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A C^*-Algebra on Schur Algebras

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Abstract. In this paper, we show the relation between the Schur algebras $S^r_{\Lambda,\Sigma}(\mathcal{B})$ and $S^{r'}_{\Lambda,\Sigma}(\mathcal{B})$, where $1 \leq r' < r < \infty$. Then we set up the involution operator in these Schur algebras and show that with this involution operator there is only one C^*-algebra among these classes of Banach algebras. Furthermore, we show the equivalence of a condition on the Schur multiplier norm and the existence of C^*-algebra.

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1. Introduction

Fix p and q with $1 \leq p, q < \infty$. The space of pth power summable sequences of complex numbers is denoted by l_p, and the space of matrices which define bounded linear transformations from l_p to l_q is denoted by $\mathcal{B}(l_p, l_q)$. Let $A = [a_{jk}]\,, B = [b_{jk}]$ be infinite matrices, not necessarily in $\mathcal{B}(l_p, l_q)$. The Schur product $A \bullet B$ of A and B is defined by $A \bullet B = [a_{jk} b_{jk}]$. This product was first defined by I. Schur in [10]. He gave a nice property of this product: That is submultiplicative with respect to the operator norm on $\mathcal{B}(l_2)$. He also proved that the space $\mathcal{B}(l_2)$ forms a commutative Banach algebra under the Schur product and the norm of the operator defined on $\mathcal{B}(l_2)$ by the matrix. From the results of Schur we see that the Schur product has some nice properties that the usual product lacks. Many areas in mathematics such as matrix theory, complex function theory, operator theory and operator algebras have made use of results from the study of the Schur product and have injected new problems in return. See [1, 4, 5] for further references to related literature. In [2] we studied algebras under the Schur product operation of matrices over a Banach algebra, for which the matrix of the norms of the entries define bounded operators. In [7] S.-C. Ong studied the operator $B \mapsto A \bullet B$ on the algebra of $n \times n$ scalar matrices whenever A is a fixed scalar matrix in this algebra and the norm of this operator, denoted by $\|A\|_m$, that is called the Schur multiplier norm of A. He characterized

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the class of all matrices in this algebra whose operator norm is equal to its Schur multiplier norm. It is noticeable that the results mentioned above concerned only matrices with scalar entries. In [5] L. Livshits studied a generalized Schur product of matrices whose entries are bounded linear operators on a Hilbert space. The product on the entries is the usual operator multiplication. In [2] P. Chaisuriya and S.-C. Ong considered the classes $S^r_{p,q}(\mathcal{B})$, where $p, q, r \geq 1$, of all infinite matrices $A = [a_{jk}]$ over a Banach algebra $\mathcal{B}$ with identity whose absolute Schur rth power $A^{[r]} = [\|a_{jk}\|^r]$ defines a bounded linear operator from l_p to l_q. It was shown in their paper that all classes $S^r_{p,q}(\mathcal{B})$ are Banach algebras under the Schur product and the norm $\|A\|_{p,q,r} := \left\|A^{[r]}\right\|^{1/r}$. Moreover they proved that the Banach algebra $S^r_{p,q}(\mathbb{C})$ contains $\mathcal{B}(l_p, l_q)$ as a proper subset and if $r = 2$ then $\mathcal{B}(l_p, l_q)$ is an ideal of $S^2_{p,q}(\mathbb{C})$. In [8] we extend the results from sequence spaces l_p, l_q to Λ, Σ, where Λ, Σ are any sequence in $\{c_0\} \cup \{l_p : 1 \leq p < \infty\}$ and we showed that the space $S^r_{\Lambda,\Sigma}(\mathcal{B})$ of infinite matrices $A = [a_{jk}]$ over Banach algebra $\mathcal{B}$ such that $A^{[r]}$ defines a bounded linear operator from Λ to Σ are Banach algebras under the Schur product and the norm $\|A\|_{\Lambda,\Sigma,r} := \left\|A^{[r]}\right\|^{1/r}_{\Lambda,\Sigma}$.

2. Preliminaries results

Let $1 \leq r < \infty$, Λ, Σ be any sequence spaces in $\{c_0\} \cup \{l_p : 1 \leq p < \infty\}$ and $\mathcal{B}$ be any Banach algebra with identity. We denote $\mathcal{M}(\mathcal{B})$ the class of all matrices $A = [a_{jk}]$, where $a_{jk} \in \mathcal{B}$ for all $j, k \geq 1$ (i.e., one side infinite matrices over $\mathcal{B}$).

Definition 2.1. *Given a matrix $A = [a_{jk}] \in \mathcal{M}(\mathcal{B})$, and for any positive real number r, the absolute Schur rth power of A is the scalar matrix $A^{[r]} = [\|a_{jk}\|^r]$ with non-negative entries.*

The following result is easy to check.

Lemma 2.1.

(a) *Let $A = [a_{jk}]$ and $B = [b_{jk}]$ be scalar matrices. If $|a_{jk}| \leq b_{jk}$ for all j, k then*
$$\|A\|_{\Lambda,\Sigma} \leq \left\|A^{[1]}\right\|_{\Lambda,\Sigma} \leq \|B\|_{\Lambda,\Sigma}.$$

(b) *For any $\alpha, \beta \geq 0$ and $t > 0$,*
$$(\alpha + \beta)^t \leq 2^t(\alpha^t + \beta^t).$$

(c) *For any matrix $A = [a_{jk}]$ over $\mathbb{C}$, $|a_{jk}| \leq \|A\|_{\Lambda,\Sigma}$ for all j, k.*

(d) *For any scalar matrices A and B with non-negative entries*
$$\|A \bullet B\|_{\Lambda,\Sigma} \leq \|A\|_{\Lambda,\Sigma} \|B\|_{\Lambda,\Sigma}.$$

The Proposition 2.1, 2.2 and Theorem 2.1 have been proved in [8].

