

Fuzzy Quasi-Ideals of Ordered Semigroups

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Abstract. In this paper, we characterize ordered semigroups in terms of fuzzy quasi-ideals. We characterize left simple, right simple and completely regular ordered semigroups in terms of fuzzy quasi-ideals. We define semiprime fuzzy quasi-ideal of ordered semigroups and characterize completely regular ordered semigroup in terms of semiprime fuzzy quasi-ideals. We also study the decomposition of left and right simple ordered semigroups having the property $a \leq a^2$ for all $a \in S$, by means of fuzzy quasi-ideals.

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1. Introduction

There has been a rapid growth worldwide in the interest of fuzzy set theory and its applications from the past several years. Evidence of this can be found in the increasing number of high-quality research articles on fuzzy sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences held every year. It seems that the fuzzy set theory deals with the applications of fuzzy technology in information processing. The information processing is already important and it will certainly increase in importance in the future. Granular computing refers to the representation of information in the form of aggregates, called granules. If granules are modeled as fuzzy sets, then fuzzy logics are used. This new computing methodology has been considered by Brageila and Pedrycz in [4]. A presentation of updated trends in fuzzy set theory and its applications has been considered by Pedrycz and Gomide in [21]. A systematic exposition of fuzzy semigroups by Mordeson, Malik and Kuroki appeared in [19], where

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one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [20] deals with the applications of fuzzy approach to the concepts of automata and formal languages. The notion of quasi-ideals play an important role in the study of ring theory, semiring theory, semigroup theory and ordered semigroup theory etc. For a detail study of quasi-ideals in rings and semigroups, we refer the reader to [23]. The fuzzy subsets in semigroups were first studied by Kuroki [16–18] and Ahsan *et al.* [1]. The fuzzy quasi-ideals in semigroups were studied in [1] and [18], where the basic properties of semigroups in terms of fuzzy quasi-ideals are given. The concept of a quasi-ideal in rings and semigroups was studied by Stienfeld in [23], and Kehayopulu extended the concept of quasi-ideals in ordered semigroups S as a non-empty subset Q of S such that [8]:

- (1) $(QS) \cap (SQ) \subseteq Q$ and
- (2) if $a \in Q$ and $S \ni b \leq a$ then $b \in Q$.

For a detail study of ideal theory and fuzzy ideal theory of ordered semigroups we refer the reader to [10–14], [7–9] and [15].

In this paper, we characterize regular, left and right simple ordered semigroups and completely regular ordered semigroups. We prove that an ordered semigroup S is regular, left and right simple if and only if every fuzzy quasi-ideal of S is a constant function. We also prove that S is completely regular if and only if for every fuzzy quasi-ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$. We define semiprime fuzzy quasi-ideal of ordered semigroups and prove that an ordered semigroup S is completely regular if and only if every fuzzy quasi-ideal f of S is semiprime. Next we characterize semilattices of left and right simple ordered semigroups in terms of fuzzy quasi-ideals of S . We prove that an ordered semigroup S is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$, for all $a, b \in S$. In the last section of this paper, we discuss ordered semigroups having the property $a \leq a^2$ for all $a \in S$ and prove that an ordered semigroup S (having the property $a \leq a^2 \forall a \in S$) is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have $f(ab) = f(ba)$, for all $a, b \in S$.

2. Some basic results and definitions

In this section, we give some basic definitions and results, which are necessary for the subsequent sections.

By an ordered semigroup we mean a structure (S, \cdot, \leq) such that:

- (OS1) (S, \cdot) is a semigroup.
- (OS2) (S, \leq) is a poset.
- (OS3) $(\forall a, b, x \in S)(a \leq b \longrightarrow ax \leq bx \text{ and } xa \leq xb)$.

Let (S, \cdot, \leq) be an ordered semigroup and $\emptyset \neq A \subseteq S$, denote

$$[A] := \{t \in S \mid t \leq h \text{ for some } h \in A\}.$$

For $A, B \subseteq S$, denote

$$AB = \{ab \mid a \in A, b \in B\}.$$

For $a \in S$, we write $(a]$ instead of $(\{a\})$.

Let S be an ordered semigroup and $A, B \subseteq S$. Then $A \subseteq (A]$, $(A][B] \subseteq (AB]$, $((A]) = (A]$ and $((A][B]) \subseteq (AB]$ (see [8]).

Let (S, \cdot, \leq) be an ordered semigroup, $\emptyset \neq A \subseteq S$. Then A is called a *subsemigroup* of S if $A^2 \subseteq A$ (see [9]).

Let (S, \cdot, \leq) be an ordered semigroup. $\emptyset \neq A \subseteq S$ is called a *right* (resp. *left*) *ideal* (see [13]) of S if:

- (1) $AS \subseteq A$ (resp. $SA \subseteq A$) and
- (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$.

If A is both a right and a left ideal of S , then it is called an *ideal* of S . A subsemigroup B of S is called a *bi-ideal* (see [9]) of S if:

- (1) $BSB \subseteq B$ and
- (2) if $a \in B$, $S \ni b \leq a$, then $b \in B$.

Let (S, \cdot, \leq) be an ordered semigroup. By a *fuzzy subset* f of S , we mean a function $f : S \rightarrow [0, 1]$.

Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *ideal* of S if:

- (1) $(\forall x, y \in S)(x \leq y \rightarrow f(x) \geq f(y))$ and
- (2) $(\forall x, y \in S)(f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$)).

If f is both a fuzzy left and a fuzzy right ideal of S , then it is called a *fuzzy ideal* of S (see [7]).

Let (S, \cdot, \leq) be an ordered semigroup and $\emptyset \neq A \subseteq S$. The *characteristic function* f_A of A is given by:

$$f_A : S \rightarrow [0, 1], a \mapsto f_A(a) := \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

Let (S, \cdot, \leq) be an ordered semigroup and $a \in S$, denote $A_a := \{(y, z) \in S \times S \mid a \leq yz\}$ (see [8]).

For two fuzzy subsets f and g of S , define

$$f \circ g : S \rightarrow [0, 1], a \mapsto f \circ g(a) \begin{cases} \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

We denote by $F(S)$ (as given in [8]) the set of all fuzzy subsets of S . We define order relation “ \preceq ” on $F(S)$ as follows:

$$f \preceq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S.$$

Then $(F(S), \circ, \preceq)$ is an ordered semigroup (see [8]).

