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Cauchy Means for Signed Measures

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Abstract. In this paper we introduce Cauchy means for signed measures of the Boas type. We show that these means are monotonic.

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1. Introduction

The following generalization of the Jensen-Steffensen inequality was obtained by Boas [4] (see also, [9, p.59]).

Theorem 1.1. *If $f : [a, b] \to \mathbb{R}$ is continuous and monotonic (either increasing or decreasing) and λ be either continuous or of bounded variation satisfying*

$$\lambda(a) \leq \lambda(t) \leq \lambda(b) \text{ for all } t \in [a, b], \quad \lambda(b) > \lambda(a), \tag{1.1}$$

then for a convex function $\phi : I \to \mathbb{R}$, where I is the range of function f, we have

$$\phi\left(\frac{1}{\lambda(b) - \lambda(a)} \int_a^b f(x) d\lambda(x)\right) \leq \frac{1}{\lambda(b) - \lambda(a)} \int_a^b \phi(f(x)) d\lambda(x). \tag{1.2}$$

The following result states the Jensen-Boas inequality (see [9, p.59]).

Theorem 1.2. *If λ is continuous or of bounded variation satisfying*

$$\lambda(a) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \ldots \leq \lambda(y_{n-1}) \leq \lambda(x_n) \leq \lambda(b), \tag{1.3}$$

for all $x_k \in (y_{k-1}, y_k)$, $y_0 = a, y_n = b$ and $\lambda(b) > \lambda(a)$ and if f is continuous and monotonic (either increasing or decreasing) in each of the $n-1$ intervals (y_{k-1}, y_k), then the inequality (1.2) is still valid under the same conditions on ϕ.

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Let $[a, b]$ be an interval equipped with a positive measure λ. Then for a positive function f for which f^r is integrable, the integral power mean of f of order $r \in \mathbb{R}$, is defined as follows:

$$M_r(f, \lambda) = \begin{cases} \left(\frac{1}{\lambda(b)-\lambda(a)} \int_a^b (f(u))^r \, d\lambda(u) \right)^{\frac{1}{r}}, & r \neq 0; \\ \exp\left(\frac{1}{\lambda(b)-\lambda(a)} \int_a^b \log(f(u)) \, d\lambda(u) \right), & r = 0. \end{cases} \tag{1.4}$$

Throughout our present investigation, we tacitly assume, without further comment, that all the integrals involved in our results exists. In [3] we define new Cauchy's means $M_{r,l}^s$ as (special case of our results in [3]):

$$M_{r,l}^s(f, \lambda) = \left(\frac{l(l-s)M_r^r(f, \lambda) - M_s^r(f, \lambda)}{r(r-s)M_l^l(f, \lambda) - M_s^l(f, \lambda)} \right)^{\frac{1}{r-l}}. \tag{1.5}$$

In this paper we define Cauchy's type means by using Jensen-Steffensen's and Jensen-Boas's inequalities.

Let us note that power means related to (1.4) in the case when the weights satisfies Jensen-Steffensen's conditions are given in [1]. In this paper we introduce means of the Cauchy type similar to (1.5) for signed measures.

2. Main results

For our results in this section we need the following useful lemma.

Lemma 2.1. *Let $f \in C^2(I)$ be such that f'' is bounded (if I is compact interval, then this is superfluous) and $m = \min f''$ ($\inf f''$ if I is not a compact interval), $M = \max f''$ ($\sup f''$ if I is not a compact interval). Consider the functions ϕ_1, ϕ_2 defined as,*

$$\phi_1(t) = \frac{M}{2}t^2 - f(t), \tag{2.1}$$

$$\phi_2(t) = f(t) - \frac{m}{2}t^2, \tag{2.2}$$

then ϕ_1 and ϕ_2 are convex functions.

