)) = T(J \setminus (F+G))$, a contradiction to the fact that T strongly preserves regularity because $J \setminus (F+G)$ is regular, while $J \setminus (E+F+G)$ is not regular by Lemma 3.1. Hence the result follows. ∎

Let A be a matrix in $\mathcal{M}_3(\mathbb{B}_1)$. If $|A| \leq 4$, then A is regular by Corollary 2.1 (i). And if $|A| \geq 7$, then $b(A) \leq 2$ and so A is regular by Lemma 2.2. Hence, if $A \in \mathcal{M}_3(\mathbb{B}_1)$ is not regular, then $|A| = 5$ or 6 and there are permutation matrices P and Q such that PAQ is of the form of following:

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Furthermore, if E is a cell with $E \leq C$, then there are permutation matrices P' and Q' such that $P'(C \setminus E)Q' = B$ and hence $C \setminus E$ is not regular.

Lemma 3.2. *For every cell E in $\mathcal{M}_3(\mathbb{B}_1)$, $T(E)$ is a cell.*

Proof. Suppose that $|T(E_1)| \geq 2$ for some cell $E_1 \in \mathcal{M}_3(\mathbb{B}_1)$. Let $A \in \mathcal{M}_3(\mathbb{B}_1)$ be a matrix that is not regular with $E_1 \leq A$ and $|A| = 5$. Then $T(A)$ is not regular and so $|T(A)| \in \{5, 6\}$. Let $B \in \mathcal{M}_3(\mathbb{B}_1)$ be a matrix with $B \leq A$ and $|B| = 4$. If $|T(B)| \geq 5$, then $T(B)$ is not regular, while B is regular by Corollary 2.1 (i), a contradiction. Hence there is not a matrix B with $B \leq A$ and $|B| = 4$ such that $|T(B)| \geq 5$.

Write $A = \sum_{i=1}^{5} E_i$ for distinct cells $E_1, \ldots, E_5$. It follows from Proposition 3.1 that

$$|T(E_1)| < |T(E_1 + E_2)| < |T(E_1 + E_2 + E_3)|$$

and hence $4 \leq |T(E_1 + E_2 + E_3)| \leq |T(A)|$ because $|T(E_1)| \geq 2$. Thus we have $|T(E_1 + E_2 + E_3)| = 4$. Since $T(\sum_{i=1}^{3} E_i) \leq T(\sum_{i=1}^{4} E_i)$ and $|T(\sum_{i=1}^{4} E_i)| \geq 5$ is impossible, we have

$$T\left(\sum_{i=1}^{3} E_i\right) = T\left(\sum_{i=1}^{4} E_i\right),$$

and hence $T(E_1 + E_2 + E_3 + E_5) = T(A)$, a contradiction because A is not regular, while $E_1 + E_2 + E_3 + E_5$ is regular by Corollary 2.1 (i). Thus we have $|T(E)| \leq 1$ and hence $|T(E)| = 1$ for every cell E by Lemma 2.3. Consequently, $T(E)$ is a cell for every cell E. ∎

For any $k \in \{1, 2, \ldots, n^2\}$, let S_k denote a sum of arbitrary distinct cells in $\mathcal{M}_n(\mathbb{B}_1)$ with $|S_k| = k$.

Proposition 3.2.

(i) *If $n = 2t$ and $t \geq 2$, then $|T(S_{tn-1})| \leq n^2 - 3$ for all $S_{tn-1} \in \mathcal{M}_n(\mathbb{B}_1)$,*

(ii) *If $n = 2t + 1$ and $t \geq 2$, then*

$$|T(S_{(t+1)n-(t+1)})| \leq n^2 - 2$$

for all $S_{(t+1)n-(t+1)} \in \mathcal{M}_n(\mathbb{B}_1)$.

Proof. (i) Let $n = 2t$ with $t \geq 2$. Suppose that $|T(S_{tn-1})| \geq n^2 - 2$ for some $S_{tn-1} \in \mathcal{M}_n(\mathbb{B}_1)$. Since $|S_{tn-1}| = tn - 1$, there must be three distinct cells E_1, E_2 and E_3 such that they are not dominated by S_{tn-1} and $b(E_1 + E_2 + E_3) = 3$. Hence there is a matrix $A \in \mathcal{M}_n(\mathbb{B}_1)$ such that $S_{tn-1} + A = J \setminus (E_1 + E_2 + E_3)$. It follows from $|T(S_{tn-1})| \geq n^2 - 2$ that $|T(J \setminus (E_1 + E_2 + E_3))| \geq n^2 - 2$ and hence $B = T(J \setminus (E_1 + E_2 + E_3))$ is regular by Lemma 2.2 because $b(B) \leq 2$. But $J \setminus (E_1 + E_2 + E_3)$ is not regular by Lemma 3.1, a contradiction. Hence the result follows.