Proposition 2.1. (Hölder-type Inequality) *Let $A, B \in \mathcal{M}(\mathcal{B})$. Then*
$$\left\|(A \bullet B)^{[1]}\right\|_{\Lambda,\Sigma} \leq \left\|A^{[1]} \bullet B^{[1]}\right\|_{\Lambda,\Sigma} \leq \left\|A^{[r]}\right\|^{\frac{1}{r}}_{\Lambda,\Sigma} \left\|B^{[r^*]}\right\|^{\frac{1}{r^*}}_{\Lambda,\Sigma},$$
for $1 < r, r^* < \infty$ with $\frac{1}{r} + \frac{1}{r^*} = 1$.

Definition 2.2. *For any $A \in \mathcal{M}(\mathcal{B})$ and $1 \le r < \infty$, the absolute Schur r-norm of A is defined by* $\|A\|_{\Lambda,\Sigma,r} := \left\|A^{[r]}\right\|_{\Lambda,\Sigma}^{\frac{1}{r}}$.

Proposition 2.2. (Minkowski-type Inequality) *For any $A, B \in \mathcal{M}(\mathcal{B})$ and $1 \le r < \infty$,*

$$\|A + B\|_{\Lambda,\Sigma,r} \le \|A\|_{\Lambda,\Sigma,r} + \|B\|_{\Lambda,\Sigma,r}.$$

Corollary 2.1. *For any $1 \le r < \infty$, $S^r_{\Lambda,\Sigma}(\mathcal{B})$ is a subspace of $\mathcal{M}(\mathcal{B})$ and is a normed space under the norm $\|.\|_{\Lambda,\Sigma,r}$.*

Theorem 2.1. *For any $1 \le r < \infty$, the normed space $S^r_{\Lambda,\Sigma}(\mathcal{B})$ is a Banach algebra under the Schur product operation and the norm $\|.\|_{\Lambda,\Sigma,r}$.*

3. Relationship between the set of absolutely bounded matrices

Theorem 3.1. *For $1 < r < \infty$, we have*

(a) *For $1 \le r' < r < \infty$, $S^{r'}_{\Lambda,\Sigma}(\mathcal{B}) \subseteq S^r_{\Lambda,\Sigma}(\mathcal{B})$ and $\|A\|_{\Lambda,\Sigma,r} \le \|A\|_{\Lambda,\Sigma,r'}$ for all $A \in S^{r'}_{\Lambda,\Sigma}(\mathcal{B})$.*

(b) *If $1 \le r' < r < \infty$ and $(\Lambda, \Sigma) \ne (l_1, c_0)$, then $S^{r'}_{\Lambda,\Sigma}(\mathcal{B}) \subsetneq S^r_{\Lambda,\Sigma}(\mathcal{B})$, and $S^{r'}_{\Lambda,\Sigma}(\mathcal{B})$ is not closed in $S^r_{\Lambda,\Sigma}(\mathcal{B})$.*

(c) *For all $r \in [1, \infty)$, and for all $A = [a_{j,k}] \in S^r_{l_1,c_0}(\mathcal{B})$, $\|A\|_{l_1,c_0,r} = \sup_{j,k} \|a_{j,k}\|$. Furthermore, $S^1_{l_1,c_0}(\mathcal{B}) = S^r_{l_1,c_0}(\mathcal{B})$.*

(d) *If $A = [a_{jk}] \in S^r_{c_0,c_0}(\mathcal{B})$, then $\|A\|_{c_0,c_0,r} = \sup_j \left(\sum_{k=1}^{\infty} \|a_{jk}\|^r\right)^{1/r}$.*

Proof. Let $A = [a_{jk}]$ be a nonzero matrix in $S^{r'}_{\Lambda,\Sigma}(\mathcal{B})$. From Lemma 2.1(c), we have

$$\frac{\|a_{jk}\|}{\|A\|_{\Lambda,\Sigma,r'}} \le 1$$

for all (j, k). Hence, for each (j, k),

$$\left(\frac{\|a_{jk}\|}{\|A\|_{\Lambda,\Sigma,r'}}\right)^r \le \left(\frac{\|a_{jk}\|}{\|A\|_{\Lambda,\Sigma,r'}}\right)^{r'},$$

that is $\|a_{jk}\|^r \le \|A\|^{r-r'}_{\Lambda,\Sigma,r'} \|a_{jk}\|^{r'}$. Thus, by Lemma 2.1(a)

$$\left\|A^{[r]}\right\|_{\Lambda,\Sigma} \le \left\|\|A\|^{r-r'}_{\Lambda,\Sigma,r'} (A^{[r']})\right\|_{\Lambda,\Sigma} = \|A\|^{r-r'}_{\Lambda,\Sigma,r'} \left\|A^{[r']}\right\|_{\Lambda,\Sigma}.$$