Theorem 2.1. *Let I be a compact real interval and $f \in C^2(I)$. Let λ be either continuous or of bounded variation satisfying $\lambda(a) \le \lambda(t) \le \lambda(b)$ for all $t \in [a, b]$, $\lambda(b) > \lambda(a)$, and let $h : [a, b] \to I$ be continuous and monotonic (either increasing or decreasing). Then there exists $\xi \in I$ such that,*

$$\begin{aligned} &\frac{1}{\lambda(b)-\lambda(a)} \int_a^b f(h(t)) d\lambda(t) - f\left(\frac{1}{\lambda(b)-\lambda(a)} \int_a^b h(t) d\lambda(t) \right) \\ &= \frac{f''(\xi)}{2} \left[\frac{1}{\lambda(b)-\lambda(a)} \int_a^b (h(t))^2 d\lambda(t) - \left(\frac{1}{\lambda(b)-\lambda(a)} \int_a^b h(t) d\lambda(t) \right)^2 \right]. \end{aligned} \tag{2.3}$$

Proof. Since f'' is continuous on I (I is compact interval), it is bounded, let $\max f'' = M$, $\min f'' = m$. Then in Theorem 1.1, by setting $\phi = \phi_1$ and $\phi = \phi_2$ respectively as defined in Lemma 2.1, we get the following inequalities

$$\frac{1}{\lambda(b)-\lambda(a)}\int_a^b f(h(t))d\lambda(t) - f\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)$$
$$\leq \frac{M}{2}\left[\frac{1}{\lambda(b)-\lambda(a)}\int_a^b (h(t))^2 d\lambda(t) - \left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)^2\right], \tag{2.4}$$

$$\frac{1}{\lambda(b)-\lambda(a)}\int_a^b f(h(t))d\lambda(t) - f\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)$$
$$\geq \frac{m}{2}\left[\frac{1}{\lambda(b)-\lambda(a)}\int_a^b (h(t))^2 d\lambda(t) - \left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)^2\right]. \tag{2.5}$$

Now by combining both inequalities and by applying the mean value theorem to the function $f \in C^2(I)$ there exists $\xi \in I$ such that

$$\frac{1}{\lambda(b)-\lambda(a)}\int_a^b f(h(t))d\lambda(t) - f\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)$$
$$= \frac{f''(\xi)}{2}\left[\frac{1}{\lambda(b)-\lambda(a)}\int_a^b (h(t))^2 d\lambda(t) - \left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)^2\right].$$ ∎

Theorem 2.2. *Let I be a compact real interval and $f, g \in C^2(I)$. Let λ be either continuous or of bounded variation satisfying $\lambda(a) \leq \lambda(t) \leq \lambda(b)$ for all $t \in [a,b]$, $\lambda(b) > \lambda(a)$, let $h : [a,b] \to I$ be continuous and monotonic (either increasing or decreasing) and let*

$$\frac{1}{\lambda(b)-\lambda(a)}\int_a^b (h(t))^2 d\lambda(t) - \left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)^2 \neq 0.$$

Then there exists $\xi \in I$ such that,

$$\frac{f''(\xi)}{g''(\xi)} = \frac{\frac{1}{\lambda(b)-\lambda(a)}\int_a^b f(h(t))d\lambda(t) - f\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)}{\frac{1}{\lambda(b)-\lambda(a)}\int_a^b g(h(t))d\lambda(t) - g\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)}, \tag{2.6}$$

provided that denominators are non-zero.

Proof. Consider the function $k \in C^2(I)$ defined as:

$$k = c_1 f - c_2 g,$$

where c_1, c_2 are defined as:

$$c_1 = \frac{1}{\lambda(b)-\lambda(a)}\int_a^b g(h(t))d\lambda(t) - g\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right), \tag{2.7}$$

$$c_2 = \frac{1}{\lambda(b)-\lambda(a)}\int_a^b f(h(t))d\lambda(t) - f\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right). \tag{2.8}$$