(ii) Similar to (i). ∎

The next lemma will be important in order to show that if E is any cell in $\mathcal{M}_n(\mathbb{B}_1)$ with $n \geq 4$, then $T(E)$ is also a cell for any linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ that strongly preserves regularity.

Lemma 3.3.

(i) *Let $n = 2t$, $t \geq 2$ and $h \in \{0, 1, 2, \ldots, tn - 2\}$. Then*

$$|T(S_{tn-1-h})| \leq n^2 - 3 - 2h$$

for all $S_{tn-1-h} \in \mathcal{M}_n(\mathbb{B}_1)$,

(ii) *Let $n = 2t + 1$, $t \geq 2$ and $h \in \{0, 1, 2, \ldots, (t+1)n - (t+2)\}$. Then*

$$|T(S_{(t+1)n-(t+1)-h})| \leq n^2 - 2 - 2h$$

for all $S_{(t+1)n-(t+1)-h} \in \mathcal{M}_n(\mathbb{B}_1)$.

Proof. (i) The proof proceeds by induction on h. If $h = 0$, the result is obvious by Proposition 3.2 (i). Next, we assume that for some $h \in \{0, 1, 2, \ldots, tn - 3\}$, the argument holds. That is,

$$|T(S_{tn-1-h})| \leq n^2 - 3 - 2h \tag{3.4}$$

for all $S_{tn-1-h} \in \mathcal{M}_n(\mathbb{B}_1)$. Now we will show that $|T(S_{tn-2-h})| \leq n^2 - 5 - 2h$ for all $S_{tn-2-h} \in \mathcal{M}_n(\mathbb{B}_1)$. Suppose that $|T(S_{tn-2-h})| \geq n^2 - 4 - 2h$ for some $S_{tn-2-h} \in \mathcal{M}_n(\mathbb{B}_1)$. By (3.4) and Proposition 3.1, we have $|T(S_{tn-2-h})| = n^2 - 4 - 2h$ and

$$|T(S_{tn-2-h} + F)| = n^2 - 3 - 2h$$

for all cells F with $F \not\leq S_{tn-2-h}$. This means that for all cell F with $F \not\leq S_{tn-2-h}$, there is only cell C_F such that

$$C_F \not\leq T(S_{tn-2-h}), \quad C_F \leq T(F) \quad \text{and} \quad T(S_{tn-2-h} + F) = T(S_{tn-2-h}) + C_F \tag{3.5}$$

because $|T(S_{tn-2-h})| = n^2 - 4 - 2h$. Let $\mathcal{E}_n$ be the set of all cells in $\mathcal{M}_n(\mathbb{B}_1)$ and let

$$\Omega = \{C_F \,|\, F \in \mathcal{E}_n \quad \text{and} \quad F \not\leq S_{tn-2-h}\}.$$

Suppose that $C_H \neq C_F$ for all distinct cells F and H that are not dominated by S_{tn-2-h}. Then we have $|\Omega| = n^2 - (tn - 2 - h)$. Since $C_F \not\leq T(S_{tn-2-h})$ for any cell F with $F \not\leq S_{tn-2-h}$, we have $|\Omega| \leq n^2 - (n^2 - 4 - 2h)$ because $|T(S_{tn-2-h})| = n^2 - 4 - 2h$. This is impossible. Hence $C_H = C_F$ for some two distinct cells F and H that are not dominated by S_{tn-2-h}. It follows from (3.5) that

$$\begin{aligned} T(S_{tn-2-h} + F + H) &= T(S_{tn-2-h} + F) + T(S_{tn-2-h} + H) \\ &= T(S_{tn-2-h}) + C_F = T(S_{tn-2-h} + F). \end{aligned}$$

But Proposition 3.1 implies that $|T(S_{tn-2-h}+F)| < |T(S_{tn-2-h}+F+H)|$ because $|S_{tn-2-h}+F+H| \leq tn \leq (n-1)n$, a contradiction. Hence the result follows.

