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Finite Groups in which Primary Subgroups have Cyclic Cofactors

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Abstract. In this paper, we prove the following theorem: Let G be a group, q be the largest prime divisor of $|G|$ and $\pi = \pi(G) \setminus \{q\}$. Suppose that the factor group $X/core_G X$ is cyclic for every p-subgroup X of G and every $p \in \pi$. Then:
(1) G is soluble and its Hall $\{2,3\}'$-subgroup is normal in G and is a dispersive group by Ore;
(2) All Hall $\{2,3\}$-subgroups of G are metanilpotent;
(3) Every Hall p'-subgroup of G is a dispersive group by Ore, for every $p \in \{2,3\}$;
(4) $l_r(G) \leq 1$, for all $r \in \pi(G)$.

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1. Introduction

All groups considered in this paper are finite.

If H is a subgroup of a group G, then the factor group $H/core_G H$ is said to be the cofactor of H in G, where $core_G H = \cap_{x \in G} H^x$ is the maximal normal subgroup of G contained in H.

The structure of the groups with given some properties of cofactors of subgroups was studied in [2–4, 11, 12]. In [2], the non-soluble groups in which the cofactors of maximal subgroups are either nilpotent groups or Schmidt groups were described. From the result in [2], we know that the groups with nilpotent cofactors of maximal subgroups are soluble. In [4], for an odd prime p, it was proved that the groups in which maximal subgroups have p-nilpotent cofactors are p-soluble. In [12], the groups in which cofactors of subgroups are simple groups have considered: Such

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groups are soluble and all Sylow subgroups of the factor group $G/Z_\infty(G)$ are abelian. In [11], the author have studied the groups in which all biprimary subgroups have primary cofactors and also described the finite non-soluble groups in which all proper subgroups have primary cofactors or biprimary cofactors. In [3], the author investigated the p-groups G in which $|H/core_G H| \in \{1, p\}$ for every subgroup H of G. In particular, it was proved that such groups are meta abelian.

As a continuation, in this paper, we study the groups in which some primary subgroups have cyclic cofactors. We obtain the following theorem.

Theorem 1.1. *Let G be a group, q be the largest prime divisor of $|G|$ and $\pi = \pi(G) \setminus \{q\}$. Suppose that the factor group $X/core_G X$ is cyclic for every p-subgroup X of G and every $p \in \pi$. Then:*

(1) *G is soluble and its Hall $\{2,3\}'$-subgroup is normal in G and is a dispersive group by Ore;*
(2) *All Hall $\{2,3\}$-subgroups of G are metanilpotent;*
(3) *Every Hall p'-subgroup of G is a dispersive group by Ore, for every $p \in \{2,3\}$;*
(4) *$l_r(G) \leq 1$, for all $r \in \pi(G)$.*

All unexplained notations and terminologies are standard. The reader is referred to [7, 9].

2. Preliminaries

Recall that a group G is called primitive if it has a maximal subgroup M such that $core_G M = 1$. In this situation, M is called a stabilizer of G (see [9]).

Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1 > p_2 > \cdots p_r$. Then, G is said to be a dispersive group by Ore if there exists $G_{p_1}, G_{p_2}, \cdots, G_{p_r}$ such that G_{p_i} is a Sylow p_i-subgroup of G and $G_{p_1} G_{p_2} \cdots G_{p_k} \trianglelefteq G$, for $k = 1, 2, \cdots, r$.

If π be a nonempty set of prime numbers, then we denote by π' the complement of π in the set of all prime numbers. We denote by $\pi(G)$ the set of all prime divisors of the order $|G|$ of a group G.

For a p-soluble group G, we denote by $l_p(G)$ the p-length of G; denote by $r_p(G)$ the p-rank of G, i.e., $r_p(G) = max\{n \mid p^n$ is the order of some p-chief factor of $G\}$. For a soluble group G, we denote by $d(G)$ the derived length of G and denotes by $r(G)$ the rank of G, i.e., $r(G) = max_{p \in \pi(G)} r_p(G)$. If $G = 1$, then let $r_p(G) = r(G) = 0$ (see [9, Definition VI.5.2])

In order to prove our theorem, we need the following a series of Lemmas.

