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Ideal Theory in the Ternary Semiring $\mathbb{Z}_0^-$

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Abstract. In this paper, we study the ideal theory in the ternary semiring $\mathbb{Z}_0^-$ of non-positive integers and obtain some results regarding the ideals of the ternary semiring $\mathbb{Z}_0^-$. Finally we show that $\mathbb{Z}_0^-$ is a Noetherian ternary semiring and also almost principal ideal ternary semiring.

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1. Introduction

In [2], we have introduced the notion of ternary semiring which generalizes the notion of ternary ring introduced by W. G. Lister [15]. The set $\mathbb{Z}_0^-$ of all non-positive integers is an example of a ternary semiring with usual binary addition and ternary multiplication. In [4, 5, 6] we have characterized respectively the prime, semiprime and maximal ideals of the ternary semiring $\mathbb{Z}_0^-$. Some works on ternary semiring may be found in [2, 3, 7, 8, 9, 10, 13, 14].

Our main purpose of this paper is to study the ideal theory in the ternary semiring $\mathbb{Z}_0^-$. In Section 2, we give some basic definitions and examples. In Section 3, we study the ideal theory in the ternary semiring $\mathbb{Z}_0^-$ and prove that $\mathbb{Z}_0^-$ is a Noetherian ternary semiring.

2. Preliminaries

Definition 2.1. *A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:*

(i) $(abc)de = a(bcd)e = ab(cde)$, (ii) $(a+b)cd = acd + bcd$,
(iii) $a(b+c)d = abd + acd$, (iv) $ab(c+d) = abc + abd$,

for all $a, b, c, d, e \in S$.

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Definition 2.2. *Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$ then '0' is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.*

Example 2.1. Let $\mathbb{Z}_0^-$ be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, $\mathbb{Z}_0^-$ forms a ternary semiring with zero.

Definition 2.3. *An additive subsemigroup T of a ternary semiring S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 2.4. *An additive subsemigroup I of a ternary semiring S is called a left (right, lateral) ideal of S if s_1s_2i (respectively is_1s_2, s_1is_2) $\in I$ for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, a right, a lateral ideal of S, then I is called an ideal of S.*

Definition 2.5. *An ideal I of a ternary semiring S is called a k-ideal if $x + y \in I$; $x \in S$, $y \in I$ imply that $x \in I$.*

Definition 2.6. *Let I be an ideal of a ternary semiring S. A subset B of I is called a basis for I if every element of I can be written in the form $\sum_{i=1}^{n} r_is_ib_i$, where $r_i, s_i \in S$ and $b_i \in B$.*

If the set B is finite, then B is called a finite basis for I.

Through out the rest of the paper, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{Z}^+$ denotes the set of all positive integers, $\mathbb{Z}^-$ denotes the set of all negative integers, $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$.

3. Ideal theory in the ternary semiring $\mathbb{Z}_0^-$

In this section we study the ideal theory in the ternary semiring of non-positive integers $\mathbb{Z}_0^-$ and classify them. The ring of integers $\mathbb{Z}$ plays a vital role in the theory of rings and it is well known that the ring of integers $\mathbb{Z}$ is a principal ideal ring (PIR) and hence a Noetherian ring. In [1], Allen and Dale proved that the semiring of nonnegative integers $\mathbb{Z}_0^+$ is a Noetherian semiring. Again we note that the semiring $\mathbb{Z}_0^+$ is not a principal ideal semiring but Allen and Dale [1] proved that $\mathbb{Z}_0^+$ is an almost principal ideal semiring. In [4], we have proved that the ternary semiring $\mathbb{Z}_0^-$ is a principal k-ideal ternary semiring but not principal ideal ternary semiring. We show that $\mathbb{Z}_0^-$ is an almost principal ideal ternary semiring. We also show that $\mathbb{Z}_0^-$ is a Noetherian ternary semiring.

Let $n \in \mathbb{Z}_0^-$ and $T_n = \{t \in \mathbb{Z}_0^- | t \leq n\} \cup \{0\}$. Then we have the following elementary results concerning T_n.

Theorem 3.1. *T_n is an ideal in $\mathbb{Z}_0^-$ such that*

(i) $T_0 = T_{-1} = \mathbb{Z}_0^-$.
(ii) $m \leq n \leq -1$ *if and only if* $T_m \subseteq T_n$.
(iii) $T_m \cup T_n = T_p$, *where* $p = \max\{m, n\}$.
(iv) $T_m \cap T_n = T_q$, *where* $q = \min\{m, n\}$.
(v) $\bigcap\{T_i : i \in \mathbb{Z}_0^-\} = \{0\}$.

