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Zero Theorems of Accretive Operators

YAN HAO
School of Mathematics, Physics and Information Science,
Zhejiang Ocean University, Zhoushan 316004, China
zjhaoyan@yahoo.cn

Abstract. In this paper, we introduce and analysis an iterative method for finding a common zero of two m-accretive operators. Under appropriate restrictions imposed on the parameters, we obtain a convergence theorem in a real Banach space.

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1. Introduction and preliminaries

Throughout this paper, we always assume that E is a real Banach space. Let E^* be the dual space of E. Let $\varphi : [0,\infty] := R^+ \to R^+$ be a continuous and strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a gauge function. The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \quad \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the case that $\varphi(t) = t$, we write J for J_φ and call J the normalized duality mapping.

Following Browder [2], we say that a Banach space E has a weakly continuous duality mapping if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to $J_\varphi x$). It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad \forall t \geq 0,$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E,$$

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where ∂ denotes the sub-differential in the sense of convex analysis.

The norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit is attained uniformly for $(x, y) \in U \times U$.

A Banach space E is said to strictly convex if and only if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for all $x, y \in E$ and $0 < \lambda < 1$ implies that $x = y$. E is said to uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x-y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

Let C be a nonempty, closed and convex subset of E. Recall that a mapping $f : C \to C$ is said to be α-contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|fx - fy\| \leq \alpha\|x-y\|, \quad \forall x, y \in C.$$

In this paper, we use Π_C to denote the collection of all contractive mappings on C. That is, $\Pi_C = \{f | f : C \to C \text{ is a contractive mapping}\}$. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(T)$ to denote the set of fixed points of T. The class of nonexpansive mapping is a kind of important nonlinear mapping which was studied by many authors; see, for example, [1–25].

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [3, 18]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C, \tag{1.1}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. That is,

$$x_t = tu + (1-t)Tx_t. \tag{1.2}$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [3] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [18] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$. Xu [22] proved that Browder's results still hold in reflexive Banach spaces which have a weakly continuous duality mapping; see [22] for more details.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is called a retraction from C onto D provided $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$

is sunny provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions are characterized as follows [9, 19]:

If E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0, \quad \forall x \in C, y \in D. \tag{1.3}$$

Reich [18] showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Theorem 1.1. *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D; that is, Q satisfies the property:*

$$\langle u - Qu, J(y - Qu)\rangle \leq 0, \quad \forall u \in C, y \in D.$$

If E is a reflexive Banach space which has a weakly continuous duality map, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J_\varphi(y - Qx)\rangle \leq 0, \quad \forall x \in C, y \in D. \tag{1.4}$$

In 2006, Xu [22] obtained an analogue of Theorem 1.1 in a reflexive Banach space. To be more precise, he proved the following result.

Theorem 1.2. *Let E be a reflexive Banach space and has a weakly continuous duality map $J_\varphi(x)$ with gauge φ. Let C be closed convex subset of E and let $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C to the equation* (1.2). *Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point of T.*

Recall that a mapping A with domain $D(A)$ and range $R(A)$ in E is accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i (i = 1, 2)$, there exists a $j_\varphi(x_2 - x_1) \in J_\varphi(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j_\varphi(x_2 - x_1)\rangle \geq 0.$$

An accretive operator A is m-accretive if $R(I+rA) = E$ for each $r > 0$. Throughout this article we always assume that A is m-accretive and has a zero (i.e., the inclusion $0 \in A(z)$ is solvable). For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I+rA)^{-1}$. Note that if A is m-accretive, then $J_r : E \to D(A)$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from E to $C := \overline{D(A)}$ which will be assumed convex.

Kim and Xu [12] studied m-accretive operators by considering the following iterative algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $u \in C$ is a fixed point and $J_{r_n} = (I + r_n A)^{-1}$. They proved that the sequence $\{x_n\}$ generated by the above iterative algorithm converges strongly to a zero point of A in the framework of uniformly smooth Banach spaces.

Recently, Qin and Su [15] studied so-called modified Mann iterations

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \end{cases} \tag{1.6}$$

where $u \in C$ is a fixed point, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $J_{r_n} = (I + r_n A)^{-1}$. They obtained that the sequence generated by the above iterative algorithm converges strongly to a zero point of A assume that E is uniformly smooth. They also proved that the conclusion still holds provided that E is a reflexive Banach space which has a weak continuous duality map.

Viscosity approximation method, which was first introduced by Moudafi [14], for the problem of finding fixed points of nonexpansive mapping has been studied by many authors.