Lemma 2.1. [8, Lemma 2.6] *Let G be a soluble primitive group with stabilizer M. Then:*

(1) *$\Phi(G) = 1$;*
(2) *$F(G) = C_G(F(G)) = O_p(G)$ and $F(G)$ is an elementary abelian p-group of order p^n for some prime p and a natural number n;*
(3) *G has a unique minimal normal subgroup which coincides with $F(G)$;*
(4) *$G = [F(G)]M$ and $O_p(M) = 1$;*
(5) *M is isomorphic with a irreducible subgroup of $GL(n, p)$.*

Lemma 2.2. *Let $\mathfrak{F}$ be a saturated formation and G a non-simple group. If G is not in $\mathfrak{F}$ but the factor group $G/N \in \mathfrak{F}$ for all nonidentity normal subgroup N of G, then G is a primitive group.*

Proof. It is obvious and hence we omit the proof. ∎

Lemma 2.3. *Let G be a p-soluble group and $r_p(G) \leq 2$. Then the following statements hold.*

(1) *If $p = 2$, then $l_2(G) \leq 2$;*
(2) *If $p = 2$ and G is S_4-free, then $l_2(G) \leq 1$;*
(3) *If $p = 2$ and G is A_4-free, then G is 2-nilpotent;*
(4) *If $p > 2$, then $l_p(G) \leq 1$;*
(5) *If $p > 2$ and p is the least prime devisor of the order of G, then G is p-nilpotent.*

Proof. We prove the lemma by induction on the order of G. It is well-known that the class of all p-soluble groups with p-length $\leq k$, where k is a natural number, is a saturated formation (see [9, Lemma VI. 6.9]) and that the class of all p-nilpotent groups is also a saturated formation (see [7, p. 98, Example 2]). Assume that the assertion of the lemma is false. By Lemma 2.2, G is a primitive group. Then, since G is p-soluble, it is easy to see that $G = [E_{p^n}]M$ such that $core_G(M) = 1$ and $C_G(E_{p^n}) = E_{p^n}$, where E_{p^n} is an elementary abelian p-group of order p^n for some prime p. Since $r_p(G) \leq 2$ by the hypothesis, $n \leq 2$.

If $n = 1$, then M is isomorphic to some subgroup of the automorphism group of the cyclic group E_p of order p. Since the automorphism group is a cyclic group of order $p - 1$, the order of M is not divided by p and so $l_p(G) \leq 1$. If p is the least prime divisor of $|G|$, then $M = 1$ and consequently $|G| = p$.

Assume that $n = 2$. Then, obviously, M is isomorphic to a nonidentity irreducible subgroup of $Aut(E_{p^2}) = GL(2, p)$ and $O_p(M) = 1$.

If $p = 2$, then $GL(2, 2) \simeq S_3$, where S_3 is the symmetric group of degree 3. If $M \simeq S_3$, then $G \simeq S_4$ and $l_2(G) = 2$. If M is not isomorphic to S_3, then $|M| = 3$ since $O_2(M) = 1$ and so $G \simeq A_4$.

Suppose that $p > 2$. Since $|GL(2, p)| = p(p^2 - 1)(p - 1)$, the order of a Sylow p-subgroup P of G is p^3. If P is abelian, then $l_p(G) = 1$ by [9, Theorem VI.6.6] . Assume that P is non-abelian. Then by [9, Theorem I.14.10], P is isomorphic to either a group of exponent p or a metacyclic group

$$M_3(p) := \langle a, b \mid a^{p^2} = b^p = 1, a^b = a^{1+p} \rangle = [\langle a \rangle]\langle b \rangle.$$

If P has exponent p, then $l_p(G) \leq 1$ by the Theorem in [1]. If $P \simeq M_3(p)$, then $P = [E_{p^2}]A$, where A is a Sylow p-subgroup of M of order p. But by [6, Theorem 5.4.3], the subgroup $\Omega_1(P)$ generated by all elements of P of order p is an elementary abelian subgroup of order p^2. Hence $\Omega_1(P) = E_{p^2}$ and hence A is not existent. The contradiction shows that $P \simeq M_3(p)$ is impossible.