Proof. We first prove that T_n is an ideal of $\mathbb{Z}_0^-$. Let $a, b \in T_n$. Then $a \leq n$ and $b \leq n$. So $a + b \leq 2n \leq n$. Again, if $r, s \in \mathbb{Z}_0^-$, where $r \neq 0, s \neq 0$; then $rsa \leq rsn \leq n$. Therefore, $a + b \in T_n$ and $rsa \in T_n$. Consequently, T_n is an ideal of $\mathbb{Z}_0^-$.

The proof of properties (i) to (v) are straightforward and therefore omitted. ∎

Remark 3.1. Note that $T_n (n \neq 0, -1)$ is not a k-ideal of the ternary semiring $\mathbb{Z}_0^-$.

Theorem 3.2. *$\mathbb{Z}_0^-$ satisfies the ascending chain condition on T_n-ideals.*

Proof. Let $\{T_{n_\alpha}\}$ be an ascending chain of T_n-ideals in $\mathbb{Z}_0^-$. Then it is finite since by Theorem 3.1, the increasing sequence $\{n_\alpha\}$ of negative integers is finite. Thus there exists $j \in \mathbb{Z}_0^-$ such that $n_i = n_j$ for each $i \leq j$. Therefore, $T_{n_i} = T_{n_j}$ for each $i \leq j$ and hence $\mathbb{Z}_0^-$ satisfies the ascending chain condition on T_n-ideals. ∎

For $a, b \in \mathbb{Z}_0^-$ the notation $S(a, b)$ will be used to denote the set $\{t \in \mathbb{Z}_0^- | a \leq t \leq b\}$.

Theorem 3.3. *If $n < -1$, then $S(2n, n)$ is a finite basis for T_n.*

Proof. Let $x \in T_n$. If $x \in S(2n, n)$ or $x = cdn$ for some $c, d \in \mathbb{Z}_0^-$, then x is generated by $S(2n, n)$. Let $x < 2n$ and $x \neq cdn$ for any $c, d \in \mathbb{Z}_0^-$. Then there exists a $k < -2$ such that $-kn < x < -(k+1)n$. However, this guarantees the existence of an $m > n$ such that $-(k+1)n + m = x$, and it follows that $n + m \in S(2n, n)$. Therefore, $x = -(k+1)n + m = -(k+2)n + n + m$, where $n \in S(2n, n)$ and $n + m \in S(2n, n)$. Thus it follows that $S(2n, n)$ is a basis for T_n. ∎

Now we study some lemmas which will be essential for the characterization of all ideals in the ternary semiring $\mathbb{Z}_0^-$. From these lemmas we have some methods by which we can determine if an ideal in $\mathbb{Z}_0^-$ contains a T_n-ideal.

Lemma 3.1. *Let I be an ideal in the ternary semiring $\mathbb{Z}_0^-$. If $a \in I, m \in \mathbb{Z}_0^-$, where $m \neq 0$, and $S(-(m-1)a, -ma) \subset I$, then there exists an $n \in \mathbb{Z}_0^-$ such that $T_n \subset I$.*

Proof. Suppose $x \in \mathbb{Z}_0^-$. If $x = cda$ for some $c, d \in \mathbb{Z}_0^-$, then clearly $x \in I$. Next suppose that $x < -(m-1)a$ and $x \neq cda$ for $c, d \in \mathbb{Z}_0^-$. Since there exists $k \leq m-1$ such that $-(k-1)a < x < -ka$, we have $-(k-1)a - b = x$ for some $b \in \mathbb{Z}_0^-$ with $b > a$. Clearly, $b > a$ implies that $-(m-1)a - b \in S(-(m-1)a, -ma) \subset I$. Therefore, $x = -(k-1)a - b = -(k-m)a + (-(m-1)a - b) \in I$. Consequently, $T_{-(m-1)a} \subset I$ and hence the proof of the lemma follows. ∎

Lemma 3.2. *Let I be an ideal in $\mathbb{Z}_0^-$. If there exists $a \in I$ such that $a + (-1) \in I$, then there exists an n such that $T_n \subset I$.*