For a non-empty family of fuzzy subsets $\{f_i\}_{i \in I}$, of an ordered semigroup S , the fuzzy subsets $\bigvee_{i \in I} f_i$ and $\bigwedge_{i \in I} f_i$ of S are defined as follows:

$$\begin{aligned} \bigvee_{i \in I} f_i : S &\rightarrow [0, 1], a \mapsto \left(\bigvee_{i \in I} f_i \right) (a) := \sup_{i \in I} \{f_i(a)\} \text{ and} \\ \bigwedge_{i \in I} f_i : S &\rightarrow [0, 1], a \mapsto \left(\bigwedge_{i \in I} f_i \right) (a) := \inf_{i \in I} \{f_i(a)\}. \end{aligned}$$

If I is a finite set, say $I = \{1, 2, \dots, n\}$, then clearly

$$\bigvee_{i \in I} f_i(a) = \max\{f_1(a), f_2(a), \dots, f_n(a)\} \text{ and}$$

$$\bigwedge_{i \in I} f_i(a) = \min\{f_1(a), f_2(a), \dots, f_n(a)\}.$$

For an ordered semigroup S , the fuzzy subsets “0” and “1” of S are defined as follows (see [8]):

$$0 : S \longrightarrow [0, 1], x \longmapsto 0(x) := 0,$$

$$1 : S \longrightarrow [0, 1], x \longmapsto 1(x) := 1.$$

Clearly, the fuzzy subset “0” (resp. “1”) of S is the least (resp. the greatest) element of the ordered set $(F(S), \preceq)$. The fuzzy subset

“0” is the zero element of $(F(S), \circ, \preceq)$ (that is, $f \circ 0 = 0 \circ f = 0$ and $0 \preceq f$ for every $f \in F(S)$).

3. Fuzzy quasi-ideals

In this section we characterize quasi-ideals of ordered semigroups by the properties of their level subsets.

Proposition 3.1. (cf. [8]) *If (S, \cdot, \leq) is an ordered semigroup and $A, B \subseteq S$. Then*

- (1) $A \subseteq B$ if and only if $f_A \preceq f_B$,
- (2) $f_A \wedge f_B = f_{A \cap B}$,
- (3) $f_A \circ f_B = f_{(AB)}$.

Lemma 3.1. *Let S be an ordered semigroup. Then every quasi-ideal of S is a subsemigroup of S .*

Lemma 3.2. (cf. [8]) *An ordered semigroup (S, \cdot, \leq) is regular if and only if for right ideal A and every left ideal B of S , we have $A \cap B = (AB)$.*

Definition 3.1. (cf. [8]) *Let (S, \cdot, \leq) be an ordered semigroup. A fuzzy subset f of S is called a fuzzy quasi-ideal of S if:*

- (1) $(f \circ 1) \wedge (1 \circ f) \preceq f$,
- (2) $(\forall x, y \in S)(x \leq y \longrightarrow f(x) \geq f(y))$.

Definition 3.2. (cf. [9]) *Let (S, \cdot, \leq) be an ordered semigroup. A fuzzy subset f of S is called a fuzzy bi-ideal of S if:*

- (1) $(\forall x, y \in S)(f(xy) \geq \min\{f(x), f(y)\})$.
- (2) $(\forall x, y, z \in S)(f(xyz) \geq \min\{f(x), f(z)\})$.
- (3) $(\forall x, y \in S)(x \leq y \longrightarrow f(x) \geq f(y))$.

Lemma 3.3. (cf. [7–9]) *Let (S, \cdot, \leq) be an ordered semigroups and $\emptyset \neq A \subseteq S$. Then A is a left (resp. right, bi- and quasi-) ideal of S if and only if the characteristic function f_A of A is a fuzzy left (resp. right, bi- and quasi-) ideal of S .*

Let (S, \cdot, \leq) be an ordered semigroup and $t \in (0, 1]$ then the set

$$U(f; t) := \{x \in S | f(x) \geq t\},$$

is called a level subset of f .

Theorem 3.1. *Let (S, \cdot, \leq) be an ordered semigroup and f be a fuzzy subset of S . Then*

$(\forall t \in (0, 1] U(f; t) \neq \emptyset$ is a quasi-ideal if and only if f is a fuzzy quasi-ideal).

Proof. (\implies). Assume that for every $t \in (0, 1]$ such that $U(f; t) \neq \emptyset$ the set $U(f; t)$ is a quasi-ideal of S . Let $x, y \in S$, $x \leq y$ be such that $f(x) < f(y)$. Then there exists $t \in (0, 1)$ such that $f(x) < t \leq f(y)$, then $y \in U(f; t)$ but $x \notin U(f; t)$. This is a contradiction. Hence $f(x) \geq f(y)$ for all $x \leq y$. Suppose that there exists $x \in S$ such that

$$f(x) \leq ((f \circ 1) \wedge (1 \circ f))(x),$$

then there exists $t \in (0, 1]$ such that

$$f(x) < t < ((f \circ 1) \wedge (1 \circ f))(x) = \min[(f \circ 1)(x), (1 \circ f)(x)]$$

and hence $(f \circ 1)(x) > t$ and $(1 \circ f)(x) > t$. Then

$$\bigvee_{(p,q) \in A_x} \min\{f(p), 1(q)\} > t \text{ and } \bigvee_{(p,q) \in A_x} \min\{1(p), f(q)\} > t.$$

This implies that there exist $b, c, d, e \in S$ with $(b, c) \in A_x$ and $(d, e) \in A_x$ such that $f(b) > t$ and $f(e) > t$. Then $b, e \in U(f; t)$ and so $bc \in U(f; t)S$ and $de \in SU(f; t)$. Hence $x \in (U(f; t)S]$ and $x \in (SU(f; t)] \implies x \in (U(f; t)S] \cap (SU(f; t)]$. By hypothesis, $(U(f; t)S] \cap (SU(f; t)] \subseteq U(f; t)$ and so $x \in U(f; t)$. Then $f(x) \geq t$. This is a contradiction. Thus $f(x) \geq ((f \circ 1) \wedge (1 \circ f))(x)$.

(\impliedby). Assume that f is a fuzzy quasi-ideal of S and $t \in (0, 1]$ such that $U(f; t) \neq \emptyset$. Let $x, y \in S$ be such that $x \leq y$ and $y \in U(f; t)$. Then $f(y) \geq t$. Since $x \leq y \implies f(x) \geq f(y)$ we have $f(x) \geq t$ and so $x \in U(f; t)$.