(ii) Similar to (i). ∎

Corollary 3.1. *$T(E)$ is a cell for all cells E.*

Proof. For $n=3$, the result was proved in Lemma 3.2. If $n=2t$ with $t \geq 2$, let $h = tn-2$ in Lemma 3.3 (i). Then $|T(S_1)| \leq 1$ for all $S_1 \in \mathcal{M}_n(\mathbb{B}_1)$. If $n = 2t+1$ with $t \geq 2$, let $h = (t+1)n-(t+2)$ in Lemma 3.3 (ii). Then $|T(S_1)| \leq 1$ for all $S_1 \in \mathcal{M}_n(\mathbb{B}_1)$. It follows from Lemma 2.3 that $|T(S_1)| = 1$ for all $S_1 \in \mathcal{M}_n(\mathbb{B}_1)$, equivalently $|T(E)| = 1$ for any cell E in $\mathcal{M}_n(\mathbb{B}_1)$. Therefore we have $T(E)$ is a cell for any cell E in $\mathcal{M}_n(\mathbb{B}_1)$. ∎

Lemma 3.4. *T is bijective on the set of cells.*

Proof. By Corollary 3.1, it suffices to show that $T(E) \neq T(F)$ for all distinct cells E and F in $\mathcal{M}_n(\mathbb{B}_1)$. Suppose $T(E) = T(F)$ for some distinct cells E and F. Then we have $T(E+F) = T(E)$. But this is impossible because $|T(E) < |T(E+F)|$ by Proposition 3.1. Thus the result follows. ∎

A matrix $L \in \mathcal{M}_n(\mathbb{B}_1)$ is called a *line matrix* if $L = \sum_{k=1}^{n} E_{i,k}$ or $L = \sum_{l=1}^{n} E_{l,j}$ for some $i, j \in \{1, \ldots, n\}$; $R_i = \sum_{k=1}^{n} E_{i,k}$ is an i^{th} *row matrix* and $C_j = \sum_{l=1}^{n} E_{l,j}$ is a j^{th} *column matrix*. Cells $E_1, E_2, \ldots, E_k$ are called *collinear* if $\sum_{i=1}^{k} E_i \leq L$ for some line matrix L.

A matrix $A \in \mathcal{M}_n(\mathbb{B}_1)$ is an *s-star matrix* if $|A| = s$ and there are cells $E_1, \ldots, E_s$ such that $A = \sum_{i=1}^{s} E_i$ and $A \leq L$ for some line matrix L. By Lemma 2.2, all line matrices and all s-star matrices are regular in $\mathcal{M}_n(\mathbb{B}_1)$.

Lemma 3.5. *T preserves all line matrices.*

Proof. By Lemma 3.4, T is bijective on the set of cells. First, we show that T preserves all 3-star matrices. If T does not preserve a 3-star matrix $A \in \mathcal{M}_n(\mathbb{B}_1)$, then we have $b(T(A)) = 2$ or 3 with $|T(A)| = 3$. By Corollary 2.1 (iii), there is a matrix $C \in \mathcal{M}_n(\mathbb{B}_1)$ with $|C| = 2$ such that $T(A) + C$ is not regular. Furthermore we can write $C = T(E_1 + E_2)$ for some distinct cells E_1 and E_2. Thus we have

$$T(A) + C = T(A + E_1 + E_2).$$

But $A + E_1 + E_2$ is regular by Corollary 2.1 (i) or (iv). This contradicts to the fact that T strongly preserves regularity. Hence T preserves all 3-star matrices.

Suppose that T does not preserve a line matrix L in $\mathcal{M}_n(\mathbb{B}_1)$. Then there are two distinct cells F_1 and F_2 dominated by L such that two cells $T(F_1)$ and $T(F_2)$ are not collinear. Let F_3 be a cell such that $F_1 + F_2 + F_3$ is a 3-star matrix. By the above result, $T(F_1 + F_2 + F_3)$ is a 3-star matrix, and hence $b(T(F_1 + F_2 + F_3)) = 1$. Thus, the three cells $T(F_1)$, $T(F_2)$ and $T(F_3)$ are collinear. This contradicts to the fact that the two cells $T(F_1)$ and $T(F_2)$ are not collinear. Therefore T preserves all line matrices. ∎

A linear operator T on $\mathcal{M}_n(\mathbb{B}_k)$ is called a (U,V)-*operator* if there are invertible matrices U and V such that $T(X) = UXV$ for all $X \in \mathcal{M}_n(\mathbb{B}_k)$ or $T(X) = UX^TV$ for all $X \in \mathcal{M}_n(\mathbb{B}_k)$.

We remind the $n \times n$ permutation matrices are the only invertible matrices in $\mathcal{M}_n(\mathbb{B}_1)$.

Now, we are ready to prove the main theorem.