Proof. If I is a T_n-ideal, then the lemma is obvious. Suppose I is not a T_n-ideal and a is the greatest element in I such that $a + (-1) \in I$. Since I is an ideal, a series of simple calculations shows that the following elements belong to I:

$-(-1)a + (-1), -(-1)a$
$-(-2)a + (-2), -(-2)a + (-1), -(-2)a$
$-(-3)a + (-3), -(-3)a + (-2), -(-3)a + (-1), -(-3)a$
. . . .
. . . .
$-(a)a + (a) = -(a-1)a, ..., -(a)a + (-3), -(a)a + (-2), -(a)a + (-1), -(a)a.$

The last row of elements is $S(-(a-1)a, -a^2)$ and hence $S(-(a-1)a, -a^2) \subset I$. Thus there exists an $n \in \mathbb{Z}_0^-$ such that $T_n \subset I$, by using Lemma 3.1. ∎

Lemma 3.3. *Let $a \in \mathbb{Z}_0^-$ and $b \in \mathbb{Z}_0^-$, where $a \neq 0$ and $b \neq 0$. If $d \in \mathbb{Z}^-$ such that $-d$ is the greatest common divisor of a and b, then there exists $s \in \mathbb{Z}_0^-$ and $t \in \mathbb{Z}_0^-$ such that $(-1)sa = (-1)tb + d$ or $(-1)tb = (-1)sa + d$.*

Proof. From elementary number theory, it is well known that $-d = s'(-a) + t'(-b)$ for some integers s' and t'. Since $0 \leq -d \leq -a$, $0 \leq -d \leq -b$ and $(-a), (-b)$ and $(-d)$ are all positive, it follows that $s' \geq 0$ and $t' \leq 0$ or $s' \leq 0$ and $t' \geq 0$. If $s' \geq 0$ and $t' \leq 0$, then $-d = s'(-a) + t'(-b) \Rightarrow (-1)s'(-a) = (-1)t'b + d \Rightarrow (-1)(-s')a = (-1)t'b + d$. Thus $(-1)sa = (-1)tb + d$, where $s = -s' \leq 0$ and $t = t' \leq 0$. On the other hand, if $s' \leq 0$ and $t' \geq 0$, then $(-1)tb = (-1)sa + d$, where $s = s' \leq 0$ and $t = -t' \leq 0$. ∎

Lemma 3.4. *Let I be an ideal in $\mathbb{Z}_0^-$, $a \in I$ and $b \in I$. If a and b are relatively prime, then there exists an n such that $T_n \subset I$.*

Proof. Since a and b are relatively prime, the Lemma 3.3 guarantees the existence of $s \in \mathbb{Z}_0^-$ and $t \in \mathbb{Z}_0^-$ such that $(-1)sa = (-1)tb + (-1)$ or $(-1)tb = (-1)sa + (-1)$. Since I is an ideal it is clear that $(-1)sa \in I$ and $(-1)tb \in I$. Consequently, $(-1)sa + (-1) \in I$ or $(-1)tb + (-1) \in I$ and the lemma follows from Lemma 3.2. ∎

Remark 3.2. It is easy to see that for $m \neq n$, T_m and T_n differ by at most a finite number of elements. Since $\mathbb{Z}_0^- = T_{-1}$, it follows that $\mathbb{Z}_0^-$ differs from a T_n-ideal by at most a finite number of elements. Consequently, if I is an ideal in $\mathbb{Z}_0^-$ containing a T_n-ideal, then $T_n \subset I \subset \mathbb{Z}_0^-$ and it follows that $\mathbb{Z}_0^-$ and I differ by at most a finite number of elements. It will be shown that an ideal I in $\mathbb{Z}_0^-$ not containing a T_n-ideal differs from the multiples of some negative integer $d < -1$ by at most a finite number of elements. Consequently, if I is an ideal in $\mathbb{Z}_0^-$ not containing a T_n-ideal, then there exist $m \in \mathbb{Z}_0^-$ and $d \in \mathbb{Z}_0^-$, where $d < -1$, such that $(-1)dT_m \subset I \subset d\mathbb{Z}_0^-\mathbb{Z}_0^- =< d >$.

In view of the above remarks, the ideals in $\mathbb{Z}_0^-$ are classified according to the following definition.