Suppose that $p > 2$ and p is the least prime divisor of $|G|$. If q is a prime divisor of G and $q \neq p$, then q divides $|GL(2, p)| = p(p^2 - 1)(p - 1)$. Since $p < q$, q divides $p + 1$. This implies that $p = 2$ and $q = 3$, a contradiction. Thus the proof is completed. ∎

Lemma 2.4. *Let H and K be subgroups of a group G. If $K \subseteq H$, then $core_G K \subseteq core_H K$.*

Proof. Since $core_G K$ is the maximal normal subgroup of G contained in K, $core_G K \subseteq K \subseteq H$ and $core_G K$ is normal in H. However, since $core_H K$ is the maximal normal subgroup of H contained in H, $core_G K \subseteq core_H K$. ∎

Lemma 2.5. *If N and H be subgroups of a group G, $N \trianglelefteq G$ and $N \subseteq H$, then $N \subseteq core_G H$ and $core_{G/N}(H/N) = (core_G H)/N$.*

Proof. Obviously, $N \subseteq core_G H$ and $(core_G H)/N$ is a normal subgroup of G/N contained in H/N. Hence

$$(core_G H)/N \subseteq core_{G/N}(H/N).$$

On the other hand, let $core_{G/N}(H/N) = K/N$. Since K/N is normal in G/N, K is normal in G and so $K \subseteq core_G H$. This implies that

$$core_{G/N}(H/N) \subseteq (core_G H)/N.$$

Therefore, $core_{G/N}(H/N) = (core_G H)/N$. ∎

Lemma 2.6. *Let G be a group and p a prime. If the cofactor of every p-subgroup of G is cyclic. Then the following statements hold.*

(1) *If H is a subgroup of G, then the cofactor of every p-subgroup of H is cyclic;*
(2) *If N is a normal subgroup of G, then the cofactor of every p-subgroup of G/N is cyclic.*

Proof. (1) Let P be an arbitrary p-subgroup of H. By Lemma 2.4, $core_G P \subseteq core_H P$. Since $P/core_G P$ is cyclic by the condition and

$$P/core_H P \simeq (P/core_G P)/(core_H P/core_G P),$$

we see that $P/core_H P$ is cyclic.

(2) Let H/N be an arbitrary p-subgroup of G/N and P is a Sylow p-subgroup of H. Then $H = PN$ and clearly $N core_G P \subseteq core_G H$. Since

$$H/core_G H = P core_G H/core_G H \simeq P/(P \cap core_G H) \simeq$$
$$\simeq (P/core_G P)/((P \cap core_G H)/core_G P)$$

and $P/core_G P$ is cyclic, we obtain that $H/core_G H$ is cyclic. Thus by Lemma 2.5, $(H/N)/(core_{G/N}(H/N)) = (H/N)/(core_G H)/N \cong H/core_G H$ is cyclic. ∎

Lemma 2.7. [10, Theorem 1.1] *Let G be a Schmidt group.Then*

(1) *$G = [P]Q$, where P is a normal Sylow p-subgroup of G, Q is a non-normal cyclic Sylow q-subgroup of G, p and q are different primes.*
(2) *$|P/P'| = p^m$, where m is the least natural number such that $q \mid p^m$.*

Example 2.1. In the symmetric group S_4 of degree 4,

$$K_4 = \{1, (12)(34), (13)(24), (14)(23)\}$$

is a minimal normal subgroup which is also an elementary abelian group of order 4.

$$P = \{1, (12)(34), (13)(24), (14)(23), (1234), (1423), (13), (24)\}$$

is a Sylow 2-subgroup of S_4. The subgroups

$$H_1 = \langle (13)(24) \rangle \times \langle (13) \rangle, \quad H_2 = \langle (13)(24) \rangle \times \langle (24 \rangle$$

are an elementary abelian subgroups of order 4 such that $core_G H_i = 1$, for $i = 1, 2$.