For a real number $t \in (0,1)$ and a contractive mapping $f \in \Pi_C$, we define a mapping $T_t x = tf(x) + (1-t)Tx$ for all $x \in C$. It is obviously that T_t is a contractive mapping on C. In fact, for any $x, y \in C$, we obtain

$$\begin{aligned} \|T_t x - T_t y\| &= \|t(f(x) - f(y)) + (1-t)(Tx - Ty)\| \\ &\leq \alpha t \|x - y\| + (1-t)\|Tx - Ty\| \\ &\leq \alpha t \|x - y\| + (1-t)\|x - y\| \\ &= (1 - t(1-\alpha))\|x - y\|. \end{aligned}$$

Let x_t be the unique fixed point of T_t. That is, x_t is the unique solution of the fixed point equation:

$$x_t = tf(x_t) + (1-t)Tx_t. \tag{1.7}$$

Theorem 1.3. [5, 7] *Let E be a reflexive Banach space which has a weakly continuous duality mapping $J_\varphi(x)$. Let C be closed convex subset of E and $T : C \to C$ be a nonexpansive mapping. Let $f : C \to C$ be a contractive mapping with $F(f) \neq \emptyset$. For any $t \in (0,1)$, let $\{x_t\}$ be defined by (1.7), where T is a nonexpansive mapping. Define a mapping $Q : \Pi_C \to F(T)$ by $Q(f) := \lim_{t\to 0} x_t, f \in \Pi_C$. Then then mapping Q is the sunny nonexpansive retraction from Π_C onto $F(T)$.*

In [5], Chen and Zhu also considered the following iterative methods:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0, \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $f : C \to C$ is an α-contraction and $J_{r_n} = (I + r_n A)^{-1}$. They proved that the sequence $\{x_n\}$ generated by the above iterative algorithm converges strongly to a zero point of A in a real Banach space which includes the corresponding results in Xu [22].

Recently, Chen *et al.* [6] further studied the following iterative method:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \end{cases} \tag{1.9}$$

where $f : C \to C$ is an α-contraction, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $J_{r_n} = (I + r_n A)^{-1}$. They also obtained a zero theorem of the operator A, see [6] for more details.

In this paper, we consider a pair of m-accretive operators instead of a single operator which was studied by Chen and Zhu [5], Chen *et al.* [6], Cho and Qin [7], Kim and Xu [12], Qin and Su [15] and Xu [22] in a reflexive Banach space which admits a weak continuous duality map. Strong convergence theorems are established. To prove the main result, we need the following results.

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [13].

Lemma 1.1. *Assume that a Banach space E has a weakly continuous duality mapping J_φ with a gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Lemma 1.2. [4] *Let C be a closed convex subset of a strictly convex Banach space E. Let T_1 and T_2 be two nonexpansive mappings on C. Suppose that $F(T_1) \cap F(T_2)$ is nonempty. Then a mapping T on C defined by*

$$Tx = \lambda T_1 x + (1-\lambda) T_2 x, \quad \forall x \in C$$

is well defined, nonexpansive and $F(T) = F(T_1) \cap F(T_2)$ holds.

Lemma 1.3. [23] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n, \quad \forall n \ge 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.4. [13] *Let E be a Banach space satisfying a weakly continuous duality map, C a nonempty closed convex subset of E and $T : C \to C$ a nonexpansive mapping with a fixed point. Then $I - T$ is demi-closed at zero, i.e., if $\{x_n\}$ is a sequence in C which converges weakly to x and if the sequence $\{(I-T)x_n\}$ converges strongly to zero, then $x = Tx$.*

2. Main results

Theorem 2.1. *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ and $f \in \Pi_C$. Let A and B be m-accretive operators in E such that $C := \overline{D(A)} \cap \overline{D(B)}$ is convex. Let $\{\beta_n\}$ be a real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n J_r x_n + (1-\beta_n) J_s x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases}$$

where $r, s > 0$, $J_r = (I+rA)^{-1}$ and $J_s = (I+sB)^{-1}$. Assume that $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. If the above control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*

(b) $\lim_{n\to\infty} \beta_n = \beta \in (0,1)$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$*,*

then the sequence $\{x_n\}$ converges strongly to some point $Q(f) \in A^{-1}(0) \cap B^{-1}(0)$, where Q is the sunny nonexpansive retraction $Q : \Pi_C \to A^{-1}(0) \cap B^{-1}(0)$.