Suppose that $x \in S$ be such that $x \in (U(f; t)S] \cap (SU(f; t)]$. Then $x \in (U(f; t)S]$ and $x \in (SU(f; t)]$ and we have $x \leq yz$ and $x \leq y'z'$ for some $y, z' \in U(f; t)$ and $z, y' \in S$. Then $(y, z) \in A_x$ and $(y', z') \in A_x$. Since $A_x \neq \emptyset$, by hypothesis

$$\begin{aligned} f(x) &\geq ((f \circ 1) \wedge (1 \circ f))(x) \\ &= \min \left[\bigvee_{(p,q) \in A_x} \min\{f(p), 1(q)\}, \bigvee_{(p_1, q_1) \in A_x} \min\{1(p_1), f(q_1)\} \right] \\ &\geq \min[\min\{f(y), 1(z)\}, \min\{1(y'), f(z')\}] \\ &= \min[\min\{f(y), 1\}, \min\{1, f(z')\}] \\ &= \min[f(y), f(z')]. \end{aligned}$$

Since $y, z' \in U(f; t)$ we have $f(y) \geq t$ and $f(z') \geq t$. Then

$$f(x) \geq \min[f(y), f(z')] \geq t,$$

and so $x \in U(f; t)$. Hence $(U(f; t)S] \cap (SU(f; t)] \subseteq U(f; t)$. Thus $U(f; t)$ is a quasi-ideal of S . \blacksquare

Example 3.1. Let $S = \{a, b, c, d, f\}$ be an ordered semigroup with the following multiplication table

\cdot	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

We define the order relation " \leq " as follows:

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (a, f), (b, b), (c, c), (d, d), (f, f)\}.$$

Quasi-ideals of S are:

$$\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, f\}, \{a, b, d\}, \\ \{a, c, d\}, \{a, b, f\}, \{a, c, f\} \text{ and } S \text{ (see [11]).}$$

Define $f : S \rightarrow [0, 1]$ by

$$f(a) = 0.8, \quad f(b) = 0.7, \quad f(d) = 0.6, \quad f(c) = f(f) = 0.5.$$

Then

$$U(f; t) := \begin{cases} S & \text{if } t \in (0, 0.5] \\ \{a, b, d\} & \text{if } t \in (0.5, 0.6] \\ \{a, b\} & \text{if } t \in (0.6, 0.7] \\ \{a\} & \text{if } t \in (0.7, 0.8] \\ \emptyset & \text{if } t \in (0.8, 1]. \end{cases}$$

Then $U(f; t)$ is a quasi-ideal and by Theorem 3.1, f is a fuzzy quasi-ideal of S .

Lemma 3.4. *Every quasi-ideal of an ordered semigroup (S, \cdot, \leq) is a bi-ideal of S .*

Lemma 3.5. *Every fuzzy quasi-ideal of an ordered semigroup S is a fuzzy bi-ideal of S .*

Proof. Let f be a fuzzy quasi-ideal of S . Let $x, y \in S$. Then $xy = x(y)$ and we have $(x, y) \in A_{xy}$. Since $A_{xy} \neq \emptyset$, we have

$$\begin{aligned} f(xy) &\geq ((f \circ 1) \wedge (1 \circ f))(xy) \\ &= \min[(f \circ 1)(xy), (1 \circ f)(xy)] \\ &= \min \left[\bigvee_{(p, q) \in A_{xy}} \min\{f(p), 1(q)\}, \bigvee_{(p_1, q_1) \in A_{xy}} \min\{1(p_1), f(q_1)\} \right] \\ &\geq \min[\min\{f(x), 1(y)\}, \min\{1(x), f(y)\}] \\ &= \min[\min\{f(x), 1\}, \min\{1, f(y)\}] \\ &= \min[f(x), f(y)]. \end{aligned}$$

Let $x, y, z \in S$. Then $(xy)z = x(yz)$ and we have $(xy, z), (x, yz) \in A_{xyz}$. Since $A_{xyz} \neq \emptyset$, we have

$$\begin{aligned} f(xyz) &\geq ((f \circ 1) \wedge (1 \circ f))(xyz) \\ &= \min[(f \circ 1)(xyz), (1 \circ f)(xyz)] \end{aligned}$$

$$\begin{aligned}
 &= \min \left[\bigvee_{(p,q) \in A_{xyz}} \min\{f(p), 1(q)\}, \bigvee_{(p_1,q_1) \in A_{xyz}} \min\{1(p_1), f(q_1)\} \right] \\
 &\geq \min[\min\{f(x), 1(yz)\}, \min\{1(xy), f(z)\}] \\
 &= \min[\min\{f(x), 1\}, \min\{1, f(z)\}] \\
 &= \min[f(x), f(z)].
 \end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $f(x) \geq f(y)$, because f is a fuzzy quasi-ideal of S . Thus f is a fuzzy bi-ideal of S . ■

Remark 3.1. The converse of Lemma 3.5, is not true in general.

Example 3.2. Consider the ordered semigroup $S = \{a, b, c, d\}$

\cdot	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$$

Then $\{a, d\}$ is a bi-ideal of S but not a quasi-ideal of S . Define $f : S \rightarrow [0, 1]$ by

$$f(a) = f(d) = 0.7, \quad f(b) = f(c) = 0.4.$$

Then

$$U(f; t) := \begin{cases} S & \text{if } t \in (0, 0.4] \\ \{a, d\} & \text{if } t \in (0.4, 0.7] \\ \emptyset & \text{if } t \in (0.7, 1]. \end{cases}$$

Then $U(f; t)$ is a bi-ideal of S and by Theorem 3.1, f is a fuzzy quasi-ideal of S . Furthermore, $U(f; t)$ is a bi-ideal of S for all $t \in (0.4, 0.7]$ but not a quasi-ideal of S , and hence by Theorem 3.1, f is a fuzzy bi-ideal of S , but not a fuzzy quasi-ideal of S .

4. Characterizations of left, right and completely regular ordered semigroup in terms of fuzzy quasi-ideals

In this section, we characterize ordered semigroups in terms of fuzzy quasi-ideals and prove that an ordered semigroup S is regular, left and right simple if and only if every fuzzy quasi-ideal f of S is a constant function. We define semiprime fuzzy quasi-ideals of ordered semigroups and prove that an ordered semigroup S is completely regular if and only if every fuzzy quasi-ideal f of S is a semiprime fuzzy quasi-ideal of S .