Lemma 2.8. *Let G be a group and p the least prime divisor of $|G|$. Suppose that the factor group $X/core_G X$ is cyclic for every p-subgroup X of G. Then:*

(1) *G is soluble and $r_p(G) \leq 2$;*
(2) *If $p > 2$, then G is p-nilpotent;*
(3) *If $p = 2$ and G is A_4-free, then G is 2-nilpotent;*
(4) *$l_p(G) = 1$.*

Proof. (1) Let P be a Sylow p-subgroup of G. Since $P/O_p(G)$ is cyclic, by [9, Theorem IV.2.8], $G/O_p(G)$ is p-nilpotent and so G is soluble. Let A/B be a p-chief factor of G. Then A/B is a minimal normal subgroup of G/B. If C/B is a proper subgroup of A/B, then obviously $core_{G/B}C/B = 1$. By Lemma 2.6, the condition of the lemma holds for all sections of G. Hence C/B is cyclic. Since C/B is elementary abelian, $|C/B| = 1$ or p. This implies that $|A/B| = p$ or p^2. Thus, $r_p(G) \leq 2$.

(2)-(3) Suppose that the assertions are false and G is a counterexample of minimal order. By Lemma 2.6, we see that all proper subgroups of G are p-nilpotent. Then by [9, Theorem IV.5.4], G is a minimal non-nilpotent with a normal Sylow p-subgroup. Hence by [7, Theorem 3.4.11] and Lemma 2.7, $G = [P]Q$, where Q is a cyclic Sylow q-subgroup of G, P is a normal Sylow p-subgroup of G and P/P' is a minimal normal subgroup of G/P' of order p^m and m is the least natural number such that $q|p^m - 1$. Since $p < q$ and $r_p(G) \leq 2$, $m = 2$. Thus $p = 2$ and $q = 3$. This implies that P/P' is an elementary abelian group of order 4 and hence $QP'/P' \leq GL(2,2) \simeq S_3$. Therefore $|Q| = 3$. Consequently $G/P' \simeq A_4$, which contradicts the fact that G is A_4-free. This contradiction show that the statements (2) and (3) hold.

(4) By (1) and Lemma 2.3, we only need to consider the case that $p = 2$ and G has a section which is isomorphic to S_4. But by Example 2.1, S_4 has a non-cyclic subgroup of order 4 with core 1. This contradicts Lemma 2.6. Thus the lemma is proved. ∎

Lemma 2.9. *Let G be a group. Suppose that $X/core_G X$ is cyclic, for every p-subgroup of G and every $p \in \{2,3\}$. Then:*

(1) *G is soluble and G is $\{2,3\}'$-closed, i.e., G has a normal Hall $\{2,3\}'$-subgroup;*
(2) *If H is a Hall $\{2,3\}$-subgroup of G, then $H/F(H)$ is cyclic.*

Proof. (1) By Lemma 2.8, we know that G is soluble. Let $\pi = \mathbb{P} \setminus \{2,3\}$ and G_π be a Hall π-subgroup of G. If $O_\pi(G) \neq 1$, then by Lemma 2.6 and by induction, $G_\pi/O_\pi(G)$ is normal in $G/O_\pi(G)$. It follows that G_π is normal in G. Now assume that $O_\pi(G) = 1$. Since the class of all π-closed groups is a saturated formation (see [13, p. 34]), By Lemma 2.2, G is primitive. Then by Lemma 2.1, we have that $G = [F(G)]M$, where $F(G)$ is the unique minimal normal subgroup of G and M is a maximal subgroup of G with $core_G M = 1$. Because $O_\pi(G) = 1$, $F(G)$ is either a 2-subgroup or a 3-subgroup. Consequently, $|F(G)| \in \{2,4,3,9\}$ since $r_p(G) \leq 2$ for $p \in \{2,3\}$ by Lemma 2.7(1). If $F(G)$ is a 2-group, then the order of $F(G)$ is 2 or 4 and M is isomorphic to some subgroup of the general linear group $GL(2,2)$. But since the order of $GL(2,2)$ is 6, we have that $G_\pi = 1$. If $F(G)$ is a 3-group, then the order of $F(G)$ is 3 or 9 and M is isomorphic to some subgroup of $GL(2,3)$. Since $|GL(2,3)| = 48$, we also have that $G_\pi = 1$. Thus, G has a normal Hall $\{2,3\}'$-subgroup.