Proof. First, we prove that $\{x_n\}$ is bounded. For any $p \in A^{-1}(0) \cap B^{-1}(0)$, we see that

$$\begin{aligned} \|y_n - p\| &= \|\beta_n J_r x_n + (1-\beta_n) J_s x_n - p\| \\ &= \|\beta_n (J_r x_n - J_r p) + (1-\beta_n)(J_s x_n - J_s p)\| \\ &\leq \beta_n \|J_r x_n - J_r p\| + (1-\beta_n)\|J_s x_n - J_s p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n) y_n - p\| \\ &= \|\alpha_n [f(x_n) - f(p)] + \alpha_n [f(p) - p] + (1-\alpha_n)(y_n - p)\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|y_n - p)\| \\ &\leq \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p)\| \\ &= [1 - \alpha_n(1-\alpha)]\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned} \tag{2.1}$$

Putting $B = \max\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\alpha}\}$, we show that $\|x_n - p\| \leq B$ for all $n \geq 0$. It is easy to see that the result holds for $n = 0$. We assume that the result holds for some $n \geq 0$. From (2.1), we can see that $\|x_{n+1} - p\| \leq B$. This shows that the sequence $\{x_n\}$ is bounded. Note that

$$\begin{aligned} & y_n - y_{n-1} \\ &= \beta_n J_r x_n + (1-\beta_n) J_s x_n - [\beta_{n-1} J_r x_{n-1} + (1-\beta_{n-1}) J_s x_{n-1}] \\ &= \beta_n (J_r x_n - J_r x_{n-1}) + (1-\beta_n)(J_s x_n - J_s x_{n-1}) + (J_r x_{n-1} - J_s x_{n-1})(\beta_n - \beta_{n-1}). \end{aligned}$$

This implies that

$$\|y_n - y_{n-1}\| \leq \beta_n \|J_r x_n - J_r x_{n-1}\| + (1-\beta_n)\|J_s x_n - J_s x_{n-1}\|$$

$$
\begin{aligned}
&+ \|J_r x_{n-1} - J_s x_{n-1}\| |\beta_n - \beta_{n-1}| \\
&\le \beta_n \|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&+ \|J_r x_{n-1} - J_s x_{n-1}\| |\beta_n - \beta_{n-1}| \\
&\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M_1, \qquad (2.2)
\end{aligned}
$$

where M_1 is an appropriate constant such that $M_1 \ge \sup_{n\ge1}\{\|J_r x_{n-1} - J_s x_{n-1}\|\}$. On the other hand, we have

$$
\begin{aligned}
x_{n+1} - x_n &= [\alpha_n f(x_n) + (1-\alpha_n) y_n] - [\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1}) y_{n-1}] \\
&= \alpha_n [f(x_n) - f(x_{n-1})] + (1-\alpha_n)(y_n - y_{n-1}) \\
&\quad + [f(x_n) - y_{n-1}](\alpha_n - \alpha_{n-1}).
\end{aligned}
$$

This gives that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\le \alpha_n \|f(x_n) - f(x_{n-1})\| + (1-\alpha_n)\|y_n - y_{n-1}\| \\
&\quad + \|f(x_n) - y_{n-1}\| |\alpha_n - \alpha_{n-1}| \\
&\le \alpha_n \alpha \|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_2, \qquad (2.3)
\end{aligned}
$$

where M_2 is an appropriate constant such that $M_2 \ge \sup_{n\ge1}\{\|f(x_n) - y_{n-1}\|\}$. Substituting (2.2) into (2.3), we see that

$$
\|x_{n+1} - x_n\| \le [1 - \alpha_n(1-\alpha)]\|x_n - x_{n-1}\| + (|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|) M_3, \qquad (2.4)
$$

where M_3 is an appropriate constant such that $M_3 = \max\{M_1, M_2\}$. From the conditions (a), (b) and applying Lemma 1.3 to (2.4), we obtain that

$$
\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \qquad (2.5)
$$

Define an operator $W : C \to C$ by

$$
Wx := \beta J_r x + (1-\beta) J_s x, \quad \forall x \in C,
$$

where $(0,1) \ni \beta = \lim_{n\to\infty} \beta_n$. From Lemma 1.2, we see that W is nonexpansive with $F(W) = F(J_r) \cap F(J_s) = A^{-1}(0) \cap B^{-1}(0)$.

Next, we prove that $\|x_n - Wx_n\| \to 0$ as $n \to \infty$. Note that

$$
\begin{aligned}
y_n - Wx_n &= \beta_n J_r x_n + (1-\beta_n) J_s x_n - [\beta J_r x_n + (1-\beta) J_s x_n] \\
&= (\beta_n - \beta) J_r x_n + (\beta - \beta_n) J_s x_n.
\end{aligned}
$$

It follows from the condition (b) that

$$
\|y_n - Wx_n\| = 0. \qquad (2.6)
$$

On the other hand, we have

$$
\begin{aligned}
\|x_n - Wx_n\| &= \|x_n - x_{n+1} + x_{n+1} - Wx_n\| \\
&= \|x_n - x_{n+1} + \alpha_n f(x_n) + (1-\alpha_n) y_n - Wx_n\| \\
&= \|x_n - x_{n+1} + \alpha_n (f(x_n) - Wx_n) + (1-\alpha_n)(y_n - Wx_n)\| \\
&= \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - Wx_n\| + (1-\alpha_n)\|y_n - Wx_n\|.
\end{aligned}
$$