This implies that $\|A\|_{\Lambda,\Sigma,r} \le \|A\|_{\Lambda,\Sigma,r'}$, so $A \in S^r_{\Lambda,\Sigma}(\mathcal{B})$. It follows that the inclusion $S^{r'}_{\Lambda,\Sigma}(\mathcal{B}) \subseteq S^r_{\Lambda,\Sigma}(\mathcal{B})$ holds. Next, we will show that $\|A\|_{l_1,c_0,r} = \sup_{j,k} \|a_{jk}\|$ for any $r \in [1, \infty)$ and $A = [a_{jk}] \in S^r_{l_1,c_0}(\mathcal{B})$. By Lemma 2.1(c) we have

$$\sup_{j,k} \|a_{jk}\| \le \left\|A^{[r]}\right\|^{1/r}_{l_1,c_0} = \|A\|_{l_1,c_0,r}, \forall j, k \tag{3.1}$$

For any $x = \{\xi_k\} \in l_1$ with $\|x\|_{l_1} \le 1$.

$$\sup_j \left|\sum_{k=1}^{\infty} \|a_{jk}\|^r \xi_k\right|^{1/r} \le \sup_j \left(\sum_{k=1}^{\infty} \|a_{jk}\|^r |\xi_k|\right)^{1/r}$$

$$\leq \sup_{j,k} \|a_{jk}\| \left(\sum_{k=1}^{\infty} |\xi_k| \right)^{1/r}$$
$$= \sup_{j,k} \|a_{jk}\| \, \|x\|_{l_1}^{1/r} \leq \sup_{j,k} \|a_{jk}\| .$$

Hence we have

$$\|A\|_{l_1,c_0,r} \leq \sup_{jk} \|a_{jk}\| . \tag{3.2}$$

From (3.1) and (3.2) we have

$$\|A\|_{l_1,c_0,r} = \sup_{j,k} \|a_{jk}\| ,$$

for all $r \in [1, \infty)$, and $A = [a_{jk}] \in S^r_{l_1,c_0}(\mathcal{B})$. Consequently, $S^1_{l_1,c_0}(\mathcal{B}) = S^r_{l_1,c_0}(\mathcal{B})$.

Let $A = [a_{jk}] \in S^r_{c_0,c_0}(\mathcal{B})$. We will show that

$$\|A\|_{c_0,c_0,r} = \sup_j \left(\sum_{k=1}^{\infty} \|a_{jk}\|^r \right)^{1/r} .$$

It is easy to see that

$$\|A\|_{c_0,c_0,r} \leq \sup_j \left(\sum_{k=1}^{\infty} \|a_{jk}\|^r \right)^{1/r} .$$

Suppose that

$$\left\| A^{[r]} \right\|_{c_0,c_0} < \sup_j \sum_{k=1}^{\infty} \|a_{jk}\|^r .$$

Then there exists a positive integer j_0 such that

$$\left\| A^{[r]} \right\|_{c_0,c_0} < \sum_{k=1}^{\infty} \|a_{j_0k}\|^r .$$

This implies that there exists a positive integer n such that

$$\left\| A^{[r]} \right\|_{c_0,c_0} < \sum_{k=1}^{n} \|a_{j_ok}\|^r .$$

Let

$$x = \{\underbrace{1, 1, 1,}_{n1's} \ldots, 1, 0, 0, \ldots\}.$$

Since

$$\|x\|_{c_0} = 1, \sum_{k=1}^{n} \|a_{j_0k}\|^r \leq \left\| A^{[r]} x \right\|_{c_0} \leq \left\| A^{[r]} \right\|_{c_0,c_0} < \sum_{k=1}^{n} \|a_{j_0k}\|^r .$$

This is a contradiction, so $\|A\|_{c_0,c_0,r} = \sup_j \left(\sum_{k=1}^{\infty} \|a_{jk}\|^r \right)^{1/r}$.

For the case where $(\Lambda, \Sigma) \neq (l_1, c_0)$, the following arguments show that each inclusion $S^{r'}_{\Lambda,\Sigma}(\mathcal{B}) \subseteq S^r_{\Lambda,\Sigma}(\mathcal{B})$ is proper, and $S^{r'}_{\Lambda,\Sigma}(\mathcal{B})$ is not closed in $S^r_{\Lambda,\Sigma}(\mathcal{B})$.

Case 1. $\Sigma = l_p$ for $1 \leq p < \infty$. Let A be the matrix with first column the sequence

$$\left\{ \left(\frac{1}{k}\right)^{(1/pr')} e \right\}_{k=1}^{\infty},$$

where e is an identity for $\mathcal{B}$ and all other columns 0. Then $A \in S^r_{\Lambda,l_p}(\mathcal{B})$. Since the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the sequence $A^{[r']}x \notin l_p$ for all nonzero element x in Λ. So $A \notin S^{r'}_{\Lambda,l_p}(\mathcal{B})$. For each n, we have

$$\left\| (A_{n\lrcorner} - A)^{[r]} \right\|_{\Lambda,l_p} = \left(\sum_{n+1}^{\infty} \left(\frac{1}{k}\right)^{r/r'} \right)^{1/p},$$

where $A_{n\lrcorner}$ denotes the matrix whose (j,k) entry is a_{jk} from A for $j,k \leq n$ and 0 otherwise. Thus $\|(A_{n\lrcorner} - A)^{[r]}\|_{\Lambda,l_p} \to 0$ as $n \to \infty$. This implies that A belongs to the closure in $S^r_{\Lambda,l_p}(\mathcal{B})$ of $S^{r'}_{\Lambda,l_p}(\mathcal{B})$. So $S^{r'}_{\Lambda,l_p}(\mathcal{B})$ is not closed in $S^r_{\Lambda,l_p}(\mathcal{B})$.