(2) Let H_p $(p = 2, 3)$ be a Sylow p-subgroup of H. Clearly $core_G(H_p) \leq O_p(H)$, where $p \in \{2, 3\}$. Then, by the hypothesis, we see that $H_2/O_2(H)$ is cyclic and $H_3/O_3(H)$ is cyclic. Since, obviously, $F(H) = O_2(H) \times O_3(H)$, $H_2F(H)/F(H) \simeq H_2/H_2 \cap F(H) = H_2/O_2(H)$ is cyclic. Similarly, $H_3F(H)/F(H)$ is cyclic. Hence by [9, Theorem IV. 2.8], $H/F(H)$ has a normal 2-complement, which is a Sylow 3-subgroup of $H/F(H)$. On the other hand, by Lemma 2.8(4), $l_2(G) \leq 1$ and consequently $l_2(H) \leq 1$. This means that $H/O_{2'}(H) = H/O_3(H)$ has a normal Sylow 2-subgroup. Thus $H/F(H) \simeq (H/O_3(H))/(F(H)/O_3(H))$ has a normal Sylow 2-subgroup. This implies that $H/F(H)$ is nilpotent. Now since every Sylow subgroup of $H/F(H)$ is cyclic, we obtain that $H/F(H)$ is cyclic. This completes the proof. ∎

Lemma 2.10. *Suppose that G is a $\{p, q\}$-group, where $p < q$. If the cofactors of all p-subgroups are cyclic, then G is metanilpotent.*

Proof. It is well-known that the class of all metanilpotent groups is a saturated formation (see [7, Theorem 3.1.20]). Assume that the assertion of the lemma is false and let G be a counterexample of minimal order. Then by Lemma 2.6, we see that every factor group of G is metanilpotent. Hence by Lemma 2.2, G is primitive. Then by Lemma 2.1, $F(G)$ is a primary group. If $F(G)$ is a p-group, then by Lemma 2.8(4), $F(G)$ is a Sylow p-subgroup of G. If $F(G)$ is a q-group, i.e., $O_p(G) = 1$, then by the hypothesis, every Sylow p-subgroup is cyclic and so G is q-closed by [9, Theorem IV.2.8]. This show that in any case, G is metanilpotent. This contradiction completes the proof. ∎

Lemma 2.11. *If G is metanilpotent group, then $l_p(G) \leq 1$, for any prime p.*

Proof. Let N be a nilpotent normal subgroup of G such that G/N is nilpotent. Then for any prime p, $N = N_p \times N_{p'}$, where N_p is the Sylow p-subgroup of N and $N_{p'}$ is the Hall p'-subgroup of N. Since $N_{p'} char N \trianglelefteq G$, $N_{p'} \trianglelefteq G$ and consequently $G_pN_{p'} = G_pN \trianglelefteq G$, where G_p is a Sylow p-subgroup of G. Therefore $l_p(G) \leq 1$. ∎

3. Proof of theorem

By using lemmas in above section, we now can prove our theorem.

Proof of Theorem 1.1. (1) If $q \leq 3$, then G is a primary group or a $\{2, 3\}$-group. In this case, obviously, the assertion holds. Now assume that $q > 3$. Then by the hypothesis of the theorem and Lemma 2.9, G is soluble and its Hall $\{2, 3\}'$-subgroup K is normal in G. If K is a q-group, then, of course, K is a dispersive group by Ore. If K is not a q-group, then by Lemma 2.8, K is r-nilpotent for the least prime divisor r of $|K|$. Let $K_{r'}$ be the normal Hall r'-subgroup of K. By Lemma 2.6, we see that $K_{r'}$ also satisfies the hypothesis of the theorem. Hence by induction, $K_{r'}$ is a dispersive group by Ore. This implies that K is a dispersive group by Ore. Therefore (1) holds.

(2) This follows directly from Lemma 2.6 and Lemma 2.10.

(3) If $|\pi(G)| = 2$, then it is trivial. Now let $|\pi(G)| \geq 3$ and H be a Hall $2'$-subgroup of G. Then by Lemma 2.6(1), we see then H satisfies the hypothesis of the theorem. Hence by Lemma 2.8(2), H is p-nilpotent for the least prime divisor p of H. Obviously, $H_{p'}$ also satisfies the hypothesis. By induction, we may assume that $H_{p'}$ is a dispersive group by Ore. It follows that H is a dispersive group by

Ore. For a Hall $3'$-subgroup of G, by using Lemma 2.8, we can similarly prove that K is a dispersive group by Ore. Thus (3) holds.