An ordered semigroup S is called left (resp. right) simple (see [12]) if for every left (resp. right) ideal A of S , we have $A = S$.

Lemma 4.1. (cf. [9, Lemma 3]) *An ordered semigroup S is left (resp. right) simple if and only if $\{Sa\} = S$ (resp. $\{aS\} = S$) for every $a \in S$.*

Theorem 4.1. *An ordered semigroup (S, \cdot, \leq) is regular, left and right simple if and only if every fuzzy quasi-ideal of S is a constant function.*

Proof. Let S be regular, left and right simple ordered semigroup. Let f be a fuzzy quasi-ideal of S and $a \in S$. We consider the set

$$E_{\Omega} := \{e \in S | e^2 \geq e\}.$$

Then E_{Ω} is non-empty. In fact, since S is regular, and $a \in S$, there exists $x \in S$ such that $a \leq axa$. Then it follows by (OS3), that

$$(ax)^2 = (axa)x \geq ax,$$

and so $ax \in E_{\Omega}$ and hence $E_{\Omega} \neq \emptyset$.

(1) Let $t \in E_{\Omega}$. Then $f(e) = f(t)$ for every $e \in E_{\Omega}$. Indeed, since S is left and right simple, we have $(St) = S$ and $(tS) = S$. Since $e \in S$, then $e \in (St)$ and $e \in (tS)$ so there exist $x, y \in S$ such that $e \leq xt$ and $e \leq ty$. Hence

$$e^2 = ee \leq (xt)(xt) = (ctx)t,$$

and we have $(ctx, t) \in A_{e^2}$ and if $e \leq ty$ then

$$e^2 = ee \leq (ty)(ty) = t(yty)$$

and hence $(t, yty) \in A_{e^2}$. Since $A_{e^2} \neq \emptyset$ and f is a fuzzy quasi-ideals of S , we have

$$\begin{aligned} f(e^2) &\geq ((f \circ 1) \wedge (1 \circ f))(e^2) \\ &= \min[(f \circ 1)(e^2), (1 \circ f)(e^2)] \\ &= \min\left[\bigvee_{(y_1, z_1) \in A_{e^2}} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2, z_2) \in A_{e^2}} \min\{1(y_2), f(z_2)\}\right] \\ &\geq \min[\min\{f(t), 1(yty)\}, \min\{1(ctx), f(t)\}] \\ &= \min[\min\{f(t), 1\}, \min\{1, f(t)\}] \\ &= \min[f(t), f(t)] = f(t). \end{aligned}$$

Since $e \in E_{\Omega}$ we have $e^2 \geq e$ and f is a fuzzy quasi-ideal of S , we have $f(e) \geq f(e^2)$. Thus $f(e) \geq f(t)$. Since S is left and right simple and $e \in S$ we have, $(Se) = S$ and $(eS) = S$. Since $t \in S$, we have $t \leq ze$ and $t \leq es$ for some $z, s \in S$. If $t \leq ze$ then

$$t^2 = tt \leq (ze)(ze) = (zez)e$$

then $(zez, e) \in A_{t^2}$. If $t \leq es$ then

$$t^2 = tt \leq (es)(es) = e(ses)$$

and we have $(e, ses) \in A_{t^2}$. Since $A_{t^2} \neq \emptyset$, we have

$$\begin{aligned} f(t^2) &\geq ((f \circ 1) \wedge (1 \circ f))(t^2) \\ &= \min[(f \circ 1)(t^2), (1 \circ f)(t^2)] \\ &= \min\left[\bigvee_{(p_1, q_1) \in A_{t^2}} \min\{f(p_1), 1(q_1)\}, \bigvee_{(p_2, q_2) \in A_{t^2}} \min\{1(p_2), f(q_2)\}\right] \\ &\geq \min[\min\{f(e), 1(ses)\}, \min\{1(zez), f(e)\}] \\ &= \min[\min\{f(e), 1\}, \min\{1, f(e)\}] \\ &= \min[f(e), f(e)] = f(e). \end{aligned}$$

Since $t \in E_\Omega$ we have $t^2 \geq t$ and since f is a fuzzy quasi-ideal of S , we have $f(t) \geq f(t^2)$. Thus $f(t) \geq f(e)$.

(2) Let $a \in S$ then $f(t) = f(a)$ for every $t \in E_\Omega$. Since $a \in S$ and S is regular, there exists $x \in S$ such that $a \leq axa$. Then from (OS3) it follows that,

$$(ax)^2 = (axa)x \geq ax \text{ and } (xa)^2 = x(axa) \geq xa.$$

Then $ax, xa \in E_\Omega$. Then by (1) we have $f(ax) = f(t)$ and $f(xa) = f(t)$. Since $(ax)(axa) \geq axa \geq a$, and $(axa)(xa) \geq axa \geq a$, so we have, $(ax, axa), (axa, xa) \in A_a$. Since $A_a \neq \emptyset$ and f is a fuzzy quasi-ideal of S , we have

$$\begin{aligned} f(a) &\geq ((f \circ 1) \wedge (1 \circ f))(a) \\ &= \min [(f \circ 1)(a), (1 \circ f)(a)] \\ &= \min \left[\bigvee_{(y_1, z_1) \in A_a} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2, z_2) \in A_a} \min\{1(y_2), f(z_2)\} \right] \\ &\geq \min [\min\{f(ax), 1(axa)\}, \min\{1(axa), f(xa)\}] \\ &= \min [\min\{f(ax), 1\}, \min\{1, f(xa)\}] \\ &= \min [f(ax), f(xa)] \\ &= \min [f(ax), f(ax)] = \min [f(t), f(t)] = f(t). \end{aligned}$$

Since S is left and right simple, we have $(Sa) = S$, $(aS) = S$. Since $t \in S$, we have $t \in (Sa)$ and $t \in (aS)$. Then $t \leq pa$ and $t \leq aq$ for some $p, q \in S$. Then $(p, a) \in A_t$ and $(a, q) \in A_t$. Since $A_t \neq \emptyset$ and f is a fuzzy ideal of S , we have

$$\begin{aligned} f(t) &\geq ((f \circ 1) \wedge (1 \circ f))(t) \\ &= \min [(f \circ 1)(t), (1 \circ f)(t)] \\ &= \min \left[\bigvee_{(y_1, z_1) \in A_t} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2, z_2) \in A_t} \min\{1(y_2), f(z_2)\} \right] \\ &\geq \min [\min\{f(a), 1(q)\}, \min\{1(p), f(a)\}] \\ &= \min [\min\{f(a), 1\}, \min\{1, f(a)\}] \\ &= \min [f(a), f(a)] = f(a). \end{aligned}$$