Case 2. $\Lambda = \Sigma = c_0$. let A be the matrix with first row the sequence

$$\left\{ \left(\frac{1}{k+1}\right)^{1/r'} e \right\}_{k=1}^{\infty}$$

and all other rows 0. Then $A \in S^r_{c_0,c_0}(\mathcal{B})$. Let

$$y = \left\{ \frac{1}{\log(k+1)} \right\}_{k=1}^{\infty}.$$

Then $y \in c_0$. Since the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)\log(k+1)}$ diverges, $A^{[r']}y$ is not defined. This means that $A \notin S^{r'}_{c_0,c_0}(\mathcal{B})$. Since

$$\left\| (A_{n\lrcorner} - A)^{[r]} \right\|_{c_0,c_0} = \sum_{k=n+1}^{\infty} \left(\frac{1}{k+1}\right)^{r/r'} \to 0 \quad \text{as} \quad n \to \infty,$$

A belongs to the closure in $S^r_{c_0,c_0}(\mathcal{B})$ of $S^{r'}_{c_0,c_0}(\mathcal{B})$. Thus $S^{r'}_{c_0,c_0}(\mathcal{B})$ is not closed in $S^r_{c_0,c_0}(\mathcal{B})$.

Case 3. $\Lambda = l_p$ for $1 < p < \infty$ and $\Sigma = c_0$. Let A be the matrix with the first row the sequence

$$\left\{ \left(\frac{1}{k+1}\right)^{1/(qr')} e \right\}_{k=1}^{\infty},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and all other rows 0. It is easy to see that $A \in S^r_{l_p,c_0}(\mathcal{B})$. Let

$$y = \left\{ \frac{1}{(k+1)^{1/p}\log(k+1)} \right\}_{k=1}^{\infty}.$$

Then $y \in l_p$. Since the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)\log(k+1)}$ diverges, $A^{[r']}y$ is not defined. This implies that $A \notin S^{r'}_{l_p,c_0}(\mathcal{B})$. Since

$$\left\|(A_{n\lrcorner} - A)^{[r]}\right\|_{l_p,c_0} = \left(\sum_{k=n+1}^{\infty}\left(\frac{1}{k+1}\right)^{r/r'}\right)^{1/q} \to 0 \quad \text{as} \quad n \to \infty,$$

A belongs to the closure in $S^r_{l_p,c_0}(\mathcal{B})$ of $S^{r'}_{l_p,c_0}(\mathcal{B})$. Hence $S^{r'}_{l_p,c_0}(\mathcal{B})$ is not closed in $S^r_{l_p,c_0}(\mathcal{B})$. The proof is complete. ∎

4. A C^*-algebra on Schur algebras

Theorem 2.1 in the above section has been proved that for $1 \le r < \infty$, all classes

$$S^r_{\Lambda,\Sigma}(\mathcal{B}) = \{A = [a_{jk}] : a_{jk} \in \mathcal{B}, A^{[r]} = [\|a_{jk}\|^r] \in \mathcal{B}(\Lambda, \Sigma)\}$$

are Banach algebras under the Schur product and the norm $\|A\|_{\Lambda,\Sigma,r} := \left\|A^{[r]}\right\|^{\frac{1}{r}}_{\Lambda,\Sigma}$. In this section we want to find the structure of the C^*-algebra from these Banach algebras. Let $\mathcal{B} = \mathfrak{C}$ be any C^*-algebra with identity and define the operator $\bigstar$ on $S^r_{\Lambda,\Sigma}(\mathfrak{C})$ by $A = [a_{jk}] \mapsto A^{\bigstar} = [a^*_{jk}]$, where the $*$ is the involution on $\mathfrak{C}$ acting on a_{jk} for each (j,k).

The following result is easy to check.

Proposition 4.1. *The operator $\bigstar$ on $S^r_{\Lambda,\Sigma}(\mathfrak{C})$ is an involution on $S^r_{\Lambda,\Sigma}(\mathfrak{C})$.*

Next we want to show that $S^r_{l_1,c_0}(\mathfrak{C})$ is a C^*-algebra.

Theorem 4.1. *Under the norm $\|.\|_{l_1,c_0,r}$, the Schur product and the above involution, one has $S^r_{l_1,c_0}(\mathfrak{C}) \cong l_\infty(\mathbb{N}\times\mathbb{N};\mathfrak{C})$ as C^*-algebra, for any $1 \le r < \infty$.*

Proof. It is not hard to check that the linear bijection $[a_{jk}] \in S^r_{l_1,c_0}(\mathfrak{C}) \mapsto (a_{jk}) \in l_\infty(\mathbb{N}\times\mathbb{N};\mathfrak{C})$ preserves the norm (because of Theorem 3.1(c)) the product and the involution. ∎

For the case $(\Lambda,\Sigma) \ne (l_1, c_0)$, we will show by examples that they are not C^*-algebras under this involution.