Definition 3.1. *An ideal I in $\mathbb{Z}_0^-$ is called a T-ideal if $T_k \subset I$ for some $k \in \mathbb{Z}_0^-$. All other ideals in $\mathbb{Z}_0^-$ are called M-ideals.*

Note 3.2. It is clear that $\mathbb{Z}_0^-$ is a T-ideal and $\{0\}$ is an M-ideal.

Lemma 3.5. *Every non-empty subset of the set of all negative integers $\mathbb{Z}^-$ has a greatest element. In particular, $\mathbb{Z}^-$ itself has the greatest element (-1).*

Remark 3.3. The above Lemma 3.5 gives the dual notion of the well-known Well-Ordering Property of the set of all natural numbers $\mathbb{Z}^+$.

The following theorem gives a characterization of T-ideals in the ternary semiring $\mathbb{Z}_0^-$ and will be used to show that $\mathbb{Z}_0^-$ is a Noetherian ternary semiring.

Theorem 3.4. *An ideal I in $\mathbb{Z}_0^-$ is a T-ideal if and only if I has a finite basis and $I = K \cup T_k$, where T_k is the maximal T_n-ideal contained in I and $K = \{t \in I : k < t < 0\}$.*

Proof. Suppose I is a T-ideal and $T_n \subset I$. Let $S = \{n \in \mathbb{Z}_0^- : T_n \subset I\}$. Since I is a T-ideal, it follows that S is a non-empty subset of $\mathbb{Z}_0^-$ and by Lemma 3.5, S contains a greatest element, say k. Now by Theorem 3.1, $T_n \subset T_k$ for each $n \in S$ and it is clear that T_k is the maximal T_n-ideal contained in I. Letting $K = \{t \in I : k < t < 0\}$, we have $I = K \cup T_k$. According to Theorem 3.3, $S(2k, k)$ is a finite basis for T_k. Since K is a finite subset of I, $S(2k, k) \cup K$ is a finite basis for I.

The converse of the theorem is obvious. ∎

Now we have the following theorem analogous to Theorem 3.1.

Theorem 3.5. *If $n \in \mathbb{Z}_0^-$ and $d \in \mathbb{Z}_0^-$, then $(-1)dT_n$ is an ideal in $\mathbb{Z}_0^-$ such that*

(i) $(-1)dT_{-1} = d\mathbb{Z}_0^-\mathbb{Z}_0^- =< d >$, $(-1)dT_n = T_n$ *if and only if* $d = -1$ *and* $(-1)dT_n = \{0\}$ *if and only if* $d = 0$.

(ii) $m \leq n$ *if and only if* $(-1)dT_m \subseteq (-1)dT_n$.

(iii) $(-1)dT_m \cup (-1)dT_n = (-1)dT_p$, *where* $p = \max\{m, n\}$.

(iv) $(-1)dT_m \cap (-1)dT_n = (-1)dT_q$, *where* $q = \min\{m, n\}$.

(v) $\bigcap\{(-1)dT_n : n \in \mathbb{Z}_0^-\} = \{0\}$.

Proof. Suppose $x \in (-1)dT_n$ and $y \in (-1)dT_n$. Then there exist $k \leq n$ and $q \leq n$ such that $x = (-1)kd$ and $y = (-1)qd$. Clearly, $k + q \leq 2n \leq n$ and hence $x+y = (-1)kd+(-1)qd = (-1)(k+q)d \in (-1)dT_n$. If $r, s \in \mathbb{Z}_0^-$, where $r \neq 0, s \neq 0$, then $rsk \leq n$ and $rsx = rs(-1)kd = (-1)(rsk)d \in (-1)dT_n$. Therefore, $(-1)dT_n$ is an ideal in $\mathbb{Z}_0^-$. ∎

The proof of properties (i) to (v) are straightforward and therefore omitted.

Note 3.3. $(-1)dT_{-1} = (-1)d\mathbb{Z}_0^- = d\mathbb{Z}_0^-\mathbb{Z}_0^- =< d >$ is a k-ideal of the ternary semiring $\mathbb{Z}_0^-$. In [6], we have proved that $d\mathbb{Z}_0^-\mathbb{Z}_0^-$ is a maximal k-ideal and hence a prime k-ideal of $\mathbb{Z}_0^-$ if and only if d is prime.