It follows from the condition (a), (2.5) and (2.6) that

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{2.7}$$

Next, we show that $\limsup_{n\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f)-x_n)\rangle \le 0$, where $Q : \Pi_C \to A^{-1}(0)\cap B^{-1}(0)$ is the sunny nonexpansive retraction. To show it, we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f)-x_n)\rangle = \lim_{i\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f)-x_{n_i})\rangle. \tag{2.8}$$

Since E is reflexive, we may further assume that $x_{n_i} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. From Lemma 1.4, we see that $\bar{x} \in F(W) = A^{-1}(0)\cap B^{-1}(0)$. Hence, we arrive at

$$\limsup_{n\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f)-x_n)\rangle = \langle (I-f)Q(f), J_\varphi(Q(f)-\bar{x})\rangle \le 0.$$

It follows from (2.5) that

$$\limsup_{n\to\infty}\langle (I-f)Q(f), J_\varphi(Q(f)-x_{n+1})\rangle \le 0. \tag{2.9}$$

Finally, we show that $x_n \to Q(f)$ as $n\to\infty$. Notice that

$$\Phi(\|y_n-Q(f)\|) = \Phi(\|\beta_n(J_r x_n - Q(f)) + (1-\beta_n)(J_s x_n - Q(f))\|) \le \Phi(\|x_n - Q(f)\|).$$

It follows from Lemma 1.1 that

$$\begin{aligned}
\Phi(\|x_{n+1}-Q(f)\|) &= \Phi(\|\alpha_n(f(x_n)-f(Q(f))) + \alpha_n(f(Q(f))-Q(f)) \\
&\quad + (1-\alpha_n)(y_n - Q(f))\|) \\
&\le \Phi(\alpha_n\|f(x_n)-f(Q(f))\| + (1-\alpha_n)\|y_n-Q(f)\|) \\
&\quad + \alpha_n\langle f(Q(f))-Q(f), J_\varphi(x_{n+1}-Q(f))\rangle \\
&\le \Phi(\alpha_n\alpha\|x_n-Q(f)\| + (1-\alpha_n)\|x_n-Q(f)\|) \\
&\quad + \alpha_n\langle f(Q(f))-Q(f), J_\varphi(x_{n+1}-Q(f))\rangle \\
&\le \Phi((1-\alpha_n(1-\alpha))\|x_n-Q(f)\|) \\
&\quad + \alpha_n\langle f(Q(f))-Q(f), J_\varphi(x_{n+1}-Q(f))\rangle \\
&\le (1-\alpha_n(1-\alpha))\Phi(\|x_n-Q(f)\|) \\
&\quad + \alpha_n\langle f(Q(f))-Q(f), J_\varphi(x_{n+1}-Q(f))\rangle.
\end{aligned}$$

From Lemma 1.3, we see that $\Phi(\|x_{n+1}-Q(f)\|)\to 0$ as $n\to\infty$. That is,

$$\|x_n - Q(f)\| \to 0 \quad \textit{as } n\to\infty.$$

This completes the proof. ∎

If $J_s = I$, then identity mapping, then we have the following results from Theorem 2.1.

Corollary 2.1. *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ and $f\in\Pi_C$. Let A be a m-accretive operator in*

E such that $C := \overline{D(A)}$ is convex. Let $\{\beta_n\}$ be a real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n J_r x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad n \geq 0, \end{cases}$$

where $r > 0$ and $J_r = (I + rA)^{-1}$. Assume that $A^{-1}(0) \neq \emptyset$. If the above control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*

(b) $\lim_{n\to\infty} \beta_n = \beta \in (0,1)$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$*,*

then the sequence $\{x_n\}$ converges strongly to some point $Q(f) \in A^{-1}(0)$, where Q is the sunny nonexpansive retraction $Q : \Pi_C \to A^{-1}(0)$.

If $A = B$ and $r = s$ in Theorem 2.1, we have the following result immediately.

Corollary 2.2. *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ and $f \in \Pi_C$. Let A be a m-accretive operator in E such that $C := \overline{D(A)}$ is convex. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_r x_n, \quad n \geq 0,$$

where $r > 0$ and $J_r = (I + rA)^{-1}$. Assume that $A^{-1}(0) \neq \emptyset$. If the above control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(b) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*,*

then the sequence $\{x_n\}$ converges strongly to some point $Q(f) \in A^{-1}(0)$, where Q is the sunny nonexpansive retraction $Q : \Pi_C \to A^{-1}(0)$.

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