(4) If $q > 3$, then by (1), we see that G is q-closed. Hence $l_q(G) = 1$. Now let $p \neq q$. If p is the least prime divisor of $|G|$, then by Lemma 2.8(4), we have that $l_p(G) = 1$. If p is not the least prime divisor of $|G|$, then $p \neq 2$ and so by Lemma 2.8(2), G is p-nilpotent. Hence $l_p(G) = 1$ again. Now assume that G is a $\{2,3\}$-group. Then by Lemma 2.10, G is metanilpotent and so by Lemma 2.11, $l_p(G) = 1$ for all $p \in \pi(G)$. Thus (4) holds. This completes the proof. ∎

4. Some applications and remarks

The following result follows directly from the statement (1) of our theorem.

Corollary 4.1. *If $q > 3$ in the conditions of Theorem, then G has a normal Sylow q-subgroup.*

Recall that a natural number n is said to be square-free if p^2 does not divide n for all prime p.

Corollary 4.2. *Let G be a group. If the order of the cofactor of every subgroup is square-free, then the following statements hold.*

(1) *G is soluble and the second commutator subgroup G'' of G is nilpotent;*
(2) $d(G) \leq 4$;
(3) $r(G) \leq 2$;
(4) *G has a normal Hall $\{2,3\}'$-subgroup which is a dispersive group by Ore;*
(5) *If G is A_4-free, then G is a dispersive group by Ore;*
(6) *$l_p(G) \leq 1$, for all primes $p \in \pi(G)$.*

Proof. If the order of the cofactor of every subgroup of G is square-free, then the order of the cofactor of every non-normal primary subgroup of G is a prime number. This shows that G satisfies the condition of Theorem 1.1. Hence, by our theorem, G is soluble and the statements (4) and (6) hold. The statements (3) and the statement (5) can be obtained by using Lemma 2.8 again and again.

(1) We first claim that the order of the factor group $\overline{G} = G/F(G)$ is square-free. In fact, let P is a Sylow p-subgroup of G. If $|P/core_G P| = 1$, then P is normal in G and so $P \subseteq F(G)$. If $|P/core_G P| = p$, then $core_G P \subseteq F(G)$ and the Sylow p-subgroup $\overline{P}$ of $G/F(G)$ is of order p. Hence our claim holds. Now, by [9, Theorem IV.2.11], we see that the commutator subgroup $\overline{G}'$ of $\overline{G}$ is a cyclic Hall subgroup of $\overline{G}$ and $\overline{G}/\overline{G}'$ is also a cyclic group. Therefore, $\overline{G}$ is a soluble meta abelian group. It follows that $\overline{G}'' = \overline{1}$ and hence $G'' \subseteq F(G)$ is nilpotent.

(2) Since $|Q/core_G Q| \in \{1, q\}$ for every q-subgroup Q of G and for every prime $q \in \pi(G)$, by [3, Lemma 1.1 and Lemma 2.1], every Sylow subgroup of G is meta abelian. Then because G'' is nilpotent by 1), we see that G'' is also meta abelian. This induces that the derived length $d(G) \leq 4$. This completes the proof. ∎

Remark 4.1. In [12], the author proved that the groups in which every subgroup has simple cofactor are soluble. However, in a soluble group, the condition "the cofactor of every subgroup is simple" is equivalent to the condition "the cofactor of every subgroup is of prime order". Hence the structure of the groups considered in [12] is contained in our Corollary 4.2.

Remark 4.2. The groups in our Theorem 1.1 and Corollary 4.2 are not necessarily supersoluble in general. For example, let p and q be two different prime numbers, q divides $p+1$ and $q>2$. By Gol'fand's theorem in [5], we know that there exists a Schmidt group $[E_{p^2}]Z_q$. Obviously, the group is not supersoluble and the orders of all cofactors of all its subgroups are square-free. This example shows that the groups of odd order in our Theorem 1.1 and Corollary 2.2 maybe is not supersoluble. The alternating group A_4 of degree 4 shows that the groups of even order in our Theorem 1.1 and Corollary 2.2 maybe is not also supersoluble.

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