It will be shown that for any ideal I in $\mathbb{Z}_0^-$ there exist $n \in \mathbb{Z}_0^-$ and $d \in \mathbb{Z}_0^-$ such that $(-1)dT_n$ is contained in I. Consequently, $(-1)dT_n$-ideal is the basic type of ideal in $\mathbb{Z}_0^-$ and the study of ideals in the ternary semiring $\mathbb{Z}_0^-$ is reduced to the problem of finding a maximal $(-1)dT_n$-ideal for each ideal in $\mathbb{Z}_0^-$. It has already observed in the previous Theorem 3.5 that $(-1)dT_n = T_n$ if $d = -1$ and $(-1)dT_n = \{0\}$ if $d = 0$. Consequently, it remains only to study the case for $d < -1$. For this purpose, in the remainder of this section it will be assumed that $d < -1$ unless otherwise stated.

The following three lemmas are analogous to the well-known properties of ideals in the ring of integers $\mathbb{Z}$.

Lemma 3.6. *If $p \in \mathbb{Z}_0^-$, $q \in \mathbb{Z}_0^-$ and p divides q, then $(-1)qT_n \subseteq (-1)pT_n$.*

Proof. Suppose $a \in (-1)qT_n$. Then there exists $k \leq n$ such that $a = (-1)kq$. Since p divides q, there exists $t \leq -1$ such that $q = (-1)tp$. Consequently, $a = (-1)kq = (-1)k(-1)tp = (-1)(-kt)p \in (-1)pT_n$, since $-kt \leq n$, and hence it follows that $(-1)qT_n \subseteq (-1)pT_n$. ∎

Lemma 3.7. *If $(-1)dT_c \subseteq (-1)bT_a$, then b divides d.*

Proof. Suppose $(-1)dT_c \subseteq (-1)bT_a$. Since $(-1)cd \in (-1)bT_a$, there exists $p \leq a$ such that $(-1)cd = (-1)pb$. This implies that b divides $(-1)cd$. Now by definition of $(-1)dT_c$, it follows that $(-1)(c+(-1))d \in (-1)bT_a$ and hence there exists $q \leq a$ such that $(-1)(c+(-1))d = (-1)qb$. Consequently, b divides $(-1)(c+(-1))d = (-1)cd+d$ and in view of the fact that b divides $(-1)cd$, it follows that b divides d. ∎

Corollary 3.1. *If $(-1)dT_c \subseteq (-1)bT_a$, then $d \leq b$.*

Proof. From the Lemma 3.7 it follows that b divides d and hence $d \leq b$, since b and d both are negative integers. ∎

Lemma 3.8. *If $(-1)bT_a \cap (-1)dT_c \neq \{0\}$, then there exist $p \in \mathbb{Z}_0^-$ and $q \in \mathbb{Z}_0^-$ such that $(-1)qT_p \subset (-1)bT_a \cap (-1)dT_c$.*

Proof. Let $x \in (-1)bT_a \cap (-1)dT_c$. Since $(-1)bT_a$ is an ideal of $\mathbb{Z}_0^-$, $x \in (-1)bT_a \Rightarrow (-1)xT_{-1} \subset (-1)bT_a$. Similarly, $(-1)xT_{-1} \subset (-1)dT_c$. Consequently, $(-1)xT_{-1} \subset (-1)bT_a \cap (-1)dT_c$ and hence the proof of the lemma follows. ∎

The following lemmas are essential to show that the ternary semiring $\mathbb{Z}_0^-$ is Noetherian on $(-1)dT_n$-ideals.

Lemma 3.9. *Any ascending sequence $\{(-1)bT_{a_j}\}$ of ideals in the ternary semiring $\mathbb{Z}_0^-$ is finite.*

Proof. Let $\{(-1)bT_{a_j}\}$ be an ascending sequence of ideals in the ternary semiring $\mathbb{Z}_0^-$. Then it is finite since by Theorem 3.5, the increasing sequence $\{a_j\}$ of negative integers is finite. Thus there exists $\alpha \in \mathbb{Z}_0^-$ such that $a_\alpha = a_n$ for each $n \leq \alpha$. Therefore, $(-1)bT_\alpha = (-1)bT_n$ for each $n \leq \alpha$ and hence the lemma follows. ∎

Lemma 3.10. *Any ascending sequence $\{(-1)b_iT_a\}$ of ideals in the ternary semiring $\mathbb{Z}_0^-$ is finite.*