Since S is left and right simple, we have $(Sa) = S$, and $(aS) = S$. Since $t \in S$, we have $t \in (Sa)$ and $t \in (aS)$. Then $t \leq pa$ and $t \leq aq$ for some $p, q \in S$. Then $(p, a) \in A_t$ and $(q, a) \in A_t$. Since $A_t \neq \emptyset$, we have

$$\begin{aligned} f(t) &\geq ((f \circ 1) \wedge (1 \circ f))(t) \\ &= \min [(f \circ 1)(t), (1 \circ f)(t)] \\ &= \min \left[\bigvee_{(y_1, z_1) \in A_t} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2, z_2) \in A_t} \min\{1(y_2), f(z_2)\} \right] \\ &\geq \min [\min\{f(a), 1(q)\}, \min\{1(p), f(a)\}] \\ &= \min [\min\{f(a), 1\}, \min\{1, f(a)\}] \end{aligned}$$

$$= \min[f(a), f(a)] = f(a).$$

Conversely, let $a \in S$. Then the set $(aS]$ is a quasi-ideal of S . In fact, $((aS]S] \cap (S(Sa]) \subseteq (aS] \cap (SaS] \subseteq (aS]$ and if $x \in (aS]$ and $S \ni y \leq x \in (aS]$, then $y \in ((S]) = (aS]$. Since $(aS]$ is a quasi-ideal of S , by Lemma 3.3, the characteristic function $f_{(aS]}$ of $(aS]$ defined by

$$f_{(aS]} : S \longrightarrow [0, 1], x \longmapsto f_{(aS]}(x) := \begin{cases} 1 & \text{if } x \in (aS] \\ 0 & \text{if } x \notin (aS] \end{cases}$$

is a fuzzy quasi-ideal of S . By hypothesis, $f_{(aS]}$ is a constant function, that is, there exists $c \in \{0, 1\}$ such that

$$f_{(aS]}(x) = c \text{ for every } x \in S.$$

Let $(aS] \subset S$ and a be an element of S such that $a \notin (aS]$, then $f_{(aS]}(x) = 0$. On the other hand, since $a^2 \in (aS]$ then $f_{(aS]}(a^2) = 1$. A contradiction to the fact that $f_{(aS]}$ is a constant function. Thus $(aS] = S$. By symmetry we can prove that $(Sa] = S$.

Since $a \in S$ and $S = (aS] = (Sa]$, we have $a(aS] = (a(Sa]) \subseteq (aSa]$, and hence S is regular. \blacksquare

An ordered semigroup S is called *left* (resp. *right*) *regular* (see [10]) if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$) or, equivalently, if (1) $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for every $a \in S$ and (2) $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S]$) for every $A \subseteq S$. An ordered semigroup S is called *completely regular* (see [12]) if it is regular, left regular and right regular.

Lemma 4.2. (cf. [12]) *An ordered semigroup (S, \cdot, \leq) is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$ or, equivalently, if $a \in (a^2Sa^2]$ for every $a \in S$.*

Lemma 4.3. *If S is an ordered semigroup and $\emptyset \neq A \subseteq S$, then the set $(A \cup (AS \cap SA))$ is the quasi-ideal of S generated by A . If $A = \{x\}$ ($x \in S$), we write $(x \cup (xS \cap Sx))$ instead of $(\{x\} \cup (\{x\}S \cap S\{x\}))$.*

Theorem 4.2. *An ordered semigroup (S, \cdot, \leq) is completely regular if and only if for each quasi-ideal f of S we have, $f(a) = f(a^2)$ for all $a \in S$.*

Proof. Let S be a completely regular ordered semigroup and f a quasi-ideal of S and let $a \in S$. Since S is left and right regular we have $a \in (Sa^2]$ and $a \in (a^2S]$. Then there exists $x, y \in S$ such that $a \leq xa^2$ and $a \leq a^2y$. Then $(x, a^2), (a^2, y) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned} f(a) &\geq ((f \circ 1) \wedge (1 \circ f))(a) \\ &= \min[(f \circ 1)(a), (1 \circ f)(a)] \\ &= \min \left[\bigvee_{(y_1, z_1)} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2, z_2)} \min\{1(y_2), f(z_2)\} \right] \\ &\geq \min [\min\{f(a^2), 1(y)\}, \min\{1(x), f(a^2)\}] \\ &= \min [\min\{f(a^2), 1\}, \min\{1, f(a^2)\}] \end{aligned}$$

$$\begin{aligned}
&= \min [f(a^2), f(a^2)] = f(a^2) \\
&= f(aa) \geq \min\{f(a), f(a)\} = f(a).
\end{aligned}$$

Thus $f(a) = f(a^2)$.

Conversely, let $a \in S$. We consider the quasi-ideal $Q(a)$ of S , generated by a^2 ($a \in S$). That is the set, $Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))$. By Lemma 3.3, $f_{Q(a^2)}$ is a fuzzy quasi-ideal of S . By hypothesis, we have

$$f_{Q(a^2)}(a) = f_{Q(a^2)}(a^2).$$

Since $a^2 \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))$, we have $f_{Q(a^2)}(a^2) := 1$, then $f_{Q(a^2)}(a) = 1$ and we have $a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))$. Then $a \leq a^2$ or $a \leq a^2x$ and $a \leq ya^2$ for some $x, y \in S$. If $a \leq a^2$ then $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2$ and so $a \in (a^2Sa^2)$. If $a \leq a^2x$ and $a \leq ya^2$ then $a \leq (a^2x)(ya^2) = a^2(yx)a^2$. Since $yx \in S$ we have $a^2(yx)a^2 \in a^2Sa^2$. Thus $a \in (a^2Sa^2)$. ■

Definition 4.1. Let (S, \cdot, \leq) be an ordered semigroup and f is a fuzzy quasi-ideal of S . Then f is called semiprime fuzzy quasi-ideal of S if

$$f(a) \geq f(a^2), \text{ for all } a \in S.$$

Theorem 4.3. An ordered semigroup (S, \cdot, \leq) is completely regular if and only if every fuzzy quasi-ideal f of S is semiprime.