Using Theorem 2.1 with $f = k$, we have

$$\left(c_1\frac{f''(\xi)}{2} - c_2\frac{g''(\xi)}{2}\right)\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b (h(t))^2 d\lambda(t)\right.$$
$$\left. - \left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)^2\right) = 0. \tag{2.9}$$

Since

$$\frac{1}{\lambda(b)-\lambda(a)}\int_a^b (h(t))^2 d\lambda(t) - \left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b h(t)d\lambda(t)\right)^2 \neq 0,$$

from (2.9) we get

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}. \tag{2.10}$$

After putting values from (2.7) and (2.8) we get (2.6). ∎

Let λ be either continuous or of bounded variation satisfying $\lambda(a) \leq \lambda(t) \leq \lambda(b)$ for all $t \in [a, b]$ and $\lambda(b) > \lambda(a)$. Then for a strictly monotone continuous function α the quasi arithmetic mean is defined as follows [1]:

$$M_\alpha(t,\lambda) = \alpha^{-1}\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b \alpha(t)d\lambda(t)\right).$$

Theorem 2.3. *Let I be a compact real interval and $\alpha, \beta, \gamma \in C^2(I)$ be strictly monotonic functions. Then*

$$\frac{\alpha(M_\alpha(t,\lambda)) - \alpha(M_\gamma(t,\lambda))}{\beta(M_\beta(t,\lambda)) - \beta(M_\gamma(t,\lambda))} = \frac{\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)}{\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)} \tag{2.11}$$

for some η in I provided that the denominators are non-zero.

Proof. Choosing $f = \alpha \circ \gamma^{-1}$, $g = \beta \circ \gamma^{-1}$ and $h(t) = \gamma(\mu(t))$, from (2.6) we get, for some ξ,

$$\frac{\alpha(M_\alpha(t,\lambda)) - \alpha(M_\gamma(t,\lambda))}{\beta(M_\beta(t,\lambda)) - \beta(M_\gamma(t,\lambda))}$$
$$= \frac{\alpha''(\gamma^{-1}(\xi))\gamma'(\gamma^{-1}(\xi)) - \alpha'(\gamma^{-1}(\xi))\gamma''(\gamma^{-1}(\xi))}{\beta''(\gamma^{-1}(\xi))\gamma'(\gamma^{-1}(\xi)) - \beta'(\gamma^{-1}(\xi))\gamma''(\gamma^{-1}(\xi))}. \tag{2.12}$$

Thus by setting $\gamma^{-1}(\xi) = \eta$ for some η in I, we have

$$\frac{\alpha(M_\alpha(t,\lambda)) - \alpha(M_\gamma(t,\lambda))}{\beta(M_\beta(t,\lambda)) - \beta(M_\gamma(t,\lambda))} = \frac{\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)}{\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)}. \tag{2.13}$$ ∎

Remark 2.1. In the case λ is a nonnegative measure, the related results are given in [3], by A. Mercer in [5, 6] and by J. Pečarić, I. Perić and H. M. Srivastava in [8].

Corollary 2.1. *Consider the function:*

$$\chi(\eta) = \frac{\alpha(M_\alpha(t,\lambda)) - \alpha(M_\gamma(t,\lambda))}{\beta(M_\beta(t,\lambda)) - \beta(M_\gamma(t,\lambda))}$$

for $t \in [a,b]$. *If* χ *has an inverse* χ^{-1}, *then*

$$a \le \eta = \chi^{-1}\left(\frac{\alpha(M_\alpha(t,\lambda)) - \alpha(M_\gamma(t,\lambda))}{\beta(M_\beta(t,\lambda)) - \beta(M_\gamma(t,\lambda))}\right) \le b,$$

that is,

$$M_{\alpha,\beta} = \chi^{-1}\left(\frac{\alpha(M_\alpha(t,\lambda)) - \alpha(M_\gamma(t,\lambda))}{\beta(M_\beta(t,\lambda)) - \beta(M_\gamma(t,\lambda))}\right)$$

is a mean.