Proof. Let $\{(-1)b_iT_a\}$ be an ascending sequence of ideals in the ternary semiring $\mathbb{Z}_0^-$. Then it is finite since by Corollary 3.1, the increasing sequence $\{b_i\}$ of negative integers is finite. Hence there exists an $\alpha \in \mathbb{Z}_0^-$ such that $b_\alpha = b_n$ for each $n \leq \alpha$. Thus it follows that $(-1)b_\alpha T_a = (-1)b_nT_a$ for each $n \leq \alpha$ and hence the lemma is proved. ∎

Theorem 3.6. *The ternary semiring $\mathbb{Z}_0^-$ satisfies the ascending chain condition on $(-1)dT_n$-ideals.*

Proof. Let $\{(-1)b_iT_{a_i}\}$ be an ascending chain of ideals in the ternary semiring $\mathbb{Z}_0^-$. Then by Lemma 3.10, it follows that there exists $\alpha \in \mathbb{Z}_0^-$ such that $b_\alpha = b_i$ for $i \leq \alpha$. Again by Lemma 3.9, it follows that there exists $\beta \in \mathbb{Z}_0^-$ such that $a_\beta = a_j$ for $j \leq \beta$. If $k = \min\{\alpha, \beta\}$, then $(-1)b_kT_{a_k} = (-1)b_pT_{a_p}$ for $p \leq k$. Hence the ternary semiring $\mathbb{Z}_0^-$ satisfies the ascending chain condition on $(-1)dT_n$-ideals. ∎

For $x \in \mathbb{Z}_0^-$, $y \in \mathbb{Z}_0^-$ and $d \in \mathbb{Z}_0^-$ where $d < -1$, we denote by $S_d(x, y)$ the set $\{k \in \mathbb{Z}_0^- : x \leq k \leq y$ and $k = (-1)md$ for some $m \in \mathbb{Z}_0^-\}$.

Theorem 3.7. *$S_d(-2nd, -nd)$ is a finite basis for $(-1)dT_n$.*

Proof. Let $p = (-1)qd \in (-1)dT_n$. If $p \in S_d(-2nd, -nd)$ or $p = and$ for some $a \in \mathbb{Z}_0^-$, then p is generated by $S_d(-2nd, -nd)$. Suppose $p \notin S_d(-2nd, -nd)$ and $p \neq and$ for $a \in \mathbb{Z}_0^-$. Since $p < -2nd$ and there exists $k < -2$ such that $knd < p < (k+1)nd$, it follows that there exists an $m = -td > -nd$ with $t \in \mathbb{Z}_0^-$ such that $(k+1)nd + m = p$. Now $m > -nd$ implies that $-nd + m = -nd - td = -(n+t)d \in$

$S_d(-2nd, -nd)$. Therefore, $p = (k+2)nd + (-nd+m)$ and hence $S_d(-2nd, -nd)$ is a finite basis for $(-1)dT_n$. ∎

Lemma 3.11. *Let I be an ideal in $\mathbb{Z}_0^-$, $a \in I$ and d divide a, where $d < -1$. If there exists $m \in \mathbb{Z}_0^-$ such that $S_d(-(m-1)a, -ma) \subset I$, then there exists $n \in \mathbb{Z}_0^-$ such that $(-1)dT_n \subset I$.*

Proof. If d divides a, then there exists $b \in \mathbb{Z}_0^-$ such that $a = -bd$ and it follows that $S_d(-(m-1)a, -ma) = S_d((mb-b)d, mbd)$. We shall show that $(-1)dT_{-(m-1)b} \subset I$. To show this we have to show that $x = -yd \in I$, where $y \leq -(m-1)b$. Clearly, if $x = -yd$, where $y = -(m-1)b$, then $x \in S_d(-(m-1)a, -ma) \subset I$. Next suppose that $x = -yd$, where $y < -(m-1)b$. Then it is clear that $x < -(m-1)a$ and there exists $k \leq -1$ such that $(k-1)ma < x < kma$. Consequently, there exists $r \in \mathbb{Z}_0^-$ such that $r > -ma$ and $kma+r = x$. Again $kma+r = -kmbd+r = x = -yd$ implies that $r = -cd$ for some $c = (y-kmb) \in \mathbb{Z}_0^-$. Also $-(m-1)a < -ma+r < -ma$ and it is easy to see that $-ma+r = mbd+(-cd) = -(-mb+c)d \in S_d(-(m-1)a, -ma) \subset I$. Therefore, $-ma+r \in I$ and $(k+1)ma \in I$ together imply that $x = -yd = kma+r = (k+1)ma + (-ma+r) \in I$. This implies that for each $y \leq -(m-1)b$, $-yd \in I$ and hence letting $n = -(m-1)b$ it is clear that $(-1)dT_n \subset I$. ∎