Example 4.1. Let $A \in \mathcal{M}(\mathbb{C})$ be the matrix with the first row the sequence $\{\frac{1}{2^{k-1}}\}_{k=1}^{\infty}$ and the other rows all zero. We first show that $A \in S^r_{c_0,c_0}(\mathbb{C})$. Let $x = \{\xi_k\}_{k=1}^{\infty} \in c_0$. Then for any positive integer n,

$$\sum_{k=1}^{n}\left|\frac{1}{2^{r(k-1)}}\xi_k\right| = \sum_{k=1}^{n}\frac{1}{2^{r(k-1)}}|\xi_k| \le \|x\|_{c_0}\sum_{k=1}^{\infty}\frac{1}{2^{r(k-1)}} < \infty.$$

This implies that the monotone increasing sequence

$$\left\{\sum_{k=1}^{n}\left|\frac{1}{2^{r(k-1)}}\xi_k\right|\right\}_{n=1}^{\infty}$$

is bounded. So, it is convergent and also

$$\sum_{k=1}^{\infty}\left|\frac{1}{2^{r(k-1)}}\xi_k\right| \le \|x\|_{c_0}\sum_{k=1}^{\infty}\frac{1}{2^{r(k-1)}}.$$

That is, the series $\sum_{k=1}^{\infty} \frac{1}{2^{r(k-1)}}\xi_k$ converges.
Thus

$$A^{[r]}x = \left\{ \sum_{k=1}^{\infty} \frac{1}{2^{r(k-1)}}\xi_k, 0, 0, \ldots \right\} \in c_0$$

for any $x = \{\xi_k\}_{k=1}^{\infty} \in c_0$. Moreover

$$\left\| A^{[r]} \right\|_{c_0,c_0} = \sup_{\|x\|\leq 1} \left\| A^{[r]}x \right\|_{c_0} \leq \sum_{k=1}^{\infty} \frac{1}{2^{r(k-1)}} < \infty \tag{4.1}$$

which means that $A^{[r]} \in \mathcal{B}(c_0, c_0)$. Thus $A \in S^r_{c_0,c_0}(\mathbb{C})$.
To see that

$$\left\| A^{[r]} \right\|_{c_0,c_0} = \sum_{k=1}^{\infty} \frac{1}{2^{r(k-1)}},$$

let

$$x_N = (\underbrace{1, 1, \ldots, 1}_{N1's}, 0, 0, 0, \ldots).$$

Then we get

$$\left\| A^{[r]}x_N \right\|_{c_0} = \sum_{k=1}^{N} \frac{1}{2^{r(k-1)}}.$$

This implies that $\left\| A^{[r]} \right\|_{c_0,c_0} \geq \sum_{k=1}^{N} \frac{1}{2^{r(k-1)}}$ for each N. By letting $N \to \infty$, we get that

$$\left\| A^{[r]} \right\|_{c_0,c_0} \geq \sum_{k=1}^{\infty} \frac{1}{2^{r(k-1)}}. \tag{4.2}$$

Therefore from (4.1) and (4.2) we have $\left\| A^{[r]} \right\|_{c_0,c_0} = \sum_{k=1}^{\infty} \frac{1}{2^{r(k-1)}} = \frac{2^r}{2^r-1}$. Thus

$$\|A\|_{c_0,c_0,r} = \left\| A^{[r]} \right\|_{c_0,c_0}^{1/r} = \left(\frac{2^r}{2^r - 1} \right)^{1/r}.$$

The matrix $A^{\bigstar} \bullet A$ will have the first row the sequence $\left\{ \frac{1}{2^{2(k-1)}} \right\}_{k=1}^{\infty}$ and all other rows zero. By the same argument, we have

$$\left\| A^{\bigstar} \bullet A \right\|_{c_0,c_0,r} = \left(\sum_{k=1}^{\infty} \frac{1}{2^{2r(k-1)}} \right)^{1/r} = \left(\frac{4^r}{4^r - 1} \right)^{1/r}.$$

Since $4^r - 1 = (2^r - 1)^2$ if and only if $r = 0$. Therefore for any $1 \leq r < \infty$,

$$\left\| A^{\bigstar} \bullet A \right\|_{c_0,c_0,r} = \left(\frac{4^r}{4^r - 1} \right)^{1/r} \neq \left(\frac{2^r}{2^r - 1} \right)^{2/r} = \|A\|^2_{c_0,c_0,r}.$$

Hence, $S^r_{c_0,c_0}(\mathfrak{C})$ are not C^*-algebras.

The next example shows that $S^r_{c_0,l_p}(\mathfrak{C})$ are not C^*-algebras.

Example 4.2. Let $1 \leq p < \infty$ and $A \in \mathcal{M}(\mathbb{C})$ be the matrix with the first row $(1, 2, 0, 0, 0, \ldots)$ and all other rows zero. We will first show that $A \in S^r_{c_0,l_p}(\mathbb{C})$. Let $x = \{\xi_k\}_{k=1}^{\infty} \in c_0$. Then we get that $A^{[r]}x = \{\xi_1 + 2^r\xi_2, 0, 0, 0, \ldots\} \in l_p$, and

$$\left\| A^{[r]}x \right\|_{l_p} = |\xi_1 + 2^r\xi_2| \leq |\xi_1| + 2^r |\xi_2| \leq \|x\|_{c_0} (1 + 2^r).$$

So, $\left\|A^{[r]}\right\|_{c_0,l_p} \leq 1 + 2^r < \infty$. This implies that $A^{[r]} \in \mathcal{B}(c_0, l_p)$. Therefore $A \in S^r_{c_0,l_p}(\mathbb{C})$.
Next we want to find $\left\|A^{\star} \bullet A\right\|_{c_0,l_p,r}$. Let $\tilde{x} = (1, 1, 0, 0, \ldots) \in c_0$, clearly $\|\tilde{x}\|_{c_0} = 1$ and $\left\|A^{[r]}\right\|_{c_0,l_p} \geq \left\|A^{[r]}\tilde{x}\right\|_{l_p} = 1 + 2^r$. Therefore $\left\|A^{[r]}\right\|_{c_0,l_p} = 1 + 2^r$. Thus

$$\|A\|_{c_0,l_p,r} = \left\|A^{[r]}\right\|^{1/r}_{c_0,l_p} = (1 + 2^r)^{1/r}.$$

By using the same argument we have $\left\|A^{\star} \bullet A\right\|_{c_0,l_p,r} = (1 + 4^r)^{1/r}$.
Therefore

$$\left\|A^{\star} \bullet A\right\|_{c_0,l_p,r} = (1 + 4^r)^{1/r} \neq (1 + 2^r)^{2/r} = \|A\|^2_{c_0,l_p,r},$$

since $1 + 4^r \neq (1 + 2^r)^2$ for any positive real number r. Hence, $S^r_{c_0,l_p}(\mathfrak{C})$ are not C^*-algebras.