Theorem 3.1. *Let T be a linear operator on $\mathcal{M}_n(\mathbb{B}_1)$ with $n \geq 3$. Then T strongly preserves regularity if and only if T is a (U, V)-operator.*

Proof. If T is a (U, V)-operator on $\mathcal{M}_n(\mathbb{B}_1)$, clearly T strongly preserves regularity by Proposition 2.1.

Conversely, assume that T strongly preserves regularity. Then T is bijective on the set of cells by Lemma 3.4 and T preserves all line matrices by Lemma 3.5. Since no combination of s row matrices and t column matrices can dominate J_n where $s + t = n$ unless $s = 0$ or $t = 0$, we have that either

(1) the image of each row matrix is a row matrix and the image of each column matrix is a column matrix, or
(2) the image of each row matrix is a column matrix and the image of each column matrix is a row matrix.

If (1) holds, then there are permutations σ and τ of $\{1, \ldots, n\}$ such that $T(R_i) = R_{\sigma(i)}$ and $T(C_j) = C_{\tau(j)}$ for all $i, j \in \{1, 2, \ldots, n\}$. Let U and V be permutation (i.e., invertible) matrices corresponding to σ and τ, respectively. Then we have

$$T(E_{i,j}) = E_{\sigma(i),\tau(j)} = UE_{i,j}V$$

for all cells $E_{i,j}$ in $\mathcal{M}_n(\mathbb{B}_1)$. Let $X = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j}E_{i,j}$ be any matrix in $\mathcal{M}_n(\mathbb{B}_1)$. By the action of T on the cells, we have $T(X) = UXV$. If (2) holds, then a parallel argument shows that there are invertible matrices U and V such that $T(X) = UX^TV$ for all $X \in \mathcal{M}_n(\mathbb{B}_1)$. ∎

Thus, as shown in Theorems 2.3 and 3.1, we have characterizations of the linear operators that strongly preserve regular matrices over the binary Boolean algebra.

4. The general Boolean cases

If T is a linear operator on $\mathcal{M}_n(\mathbb{B}_k)$ with $k \geq 1$, for each $p \in \{1, 2, \ldots, k\}$, define its p^{th} *constituent operator*, T_p, by $T_p(B) = (T(B))_p$ for all $B \in \mathcal{M}_n(\mathbb{B}_1)$. By the linearity of T, we have

$$T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p)$$

for all $A \in \mathcal{M}_n(\mathbb{B}_k)$.

Lemma 4.1. *If T is a linear operator on $\mathcal{M}_n(\mathbb{B}_k)$ that strongly preserves regularity, then its all constituent operators on $\mathcal{M}_n(\mathbb{B}_1)$ strongly preserve regularity.*

Proof. Let A be any matrix in $\mathcal{M}_n(\mathbb{B}_1)$. Obviously, A is the matrix in $\mathcal{M}_n(\mathbb{B}_k)$ such that $A_p = A$ for all $p = 1, \ldots, k$. If A is regular in $\mathcal{M}_n(\mathbb{B}_1)$, then A is regular in $\mathcal{M}_n(\mathbb{B}_k)$ by Theorem 2.1. Since T preserves regularity, we have $T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p)$ is also regular in $\mathcal{M}_n(\mathbb{B}_k)$. Again by Theorem 2.1, each $T_p(A_p)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ so that $T_p(A)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$.

Conversely, if $T_p(A)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$, then $T(A) = \sum_{p=1}^{k} \sigma_p T_p(A_p)$ is regular in $\mathcal{M}_n(\mathbb{B}_k)$ by Theorem 2.1. Since T strongly preserves regularity, we have A is regular in $\mathcal{M}_n(\mathbb{B}_k)$. Hence by Theorem 2.1, we have $A(= A_p)$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$. ∎

Example 4.1. Let $n \geq 3$. Define an operator T on $\mathcal{M}_n(\mathbb{B}_3)$ by

$$T(X) = \sigma_1 X_1 + \sigma_2 X_2^T + \sigma_3 X_3$$

for all $X = \sum_{p=1}^{3} \sigma_p X_p$ in $\mathcal{M}_n(\mathbb{B}_3)$. Then we can easily show that T is not a (U, V)-operator on $\mathcal{M}_n(\mathbb{B}_3)$ while its all constituent operators are (U, V)-operators on $\mathcal{M}_n(\mathbb{B}_1)$. Furthermore the theorem below shows that T strongly preserve regularity.