Proof. (\implies). Let S be completely regular and f a fuzzy quasi-ideal of S . Let $a \in S$. Then $f(a^2) \geq f(a)$. In fact, since S is left and right regular, and $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2$ and $a \leq a^2y$ then $(x, a^2) \in A_a$ and $(a^2, y) \in A_a$. Since $A_a \neq \emptyset$, then

$$\begin{aligned}
f(a) &\geq ((f \circ 1) \wedge (1 \circ f))(a) \\
&= \min[(f \circ 1)(a) \wedge (1 \circ f)(a)] \\
&= \min \left[\bigvee_{(p,q) \in X_a} \min \{f(p), 1(q)\}, \bigvee_{(p_1,q_1) \in X_a} \min \{1(p_1), f(q_1)\} \right] \\
&\geq \min [\min \{f(a^2), 1(y)\}, \min \{1(x), f(a^2)\}] \\
&= \min [\min \{f(a^2), 1\}, \min \{1, f(a^2)\}] \\
&= \min [f(a^2), f(a^2)] \\
&= f(a^2),
\end{aligned}$$

(\impliedby). Let f be a fuzzy quasi-ideal of S , such that $f(a) \geq f(a^2)$ for all $a \in S$. We consider the quasi-ideal $Q(a^2)$ of S generated by a^2 . That is, the set $Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))$. Then by Lemma 3.3, $f_{Q(a^2)}$ is a fuzzy quasi-ideal of S . By hypothesis,

$$f_{Q(a^2)}(a) \geq f_{Q(a^2)}(a^2).$$

Since $a^2 \in Q(a^2)$, we have $f_{Q(a^2)}(a^2) = 1$ and $f_{Q(a^2)}(a) = 1$ and we have $a \in Q(a^2) = (a^2 \cup (a^2S \cap Sa^2))$. Then $a \leq a^2$ or $a \leq a^2x$ and $a \leq za^2$ for some $x, z \in S$. If $a \leq a^2$ then $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2$, then $a \in (a^2Sa^2)$. If $a \leq a^2x$ and $a \leq za^2$, then $a \leq (a^2x)(za^2) = a^2(xz)a^2$, since $xz \in S$, we have $a^2(xz)a^2 \in a^2Sa^2$ and so $a \in (a^2Sa^2)$. ■

5. Some semilattices of left and right simple ordered semigroups in terms of fuzzy quasi-ideals

In this section, we characterize semilattices of left and right simple ordered semigroups in terms of fuzzy quasi-ideals of S . We prove that an ordered semigroup S is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$, for all $a, b \in S$. We also discuss the semilattice of ordered semigroups having the property $a \leq a^2$ for all $a \in S$ and prove that an ordered semigroup S (having the property $a \leq a^2 \forall a \in S$) is a semilattice of left and right simple ordered semigroups if and only if for every fuzzy quasi-ideal f of S we have $f(ab) = f(ba)$, for all $a, b \in S$.

A subset T of an ordered semigroup S is called semiprime if for every $a \in S$ such that $a^2 \in T$ we have $a \in T$. Equivalently, $A^2 \subseteq T$ we have $A \subseteq T$ (see [9]).

Lemma 5.1. (cf. [9]) *Let S be an ordered semigroup. Then the following are equivalent:*

- (i) $(x)_{\mathcal{N}}$ is a left (resp. right) simple subsemigroup of S , for every $x \in S$.
- (ii) Every left (resp. right) ideal of S is a right (resp. left) ideal of S and semiprime.

Let (S, \cdot, \leq) be an ordered semigroup. A subsemigroup F of S is called a *filter* (see [13]) of S if: (1) $(\forall a, b \in S) (ab \in F \rightarrow a \in F \text{ and } b \in F)$, and (2) $(\forall a \in S)(b \in F)(a \geq b \rightarrow a \in F)$. For $x \in S$, we denote by $N(x)$ the least filter of S generated x ($x \in S$). By \mathcal{N} we mean the equivalence relation on S defined by $\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\}$ (see [14]). An equivalence relation σ on S is called congruence if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called semilattice congruence on S , if $(a, a^2) \in \sigma$ and $(ab, ba) \in \sigma$ for each $a, b \in S$ (see [13]). If σ is a semilattice congruence on S then the σ -class $(x)_{\sigma}$ of S containing x is a subsemigroup of S for every $x \in S$ (see [14]). An ordered semigroup S is called a semilattice of left and right simple semigroups if there exists a semilattice congruence σ on S such that the σ -class $(x)_{\sigma}$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$. Equivalently, there exists a semilattice Y and a family $\{S_{\alpha}\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that:

- (i) $S_{\alpha} \cap S_{\beta} = \emptyset \quad \forall \alpha, \beta \in Y, \alpha \neq \beta$,
- (ii) $S = \bigcup_{\alpha \in Y} S_{\alpha}$,
- (iii) $S_{\alpha} S_{\beta} \subseteq S_{\alpha\beta} \quad \forall \alpha, \beta \in Y$.

The semilattice congruences in ordered semigroups are defined exactly as in semigroups without ordered—so the two definitions are equivalent (see [14]).

Lemma 5.2. (cf. [9]) *An ordered semigroup (S, \cdot, \leq) is a semilattice of left and right simple semigroups if and only if for all bi-ideals A, B of S , we have*

$$[A] = A \text{ and } (AB) = (BA).$$

Lemma 5.3. (cf. [9]) *An ordered semigroup (S, \cdot, \leq) is a semilattice of left and right simple semigroups if and only if for all quasi-ideals A, B of S , we have*

$$[A] = A \text{ and } (AB) = (BA).$$

Proof. Follows from Lemmas 5.1 and 5.2. ■

Theorem 5.1. *An ordered semigroup (S, \cdot, \leq) is a semilattice of left and right simple semigroups if and only if for every fuzzy quasi-ideal f of S , we have*

$$f(a) = f(a^2) \text{ and } f(ab) = f(ba) \text{ for all } a, b \in S.$$

Proof. (\implies). Suppose that S is a semilattice of left and right simple semigroups. Then by hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that

- (i) $S_\alpha \cap S_\beta = \emptyset \quad \forall \alpha, \beta \in Y, \alpha \neq \beta$,
- (ii) $S = \bigcup_{\alpha \in Y} S_\alpha$,
- (iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta} \quad \forall \alpha, \beta \in Y$.