Now, from the results given above, we deduce the corresponding results for integral power means. Indeed for $r \in \mathbb{R}$, the integral power mean is defined as follows:

$$M_r(g,\lambda) = \begin{cases} \left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b g(t)d\lambda(t)\right)^{\frac{1}{r}}, & r \neq 0; \\ \exp\left(\frac{1}{\lambda(b)-\lambda(a)}\int_a^b \log g(t)d\lambda(t)\right), & r = 0, \end{cases} \tag{2.14}$$

where g is positive function for which g^r and $\log g$ are integrable.

Corollary 2.2. *Let* $r, s, l \in \mathbb{R}$ *and* λ *be either continuous or of bounded variation satisfying* $\lambda(a) \le \lambda(t) \le \lambda(b)$ *for all* $t \in [a,b]$ *and* $\lambda(b) > \lambda(a)$. *Then*

$$\frac{M_r^r(g,\lambda) - M_s^r(g,\lambda)}{M_l^l(g,\lambda) - M_s^l(g,\lambda)} = \frac{r(r-s)}{l(l-s)}\eta^{r-l}. \tag{2.15}$$

Proof. If we set $\alpha(t) = t^r, \beta(t) = t^l, \gamma(t) = t^s$ in equation (2.11) we get the required result. ∎

Remark 2.2. Let us note that (2.15) in the case of positive measure is obtained in [3].

Now from (2.15) we have

$$\eta = \left(\frac{l(l-s)}{r(r-s)}\frac{M_r^r(g,\lambda) - M_s^r(g,\lambda)}{M_l^l(g,\lambda) - M_s^l(g,\lambda)}\right)^{\frac{1}{r-l}}, \tag{2.16}$$

so we can again introduce new means similar to (1.5) but for negative measure λ.

3. Cauchy's means

Similarly to (2.4) in [3] for positive measure λ, we define, by using (2.1) a new mean $M_{r,l}^s$ for signed measure as follows:

$$M_{r,l}^s(f,\lambda) = \left(\frac{l(l-s)}{r(r-s)}\frac{M_r^r(f,\lambda) - M_s^r(f,\lambda)}{M_l^l(f,\lambda) - M_s^l(f,\lambda)}\right)^{\frac{1}{r-l}}, \quad l \neq r \neq s, l, r \neq 0;$$

under the same conditions as in Corollary 2.2.

By calculating appropriate limits the complete definition of $M_{r,l}^s(f,\lambda)$ is as follows:

$$
\begin{aligned}
&M^s_{r,l}(f,\lambda) = \left(\frac{l(l-s)}{r(r-s)}\frac{M^r_r(f,\lambda)-M^r_s(f,\lambda)}{M^l_l(f,\lambda)-M^l_s(f,\lambda)}\right)^{\frac{1}{r-l}},\\
&l \neq r \neq s, l, r \neq 0;\\
&M^s_{r,0}(f,\lambda) = M^s_{0,r}(f,\lambda) = \left(\frac{s[M^r_r(f,\lambda)-M^r_s(f,\lambda)]}{r(r-s)[\log M_s(f,\lambda)-\log M_0(f,\lambda)]}\right)^{\frac{1}{r}},\\
&r \neq s, r, s \neq 0;\\
&M^s_{s,l}(f,\lambda) = M^s_{l,s}(f,\lambda)\\
&= \left(\frac{l(l-s)}{s}\frac{\frac{1}{\lambda(b)-\lambda(a)}\int f(u)^s \log f(u) d\lambda(u) - M^s_s(f,\lambda)\log M_s(f,\lambda)}{M^l_l(f,\lambda)-M^l_s(f,\lambda)}\right)^{\frac{1}{s-l}},\\
&l \neq s, l, s \neq 0;\\
&M^s_{s,0}(f,\lambda) = M^s_{0,s}(f,\lambda)\\
&= \left(\frac{\frac{1}{\lambda(b)-\lambda(a)}\int f(u)^s \log f(u) d\lambda(u) - M^s_s(f,\lambda)\log M_s(f,\lambda)}{\log M_s(f,\lambda)-\log M_0(f,\lambda)}\right)^{\frac{1}{s}}, s \neq 0;\\
&M^0_{r,l}(f,\lambda) = \left(\frac{l^2(M^r_r(f,\lambda)-M^r_0(f,\lambda))}{r^2(M^l_l(f,\lambda)-M^l_0(f,\lambda))}\right)^{\frac{1}{r-l}}, \quad l, r \neq 0;\\
&M^0_{r,0}(f,\lambda) = M^0_{0,r}(f,\lambda) = \left(\frac{2[M^r_r(f,\lambda)-M^r_0(f,\lambda)]}{r^2[M^2_2(\log f,\lambda)-M^2_1(\log f,\lambda)]}\right)^{\frac{1}{r}}, \quad r \neq 0.\\
&M^s_{t,t} = \exp\left(-\frac{2t-s}{t(t-s)} + \frac{\frac{1}{\lambda(b)-\lambda(a)}\int f^t \log f d\lambda(u) - M^t_s(f,\lambda)\log M_s(f,\lambda)}{M^t_t(f,\lambda)-M^t_s(f,\lambda)}\right), \quad t \neq s;\\
&M^0_{t,t} = \exp\left(-\frac{2}{t} + \frac{\frac{1}{\lambda(b)-\lambda(a)}\int f^t \log f d\lambda(u) - M^t_0(f,\lambda)\log M_0(f,\lambda)}{M^t_t(f,\lambda)-M^t_0(f,\lambda)}\right), t \neq 0;\\
&M^0_{0,0} = \exp\left(\frac{1}{3}\frac{\int(\log f)^3 d\lambda(u) - (\log M_0(f,\lambda))^3}{\int(\log f)^2 d\lambda(u) - (\log M_0(f,\lambda))^2}\right),\\
&M^s_{s,s} = \exp\left(-\frac{1}{s} + \frac{\frac{1}{\lambda(b)-\lambda(a)}\int f^s(\log f)^2 d\lambda(u) - M^s_s(f,\lambda)(\log M_s(f,\lambda))^2}{2(\int f^s \log f d\lambda(u) - (M^s_s(f,\lambda)\log M_s(f,\lambda)))}\right), \quad s \neq 0;\\
&M^s_{0,0} = \exp\left(\frac{1}{s} + \frac{\frac{1}{\lambda(b)-\lambda(a)}\int(\log f)^2 d\lambda(u) - (\log M_s(f,\lambda))^2}{2(\int \log f d\lambda(u) - \log M_s(f,\lambda))}\right), \quad s \neq 0.
\end{aligned} \tag{3.1}
$$

4. Monotonicity

In this section we prove the monotonicity of (3.1). The following two lemmas can be obtained similar to Theorem 1 in [2] (see also Remark 2 in [2]):

Lemma 4.1. *Let Λ_t be defined as:*

$$
\Lambda_t(g,\lambda) = \begin{cases} \frac{M^t_t(g,\lambda)-M^t_1(g,\lambda)}{t(t-1)}, & t \neq 0, 1; \\ \log M_1(g,\lambda) - M_1(\log g,\lambda), & t = 0; \\ M_1(g\log g,\lambda) - M_1(g,\lambda)M_1(\log g,\lambda), & t = 1, \end{cases} \tag{4.1}
$$

and let Λ_t be positive. Then Λ_t is a log-convex function.

Lemma 4.2. *Let Ω_t be defined as*

$$
\Omega_t = \begin{cases} \frac{1}{t^2}(M^t_t(f,\lambda) - M^t_0(f,\lambda)), & t \neq 0; \\ \frac{1}{2}\left(M^2_2(\log f,\lambda) - M^2_1(\log f,\lambda)\right), & t = 0, \end{cases} \tag{4.2}
$$

and let Ω_t be positive. Then Ω_t is a log-convex function.