Theorem 4.1. *Let T be a linear operator on $\mathcal{M}_n(\mathbb{B}_k)$ with $n \geq 3$. Then the following statements are equivalent:*

(i) *T strongly preserves regularity on $\mathcal{M}_n(\mathbb{B}_k)$;*
(ii) *All constituent operators of T strongly preserve on $\mathcal{M}_n(\mathbb{B}_1)$;*
(iii) *There are invertible matrices U and V such that*

$$T(X) = UXV \quad \text{for all } X \in \mathcal{M}_n(\mathbb{B}_k), \quad \text{or} \tag{4.1}$$

$$T(X) = U\left(\sum_{p=1}^{k} \sigma_p Y_p\right)V \quad \text{for all } X \in \mathcal{M}_n(\mathbb{B}_k), \tag{4.2}$$

where $Y_p = X_p$ or X_p^T for all $p = 1, \ldots, k$.

Proof. It follows from Lemma 4.1 that (i) implies (ii). Assume (ii) holds. That is, each constituent operator T_p of T strongly preserves regularity on $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$. Let $X = \sum_{p=1}^{k} \sigma_p X_p$ be any matrix in $\mathcal{M}_n(\mathbb{B}_k)$. Then we have $T(X) = \sum_{p=1}^{k} \sigma_p T_p(X_p)$. By Theorem 3.1, each T_p has the form

$$T_p(X_p) = U_p X_p V_p, \tag{4.3}$$

or

$$T_p(X_p) = U_p X_p^T V_p, \tag{4.4}$$

where U_p and V_p are permutation matrices for all $p = 1, \ldots, k$.

Assume that only (4.3) are possible for all $p = 1, \ldots, k$. Then we have

$$T(X) = \sum_{p=1}^{k} \sigma_p U_p X_p V_p = \left(\sum_{p=1}^{k} \sigma_p U_p\right)\left(\sum_{p=1}^{k} \sigma_p X_p\right)\left(\sum_{p=1}^{k} \sigma_p V_p\right).$$

If we let $U = \left(\sum_{p=1}^{k} \sigma_p U_p\right)$ and $V = \left(\sum_{p=1}^{k} \sigma_p V_p\right)$, then U and V are invertible matrices in $\mathcal{M}_n(\mathbb{B}_k)$ by Lemma 1.1, and hence (4.1) is satisfied.

If both (4.3) and (4.4) are possible, then $T(X) = \sum_{p=1}^{k} \sigma_p U_p Y_p V_p$, where $Y_p = X_p$ or X_p^T for each $p \in \{1, \ldots, k\}$, equivalently

$$T(X) = \left(\sum_{p=1}^{k} \sigma_p U_p\right)\left(\sum_{p=1}^{k} \sigma_p Y_p\right)\left(\sum_{p=1}^{k} \sigma_p V_p\right).$$

If we let $U = (\sum_{p=1}^{k} \sigma_p U_p)$ and $V = (\sum_{p=1}^{k} \sigma_p V_p)$, then (4.2) is satisfied. Therefore (ii) implies (iii).

Assume (iii) holds. If T has a form (4.1), then we are done by Proposition 2.1. Thus we assume (4.2). If $X = \sum_{p=1}^{k} \sigma_p X_p$ is regular in $\mathcal{M}_n(\mathbb{B}_k)$, then so is X_p in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$ by Theorem 2.1. Thus there are matrices $G_p \in \mathcal{M}_n(\mathbb{B}_1)$ such that $X_p G_p X_p = X_p$ for all $p = 1, \ldots, k$. Let $G = V^T(\sum_{p=1}^{k} \sigma_p H_p)U^T$, where $H_p = G_p$ or G_p^T according as $Y_p = X_p$ or X_p^T. Then we can easily show that $T(X)GT(X) = T(X)$ so that $T(X)$ is regular in $\mathcal{M}_n(\mathbb{B}_k)$. Conversely, if $T(X)$ is regular in $\mathcal{M}_n(\mathbb{B}_k)$, then each constituent $T_p(X_p) = U_p Y_p V_p$ is regular in $\mathcal{M}_n(\mathbb{B}_1)$ for all $p = 1, \ldots, k$. By Proposition 2.1, each X_p is regular in $\mathcal{M}_n(\mathbb{B}_1)$ because $Y_p = X_p$ or X_p^T for all $p = 1, \ldots, k$. Hence X is regular in $\mathcal{M}_n(\mathbb{B}_k)$ by Theorem 2.1. Therefore (i) is satisfied. ∎

Thus, as shown in Theorems 2.3 and 4.1, we have characterizations of the linear operators that strongly preserve regular matrices over general Boolean algebras.

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