Lemma 3.12. *Let I be an ideal in $\mathbb{Z}_0^-$, $a \in I$ and $b \in I$. If a and b are not relatively prime, then there exists $n \in \mathbb{Z}_0^-$ and $d \in \mathbb{Z}^-$ such that $(-1)dT_n \subset I$, where $-d$ is the greatest common divisor of a and b.*

Proof. Since $-d$ is the greatest common divisor of a and b, $b = -cd$ for some $c \in \mathbb{Z}_0^-$ and by Lemma 3.3, it follows that there exist $s \in \mathbb{Z}_0^-$ and $t \in \mathbb{Z}_0^-$ such that $(-1)sa = (-1)tb + d$ or $(-1)tb = (-1)sa + d$. Since I is an ideal of $\mathbb{Z}_0^-$, it is clear that $(-1)sa \in I$ and $(-1)tb \in I$. Consequently, if $(-1)sa = (-1)tb + d$, a series of simple calculations show that the following elements belong to I:

$(-1)tb + d$, $(-1)tb$.
$(-2)tb + 2d$, $(-2)tb + d$, $(-2)tb$.
$(-3)tb + 3d$, $(-3)tb + 2d$, $(-3)tb + d$, $(-3)tb$.
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$-c^2td + (-c)d$, ..., $-c^2td + 3d$, $-c^2td + 2d$, $-c^2td + d$, $-c^2td$.

Substituting $b = -cd$ in the last row, we have
$ctb + (-c)d$, ..., $ctb + 3d$, $ctb + 2d$, $ctb + d$, ctb.

Since $-c^2td = (-ct)cd$ and $-c^2td + (-c)d = (-ct)cd + (-c)d = ((-ct)c - c)d$, the last row is $S_d([(-ct)c - c]d, (-ct)cd) = S_d(-(-ct-1)b, -(-ct)b) = S_d(-(m-1)b, -mb)$, where $m = -ct$. Consequently, by Lemma 3.11, it follows that there exists $n \in \mathbb{Z}_0^-$ such that $(-1)dT_n \subset I$. On the other hand, if $(-1)tb = (-1)sa + d$, then by similar argument we have the same result. ∎

Theorem 3.8. *If I is an M-ideal in $\mathbb{Z}_0^-$, then there exist $n \in \mathbb{Z}_0^-$ and $d \in \mathbb{Z}^-$ such that $(-1)dT_n \subset I$.*

Proof. If $a \in I$ and $b \in I$ where a and b are relatively prime, then Lemma 3.4 implies that $T_k \subset I$ for some k, which is a contradiction to the fact that I is an M-ideal. Consequently, if $a \in I$ and $b \in I$ and they are not relatively prime, then by Lemma

3.12, there exist $n \in \mathbb{Z}_0^-$ and $d \in \mathbb{Z}^-$ such that $(-1)dT_n \subset I$, where $-d$ is the greatest common divisor of a and b. ∎

The following theorem gives a structure and characterization of M-ideals in the ternary semiring $\mathbb{Z}_0^-$ and is necessary to show that the ternary semiring $\mathbb{Z}_0^-$ is Noetherian.

Theorem 3.9. *An ideal I in $\mathbb{Z}_0^-$ is an M-ideal if and only if I has a finite basis and $I = L \cup (-1)qT_p$, where $q < -1$, $(-1)qT_p$ is a maximal $(-1)dT_n$-ideal contained in I, and $L = \{t \in I : -pq < t < 0\}$.*