We need the following lemma (see [3]).

Lemma 4.3. *Let f be a log-convex function. Then for any $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$ the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{4.3}$$

Theorem 4.1. *Let $t, r, u, v \in \mathbb{R}$ be such that, $t \leq v, r \leq u$. Then for* (3.1) *we have*

$$M^s_{t,r} \leq M^s_{v,u}. \tag{4.4}$$

Proof.

Case I: ($s \neq 0$). Consider Λ_t defined as in Lemma 4.1. Λ_t is continuous and log-convex. So Lemma 4.3 implies that for $t, r, u, v \in \mathbb{R}$ such that $t \leq v, r \leq u, t \neq r, v \neq u$, we have

$$\left(\frac{\Lambda_t}{\Lambda_r}\right)^{\frac{1}{t-r}} \leq \left(\frac{\Lambda_v}{\Lambda_u}\right)^{\frac{1}{v-u}}. \tag{4.5}$$

For $s > 0$, by substituting $g = f^s, t = t/s, r = r/s, u = u/s, v = v/s \in \mathbb{R}$ such that $t/s \leq v/s, r/s \leq u/s, t \neq r, v \neq u$ in (4.1) we get

$$\Lambda_{t,s}(f,\lambda) = \begin{cases} \frac{s^2}{t(t-s)}[M_t^t(f,\lambda) - M_s^t(f,\lambda)], & t \neq 0, s; \\ \log M_s(f,\lambda) - M_s(\log f, \lambda), & t = 0; \\ M_s(f \log f, \lambda) - M_s(f,\lambda)\log M_s(\log f, \lambda)), & t = s, \end{cases} \tag{4.6}$$

and (4.5) becomes,

$$\left(\frac{\Lambda_{t,s}}{\Lambda_{r,s}}\right)^{\frac{1}{t-r}} \leq \left(\frac{\Lambda_{v,s}}{\Lambda_{u,s}}\right)^{\frac{1}{v-u}}. \tag{4.7}$$

From (4.7) we get the required result.

Now when $s < 0$, by substituting $g = f^s, t = t/s, r = r/s, u = u/s, v = v/s \in \mathbb{R}$ such that $v/s \leq t/s, u/s \leq r/s, t \neq r, v \neq u$ in (4.1) we get (4.6), and (4.5) becomes,

$$\left(\frac{\Lambda_{v,s}}{\Lambda_{u,s}}\right)^{\frac{s}{v-u}} \leq \left(\frac{\Lambda_{t,s}}{\Lambda_{r,s}}\right)^{\frac{s}{t-r}}. \tag{4.8}$$

From (4.8), by raising to the power of $-\frac{1}{s}$, we get

$$\left(\frac{\Lambda_{t,s}}{\Lambda_{r,s}}\right)^{\frac{1}{t-r}} \leq \left(\frac{\Lambda_{v,s}}{\Lambda_{u,s}}\right)^{\frac{1}{v-u}}. \tag{4.9}$$

From (4.9) we get the required result.

Case II: ($s = 0$). In this case we can get our result by taking limit $s \to 0$ in (4.6) and also in this case we can consider Ω_t defined as in Lemma 4.2.

Ω_t is log-convex function. So Lemma 4.3 implies that for $t, r, u, v \in \mathbb{R}$ such that $t \leq v, r \leq u, t \neq r, v \neq u$, we have

$$\left(\frac{\Omega_t}{\Omega_r}\right)^{\frac{1}{t-r}} \leq \left(\frac{\Omega_v}{\Omega_u}\right)^{\frac{1}{v-u}}.$$

Therefore we have for $t, r, u, v \in \mathbb{R}$ such that, $t \leq v, r \leq u, t \neq r, v \neq u$,

$$M^0_{t,r} \leq M^0_{v,u}.$$

This completes the proof. ∎

Remark 4.1. Similar results can be proved by using Theorem 1.2 instead of Theorem 1.1.

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