(1) Let f be a fuzzy quasi-ideal of S and $a \in S$. By Theorem 4.2 and Lemma 4.2, it is enough to prove that $a \in (a^2Sa^2]$ for every $a \in S$. Since $a \in S = \bigcup_{\alpha \in Y} S_\alpha$,

then there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left and right simple we have $S_\alpha = (S_\alpha a]$ and $S_\alpha = (aS_\alpha]$. Then we have $(aS_\alpha] = (a(S_\alpha a]) = (aS_\alpha a]$. Since $a \in S_\alpha$ we have $a \in (aS_\alpha a]$, then there exists $x \in S_\alpha$ such that $a \leq axa$. Since $x \in (aS_\alpha a]$ there exists $y \in S_\alpha$ such that, $x \leq aya$. Thus $a \leq axa \leq a(aya)a = a^2ya^2$. Since $y \in S_\alpha$, we have $a^2ya^2 \in a^2S_\alpha a^2 \subseteq a^2Sa^2$ and we have $a \in (a^2Sa^2]$.

(2) Let $a, b \in S$. By (1), we have

$$f(ab) = f((ab)^2) = f((ab)^4).$$

Also we have

$$\begin{aligned} (ab)^4 &= (aba)(babab) \in Q(aba)Q(babab) \\ &\subseteq (Q(aba)Q(babab)) \\ &= (Q(babab)Q(aba)) \text{ (by Lemma 5.3)} \\ &= ((babab \cup (bababS \cap Sbabab))(aba \cup (abaS \cap Saba))] \\ &\subseteq ((babab \cup (bababS \cap Sbabab)(aba \cup (abaS \cap Saba))] \text{ (as } ([A](B)] = [AB]) \\ &\subseteq ((babab \cup bababS)(aba \cup Saba))] \\ &= ((babab \cup bababS)(aba \cup Saba)] \text{ (as } ([A]] = [A]) \\ &\subseteq ((baS)(Sba)] \subseteq ((baS](Sba)] \\ &= ((baS] \cap (Sba)] \text{ (by Lemma 3.2 since } S \text{ is obviously regular).} \end{aligned}$$

Then $(ab)^4 \leq (ba)x$ and $(ab)^4 \leq y(ba)$ for some $x, y \in S$. Then $(ba, x) \in A_{(ab)^4}$ and $(y, ba) \in A_{(ab)^4}$. Since $A_{(ab)^4} \neq \emptyset$, we have

$$\begin{aligned} f((ab)^4) &= ((f \circ 1) \wedge (1 \circ f))((ab)^4) \\ &= \min[(f \circ 1)((ab)^4), (1 \circ f)((ab)^4)] \\ &= \min \left[\bigvee_{(y_1, z_1) \in A_{(ab)^4}} \min\{f(y_1), 1(z_1)\}, \bigvee_{(y_2, z_2) \in A_{(ab)^4}} \min\{1(y_2), f(z_2)\} \right] \\ &\geq \min[\min\{f(ba), 1(x)\}, \min\{1(y), f(ba)\}] \end{aligned}$$

$$\begin{aligned}
&= \min [\min\{f(ba), 1\}, \min\{1, f(ba)\}] \\
&= \min [f(ba), f(ba)] \\
&= f(ba).
\end{aligned}$$

Hence $f((ab)^4) \geq f(ab)$. Since $f(ab) = f((ab)^4)$ we have $f(ab) \geq f(ba)$. By symmetry we can prove that $f(ba) \geq f(ab)$.

(\Leftarrow). Since \mathcal{N} is a semilattice congruence on S , by Lemma 5.1, it is enough to prove that every one-sided ideal of S is a two-sided ideal of S and semiprime. Let R be a right ideal of S and hence a quasi-ideal of S . Let $a \in R$ and $s \in S$. Since R is a quasi-ideal of S , by Lemma 3.3, f_R is a fuzzy quasi-ideal of S . By hypothesis,

$$f(as) = f(sa).$$

Since $as \in RS \subseteq R$, then $f(as) = 1 \implies f(sa) = 1$. Thus $sa \in R \implies SR \subseteq R$ and if $a \in R$, $S \ni b \leq a$ then $b \in R$. Thus R is a left ideal of S . Hence R is an ideal of S . Let $x \in S$ such that $x^2 \in R$. Then $x \in R$. In fact: Since R is a quasi-ideal of S by Lemma 3.6, f_R is a fuzzy quasi-ideal of S . By hypothesis,

$$f_R(x^2) = f_R(x).$$

Since $x^2 \in R$, we have $f_R(x^2) = 1$ then $f_R(x) = 1$, and we have $x \in R$. Thus R is semiprime. Similarly, we can prove that every left ideal of S is an ideal and semiprime. \blacksquare

Lemma 5.4. *Let (S, \cdot, \leq) be an ordered semigroup such that $a \leq a^2$ for all $a \in S$. Then for every fuzzy quasi-ideal f of S we have,*

$$f(a) = f(a^2) \text{ for every } a \in S.$$

Proof. Let $a \in S$ such that $a \leq a^2$. Let f be a fuzzy quasi-ideal of S . By Proposition 3.1, f is a fuzzy subsemigroup of S , then we have,

$$f(a) \geq f(a^2) \geq \min\{f(a), f(a)\} = f(a). \quad \blacksquare$$

Theorem 5.2. *Let (S, \cdot, \leq) be an ordered semigroup and $a \in S$ such that $a \leq a^2$ for all $a \in S$. Then the following are equivalent:*