The next example shows that $S^r_{l_p,c_0}(\mathfrak{C})$ are not C^*-algebras for $p \neq 1$.

Example 4.3. Let $1 < p < \infty$ and $1 \leq q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $A \in \mathcal{M}(\mathbb{C})$ be the matrix with the first row $(1, \frac{1}{2}, 0, 0, \ldots)$ and all other rows zero. First we will show that $A \in S^r_{l_p,c_0}(\mathbb{C})$, let $x = \{\xi_k\}_{k=1}^{\infty} \in l_p$. Then $A^{[r]}x = \{\xi_1 + \frac{1}{2^r}\xi_2, 0, 0, \ldots\} \in c_0$. Clearly the sequence $\{1, \frac{1}{2^r}, 0, 0, \ldots\} \in l_q$, and by using the fact that $l_q \cong (l_p)^*$, we have

$$\left\|A^{[r]}\right\|_{l_p,c_0} = \sup_{\|x\|_{l_p} \leq 1} \left\|A^{[r]}x\right\|_{c_0} = \sup_{\|\{\xi_k\}\|_{l_p} \leq 1} \left|\xi_1 + \frac{1}{2^r}\xi_2\right| = \left(1 + \frac{1}{2^{qr}}\right)^{1/q} < \infty.$$

This implies that $A^{[r]} \in \mathcal{B}(l_p, c_0)$ and

$$\|A\|_{l_p,c_0,r} = \left\|A^{[r]}\right\|^{1/r}_{l_p,c_0} = \left(1 + \frac{1}{2^{qr}}\right)^{1/qr}.$$

Similarly, we have

$$\left\|A^{\star} \bullet A\right\|_{l_p,c_0,r} = \left(1 + \frac{1}{4^{qr}}\right)^{1/qr}.$$

Since for any $1 \leq q, r < \infty$,

$$\left(1 + \frac{1}{4^{qr}}\right)^{1/qr} \neq \left(1 + \frac{1}{2^{qr}}\right)^{2/qr},$$

therefore $\left\|A^{\star} \bullet A\right\|_{l_p,c_0,r} \neq \|A\|^2_{l_p,c_0,r}$. Hence $S^r_{l_p,c_0}(\mathfrak{C})$ are not C^*-algebras for $p \neq 1$.

Example 4.4. Let $1 \le p, q < \infty$ and $A \in \mathcal{M}(\mathbb{C})$ be the matrix with first two entries of the first column 1 and 3 respectively, and all other entries zero.
We will show that $A \in S^r_{l_p,l_q}(\mathbb{C})$. Let $x = \{\xi_k\}_{k=1}^{\infty} \in l_p$, then $A^{[r]}x = (\xi_1, 3^r\xi_1, 0, 0, \ldots) \in l_q$ and also

$$\left\| A^{[r]}x \right\|_{l_q} = (|\xi_1|^q + 3^{qr}|\xi_1|^q)^{1/q} \le \left[\|x\|_{l_p}^q (1 + 3^{qr}) \right]^{1/q} = \|x\|_{l_p} (1 + 3^{qr})^{1/q}.$$

Hence $\|A^{[r]}\|_{l_p,l_q} \le (1+3^{qr})^{1/q} < \infty$. This implies that $A^{[r]} \in \mathcal{B}(l_p, l_q)$ and $\|A\|_{l_p,l_q,r} \le (1 + 3^{qr})^{1/qr}$.
Let $\tilde{x} = (1, 0, 0, \ldots)$. Then $\tilde{x} \in l_p$ and $\|\tilde{x}\|_{l_p} = 1$. We have

$$\left\| A^{[r]} \right\|_{l_p,l_q} \ge \left\| A^{[r]}\tilde{x} \right\|_{l_q} = (1 + 3^{qr})^{1/q}.$$

So, $\|A\|_{l_p,l_q,r} = (1 + 3^{qr})^{1/qr}$. Similarly, we have $\left\| A^{\bigstar} \bullet A \right\|_{l_p,l_q,r} = (1 + 9^{qr})^{1/qr}$. Since for any $1 \le q, r < \infty, 1 + 9^{qr} \ne (1 + 3^{qr})^2$, therefore

$$\left\| A^{\bigstar} \bullet A \right\|_{l_p,l_q,r} = (1 + 9^{qr})^{1/qr} \ne (1 + 3^{qr})^{2/qr} = \|A\|_{l_p,l_q,r}^2 .$$

Hence, $S^r_{l_p,l_q}(\mathfrak{C})$ are not C^*-algebras.

Therefore there is only one class C^*algebra $S^r_{l_1,c_0}(\mathfrak{C})$ of all classes of Banach algebras $S^r_{\Lambda,\Sigma}(\mathfrak{C})$.