Proof. Let I be an M-ideal in $\mathbb{Z}_0^-$. Then by Theorem 3.8, it follows that there exists $n \in \mathbb{Z}_0^-$ such that $(-1)dT_n \subset I$. Let $S = \{d \in \mathbb{Z}_0^- : -d$ is the greatest common divisor of some $a \in I$ and $b \in I\}$ and q be the greatest element in S. Then Lemma 3.12 guarantees that $W = \{n \in \mathbb{Z}_0^- : (-1)qT_n \subset I\}$ is a non-empty subset of $\mathbb{Z}_0^-$. Consequently, by Lemma 3.5, it follows that W has a greatest element and if p is the greatest element of W, then it is clear that $(-1)qT_p \subset I$. Suppose there exists $(-1)bT_a \subset I$ such that $(-1)qT_p \subseteq (-1)bT_a$. Now it follows from Lemma 3.7 that b divides q and hence $b \geq q$. Since $-b$ is the greatest common divisor of $(-1)ba$ and $(-1)b(a-1)$, we have $b \in S$ and it follows that $b \leq q$. Consequently, $b = q$. Again by using Theorem 3.5, we have $p \leq a$ and since $a \in W$ it follows that $p \geq a$. Consequently, $a = p$ and hence $(-1)qT_p = (-1)bT_a$. Therefore, $(-1)qT_p$ is a maximal ideal in I. Let $x \in I$, $x < -pq$ and $k \in \mathbb{Z}_0^-$ such that $-k$ be the greatest common divisor of x and $-pq$. Then $x = -ky$ for some $y \in \mathbb{Z}_0^-$. Now $k \in S$ and it can be shown that q divides k. Thus there exists $r \in \mathbb{Z}_0^-$ such that $k = -rq$. Consequently, $x = -ky = -(-rq)y = -(-ry)q < -pq$ implies that $-ry < p$ and hence $x \in (-1)qT_p$. Now if $L = \{t \in I : -pq < t < 0\}$, then it is clear that $I = L \cup (-1)qT_p$. Again from Theorem 3.7, it follows that $S_q(-2pq, -pq)$ is a finite basis for $(-1)qT_p$. Since L is a finite subset of I, we have $L \cup S_q(-2pq, -pq)$ is a finite basis for I.

The converse of the theorem is obvious. ∎

Definition 3.4. *An ideal I in a ternary semiring S is called almost principal if there exists a finite set $J \subset S$ such that $I \cup J = P$, where P is a principal ideal in S. A ternary semiring S is called an almost principal ideal ternary semiring if every ideal in S is almost principal.*

Theorem 3.10. *The ternary semiring $\mathbb{Z}_0^-$ is an almost principal ideal ternary semiring.*

Proof. Let I be an ideal in $\mathbb{Z}_0^-$. If I is a T-ideal, then by Theorem 3.4, $I = K \cup T_n$, where $K = \{t \in I : n < t < 0\}$. Let $S_1 = \{t \in \mathbb{Z}_0^- : t \notin I\}$. Then from Remark 3.2, it follows that S_1 is a finite subset of $\mathbb{Z}_0^-$ and $I \cup S_1 = \mathbb{Z}_0^- = < -1 >$, is a principal ideal. If I is an M-ideal, then by Theorem 3.9, $I = L \cup (-1)dT_n$, where $L = \{t \in I : -nd < t < 0\}$. Let $S_2 = \{(-1)td : t \in \mathbb{Z}_0^-$ and $(-1)td \notin I\}$. Then from Remark 3.2, it follows that S_2 is a finite subset of $\mathbb{Z}_0^-$ and $I \cup S_2 = d\mathbb{Z}_0^-\mathbb{Z}_0^- = < d >$, is a principal ideal. In either case I is an almost principal ideal and hence the theorem follows. ∎

Definition 3.5. *A ternary semiring S which satisfies the ascending chain condition for ideals is called a Noetherian ternary semiring.*

The following is the characterization theorem for Noetherian ternary semiring.

Theorem 3.11. *Let S be a ternary semiring. Then S is Noetherian if and only if every ideal of S has a finite basis.*

Proof. The proof of the theorem is similar to that of ring theory and therefore we omit it. ∎

Since any ideal in the ternary semiring $\mathbb{Z}_0^-$ is either a T-ideal or an M-ideal, Theorem 3.4 and Theorem 3.9 give a classification and structure for all ideals in the ternary semiring $\mathbb{Z}_0^-$. These results can now be used to obtain the following theorem:

Theorem 3.12. *The ternary semiring $\mathbb{Z}_0^-$ is a Noetherian ternary semiring.*

Proof. In view of Theorem 3.4 and Theorem 3.9, any ideal in the ternary semiring $\mathbb{Z}_0^-$ has a finite basis and it follows from Theorem 3.11 that the ternary semiring $\mathbb{Z}_0^-$ is a Noetherian ternary semiring. ∎

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