- (1) $ab \in (baS] \cap (Sba]$ for each $a, b \in S$.
- (2) For every fuzzy quasi-ideal f of S , we have,

$$f(ab) = f(ba) \text{ for every } a, b \in S.$$

Proof. (1) \implies (2). Let f be a fuzzy quasi-ideal of S . Since $ab \in (baS] \cap (Sba]$, then $ab \in (baS]$ and we have $ab \leq (ba)x$ for some $x \in S$. By (1), we have $(ba)x \in (xbaS] \cap (Sxba]$. Then $(ba)x \in (Sxba]$ and we have $(ba)x \leq (yx)(ba)$ and so, $ab \leq (yx)(ba) \implies (yx, ba) \in A_{ab}$. Again, since $ab \in (Sba]$, then $ab \leq z(ba)$ for some $z \in S$ and by (1) we have $z(ba) \in (bazS]$, then $z(ba) \leq (ba)(zt)$ for some $t \in S$. So we have $ab \leq (ba)(zt) \implies (ba, zt) \in A_{ab}$. Since f is a fuzzy quasi-ideal of S and $A_{ab} \neq \emptyset$, we have

$$\begin{aligned}
f(ab) &\geq ((f \circ 1) \wedge (1 \circ f))(ab) \\
&= \min[(f \circ 1)(ab), (1 \circ f)(ab)]
\end{aligned}$$

$$\begin{aligned}
&= \min\left[\bigvee_{(p_1, q_1) \in A_{ab}} \min\{f(p_1), 1(q_1)\}, \bigvee_{(p_2, q_2) \in X_{ab}} \min\{1(p_2), f(q_2)\}\right] \\
&\geq \min\{\min\{f(ba), 1(zt)\}, \min\{1(yx), f(ba)\}\} \\
&= \min\{\min\{f(ba), 1\}, \min\{1, f(ba)\}\} \\
&= \min\{f(ba), f(ba)\} = f(ba).
\end{aligned}$$

By symmetry we can prove that $f(ba) \geq f(ab)$.

(2) \implies (1). Let f be a fuzzy quasi-ideal of S . Since $a \leq a^2$ for all $a \in S$, by Lemma 5.4, we have $f(a) = f(a^2)$ for every $a, b \in S$. By (2), we have $f(ba) = f(ab)$ for every $a, b \in S$. By Theorem 5.1, it follows that S is a semilattice of left and right simple semigroups. Thus by hypothesis, there exist a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups such that

- (1) $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha, \beta \in Y$ and $\alpha \neq \beta$,
- (2) $S = \bigcup_{\alpha \in Y} S_\alpha$, and
- (3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

Let $a, b \in S$, then there exist $\alpha, \beta \in Y$ such that $a \in S_\alpha$ and $b \in S_\beta$. Then $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is left simple we have $S_{\alpha\beta} = (S_{\alpha\beta}c]^*$ and S is right simple, by Lemma 4.1, we have $S_{\alpha\beta} = (cS_{\alpha\beta}]$ for each $c \in S_{\alpha\beta}$. Since $ab, ba \in S_{\alpha\beta}$, we have $ab \in (baS_{\alpha\beta}] \cap (S_{\alpha\beta}ba] \subseteq (baS] \cap (Sba]$. \blacksquare

References

- [1] J. Ahsan, R. M. Latif and M. Shabir, Fuzzy quasi-ideals in semigroups, *J. Fuzzy Math.* **9** (2001), no. 2, 259–270.
- [2] J. Ahsan, K. Y. Li and M. Shabir, Semigroups characterized by their fuzzy bi-ideals, *J. Fuzzy Math.* **10** (2002), no. 2, 441–449.
- [3] J. Ahsan, K. Saifullah and M. Farid Khan, Semigroups characterized by their fuzzy ideals, *Mohu Xitong yu Shuxue* **9** (1995), no. 1, 29–32.
- [4] A. Bargiela and W. Pedrycz, *Granular Computing*, The Kluwer International Series in Engineering and Computer Science, 717, Kluwer Acad. Publ., Boston, MA, 2003.
- [5] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups. Vol. I*, Mathematical Surveys, No. 7 Amer. Math. Soc., Providence, RI, 1961.
- [6] N. Kehayopulu and M. Tsingelis, A note on fuzzy sets in semigroups, *Sci. Math.* **2** (1999), no. 3, 411–413 (electronic).
- [7] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered groupoids, *Semigroup Forum* **65** (2002), no. 1, 128–132.
- [8] N. Kehayopulu and M. Tsingelis, Regular ordered semigroups in terms of fuzzy subsets, *Inform. Sci.* **176** (2006), no. 24, 3675–3693.
- [9] N. Kehayopulu and M. Tsingelis, Fuzzy bi-ideals in ordered semigroups, *Inform. Sci.* **171** (2005), no. 1-3, 13–28.
- [10] N. Kehayopulu, On left regular ordered semigroups, *Math. Japon.* **35** (1990), no. 6, 1057–1060.
- [11] N. Kehayopulu, On regular duo ordered semigroups, *Math. Japon.* **37** (1992), no. 3, 535–540.
- [12] N. Kehayopulu, On completely regular ordered semigroups, *Sci. Math.* **1** (1998), no. 1, 27–32 (electronic).
- [13] N. Kehayopulu, Remark on ordered semigroups, *Math. Japon.* **35** (1990), no. 6, 1061–1063.
- [14] N. Kehayopulu and M. Tsingelis, On the decomposition of prime ideals of ordered semigroups into their \mathcal{N} -classes, *Semigroup Forum* **47** (1993), no. 3, 393–395.
- [15] A. Khan, Y. B. Jun and M. Shabir, Fuzzy ideals in ordered semigroups. I, *Quasigroups Related Systems* **16** (2008), no. 2, 207–220.

- [16] N. Kuroki, Fuzzy bi-ideals in semigroups, *Comment. Math. Univ. St. Paul.* **28** (1980), no. 1, 17–21.
- [17] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, *Fuzzy Sets and Systems* **5** (1981), no. 2, 203–215.
- [18] N. Kuroki, Fuzzy semiprime quasi-ideals in semigroups, *Inform. Sci.* **75** (1993), no. 3, 201–211.
- [19] J. N. Mordeson, D. S. Malik and N. Kuroki, *Fuzzy Semigroups*, Studies in Fuzziness and Soft Computing, 131, Springer, Berlin, 2003.
- [20] J. N. Mordeson and D. S. Malik, *Fuzzy Automata and Languages*, Computational Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [21] W. Pedrycz and F. Gomide, *An Introduction to Fuzzy Sets*, Complex Adaptive Systems, MIT Press, Cambridge, MA, 1998.
- [22] M. Shabir and A. Khan, Characterizations of ordered semigroups by the properties of their fuzzy ideals, *Comput. Math. Appl.* **59** (2010), no. 1, 539–549.
- [23] O. Steinfeld, *Quasi-Ideals in Rings and Semigroups*, Disquisitiones Mathematicae Hungaricae, 10, Akad. Kiadó, Budapest, 1978.
- [24] L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.