5. Schur multiplier norm with C^*-algebras

We see from the last section that for $S^r_{\Lambda,\Sigma}(\mathfrak{C})$ to be a C^*-algebra it suffices to show that $\left\| A^{\bigstar} \bullet A \right\|_{\Lambda,\Sigma,r} = \|A\|_{\Lambda,\Sigma,r}^2$. In this section we want to find the relationship between this property and the Schur multiplier norm that is mentioned in [7].

Definition 5.1. *For each $A \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$, the operator $S_A : S^r_{\Lambda,\Sigma}(\mathfrak{C}) \to S^r_{\Lambda,\Sigma}(\mathfrak{C})$ is defined by $B \mapsto S_A(B) = A \bullet B$. The norm of the operator S_A is called the Schur multiplier norm of A and denoted by $\|A\|_m$. That is*

$$\|A\|_m = \|S_A\| = \sup_{\|B\| \le 1} \|A \bullet B\|_{\Lambda,\Sigma,r} .$$

Since we have $\|A \bullet B\|_{\Lambda,\Sigma,r} \le \|A\|_{\Lambda,\Sigma,r} \|B\|_{\Lambda,\Sigma,r}$, this implies that $\|A\|_m \le \|A\|_{\Lambda,\Sigma,r}$. The next example shows that there exists $A \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$ such that $\left\| A^{\bigstar} \bullet A \right\|_{\Lambda,\Sigma,r} = \|A\|_{\Lambda,\Sigma,r}^2$ for any case of (Λ, Σ).

Example 5.1. Let A be the matrix with (1, 1)-entry αe, where $\alpha \in \mathbb{C}$ and e is the identity of C^*-algebra $\mathfrak{C}$, and all other entries zero. Clearly, $A \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$ for any case of (Λ, Σ) and $\left\| A^{\bigstar} \bullet A \right\|_{\Lambda,\Sigma,r} = |\alpha|^2 = \|A\|_{\Lambda,\Sigma,r}^2$.

Lemma 5.1. *For any $A = [a_{jk}], B = [b_{jk}] \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$,*

$$\|A \bullet B\|_{\Lambda,\Sigma,r} \le \left\| A^{\bigstar} \bullet A \right\|_{\Lambda,\Sigma,r}^{1/2} \left\| B^{\bigstar} \bullet B \right\|_{\Lambda,\Sigma,r}^{1/2} .$$

Proof. Let $A = [a_{jk}], B = [b_{jk}] \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$. By Theorem 2.1, $A \bullet B \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$. To show that $\|A \bullet B\|_{\Lambda,\Sigma,r} \leq \left\|A^\star \bullet A\right\|^{1/2}_{\Lambda,\Sigma,r} \left\|B^\star \bullet B\right\|^{1/2}_{\Lambda,\Sigma,r}$ we let $x = \{\xi_k\}_{k=1}^\infty \in \Lambda$ with $\|x\|_\Lambda \leq 1$.

Case I. $\Sigma = c_0$. For each positive integer j, we have that

$$\begin{aligned}
\left|\sum_{k=1}^\infty \|a_{jk}b_{jk}\|^r \xi_k\right| &\leq \sum_{k=1}^\infty \|a_{jk}\|^r \|b_{jk}\|^r |\xi_k|^{1/2}|\xi_k|^{1/2} \\
&\leq \sup_j \left(\sum_{k=1}^\infty \left\|a^*_{jk}a_{jk}\right\|^r |\xi_k|\right)^{1/2} \sup_j \left(\sum_{k=1}^\infty \left\|b^*_{jk}b_{jk}\right\|^r |\xi_k|\right)^{1/2} \\
&= \left\|(A^\star \bullet A)^{[r]}|x|\right\|^{1/2}_{c_0} \left\|(B^\star \bullet B)^{[r]}|x|\right\|^{1/2}_{c_0} \\
&\leq \left\|(A^\star \bullet A)^{[r]}\right\|^{1/2}_{\Lambda,c_0} \left\|(B^\star \bullet B)^{[r]}\right\|^{1/2}_{\Lambda,c_0}.
\end{aligned}$$

Hence,

$$\left\|(A \bullet B)^{[r]}x\right\|_{c_0} = \sup_j \left|\sum_{k=1}^\infty \|a_{jk}b_{jk}\|^r \xi_k\right| \leq \left\|(A^\star \bullet A)^{[r]}\right\|^{1/2}_{\Lambda,c_0} \left\|(B^\star \bullet B)^{[r]}\right\|^{1/2}_{\Lambda,c_0}.$$

This implies that

$$\left\|(A \bullet B)^{[r]}\right\|_{\Lambda,c_0} \leq \left\|(A^\star \bullet A)^{[r]}\right\|^{1/2}_{\Lambda,c_0} \left\|(B^\star \bullet B)^{[r]}\right\|^{1/2}_{\Lambda,c_0}.$$

Thus,

$$\|A \bullet B\|_{\Lambda,c_0,r} \leq \left\|A^\star \bullet A\right\|^{1/2}_{\Lambda,c_0,r} \left\|B^\star \bullet B\right\|^{1/2}_{\Lambda,c_0,r}.$$

Case II. $\Sigma = l_p$. By using the Cauchy-Schwarz inequality twice we have,

$$\begin{aligned}
\sum_{j=1}^\infty \left|\sum_{k=1}^\infty \|a_{jk}b_{jk}\|^r \xi_k\right|^p &\leq \sum_{j=1}^\infty \left(\sum_{k=1}^\infty \|a_{jk}\|^r \|b_{jk}\|^r |\xi_k|^{1/2}|\xi_k|^{1/2}\right)^p \\
&\leq \sum_{j=1}^\infty \left(\sum_{k=1}^\infty \|a_{jk}\|^{2r} |\xi_k|\right)^{p/2} \left(\sum_{k=1}^\infty \|b_{jk}\|^{2r} |\xi_k|\right)^{p/2} \\
&\leq \left\{\sum_{j=1}^\infty \left(\sum_{k=1}^\infty \left\|a^*_{jk}a_{jk}\right\|^r |\xi_k|\right)^p\right\}^{1/2} \\
&\quad \times \left\{\sum_{j=1}^\infty \left(\sum_{k=1}^\infty \left\|b^*_{jk}b_{jk}\right\|^r |\xi_k|\right)^p\right\}^{1/2} \\
&= \left\|(A^\star \bullet A)^{[r]}|x|\right\|^{p/2}_{l_p} \left\|(B^\star \bullet B)^{[r]}|x|\right\|^{p/2}_{l_p}
\end{aligned}$$

$$\leq \left\|(A^\bigstar \bullet A)^{[r]}\right\|_{\Lambda,l_p}^{p/2} \left\|(B^\bigstar \bullet B)^{[r]}\right\|_{\Lambda,l_p}^{p/2}.$$

So,

$$\left\|(A \bullet B)^{[r]}x\right\|_{l_p} = \left\{\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\|a_{jk}b_{jk}\|^r \xi_k\right|^p\right\}^{1/p} \leq \left\|(A^\bigstar \bullet A)^{[r]}\right\|_{\Lambda,l_p}^{1/2} \left\|(B^\bigstar \bullet B)^{[r]}\right\|_{\Lambda,l_p}^{1/2}.$$

It follows that

$$\left\|(A \bullet B)^{[r]}\right\|_{\Lambda,l_p} \leq \left\|(A^\bigstar \bullet A)^{[r]}\right\|_{\Lambda,l_p}^{1/2} \left\|(B^\bigstar \bullet B)^{[r]}\right\|_{\Lambda,l_p}^{1/2}.$$

Hence

$$\|A \bullet B\|_{\Lambda,l_p,r} \leq \left\|A^\bigstar \bullet A\right\|_{\Lambda,l_p,r}^{1/2} \left\|B^\bigstar \bullet B\right\|_{\Lambda,l_p,r}^{1/2}.$$

Therefore in all cases

$$\|A \bullet B\|_{\Lambda,\Sigma,r} \leq \left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r}^{1/2} \left\|B^\bigstar \bullet B\right\|_{\Lambda,\Sigma,r}^{1/2}.$$

The proof is complete. ∎

Theorem 5.1. *Let $A \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$. Then $\|A\|_m = \|A\|_{\Lambda,\Sigma,r}$ if and only if*

$$\left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r} = \|A\|_{\Lambda,\Sigma,r}^2.$$

Proof. First, we assume that $\left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r} = \|A\|_{\Lambda,\Sigma,r}^2$. We already have $\|A\|_m \leq \|A\|_{\Lambda,\Sigma,r}$. To show $\|A\|_m = \|A\|_{\Lambda,\Sigma,r}$, suppose that $\|A\|_m < \|A\|_{\Lambda,\Sigma,r}$. Then we get that $\|A \bullet B\|_{\Lambda,\Sigma,r} < \|A\|_{\Lambda,\Sigma,r}$, for all $B \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$ with $\|B\|_{\Lambda,\Sigma,r} \leq 1$. From the definition of C^*-algebra, we have $\|a_{jk}\| = \left\|a^*_{jk}\right\|$ for all $a_{jk} \in \mathfrak{C}$. Then we get $\|A\|_{\Lambda,\Sigma,r} = \left\|A^\bigstar\right\|_{\Lambda,\Sigma,r}$ and so

$$\left\|A \bullet \frac{A^\bigstar}{\|A\|_{\Lambda,\Sigma,r}}\right\|_{\Lambda,\Sigma,r} < \|A\|_{\Lambda,\Sigma,r}.$$

Thus

$$\left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r} = \left\|A \bullet A^\bigstar\right\|_{\Lambda,\Sigma,r} < \|A\|_{\Lambda,\Sigma,r}^2,$$

which is a contradiction. So, $\|A\|_m = \|A\|_{\Lambda,\Sigma,r}$.

Conversely, assume that $\|A\|_m = \|A\|_{\Lambda,\Sigma,r}$. Since we already have $\left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r} \leq \|A\|_{\Lambda,\Sigma,r}^2$, it suffices to show that

$$\left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r} \geq \|A\|_{\Lambda,\Sigma,r}^2.$$

Let $B \in S^r_{\Lambda,\Sigma}(\mathfrak{C})$ with $\|B\|_{\Lambda,\Sigma,r} \leq 1$. Now by using Lemma 5.1 we have

$$\|A \bullet B\|_{\Lambda,\Sigma,r} \leq \left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r}^{1/2} \left\|B^\bigstar \bullet B\right\|_{\Lambda,\Sigma,r}^{1/2} \leq \left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r}^{\frac{1}{2}}.$$

This implies that $\|A\|_m \leq \left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r}^{\frac{1}{2}}$. But by assumption we have $\|A\|_m = \|A\|_{\Lambda,\Sigma,r}$, hence $\|A\|_{\Lambda,\Sigma,r}^2 \leq \left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r}$ then we have $\left\|A^\bigstar \bullet A\right\|_{\Lambda,\Sigma,r} = \|A\|_{\Lambda,\Sigma,r}^2$. The proof is complete. ∎

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