

## Positive Implicative Ideals of BCK-Algebras Based on a Soft Set Theory

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**Abstract.** Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. In this paper we apply the notion of soft sets by Molodtsov to the theory of BCK-algebras. The notions of positive implicative soft ideals and positive implicative idealistic soft BCK-algebras are introduced, and their basic properties are derived.

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### 1. Introduction

To solve complicated problem in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [8]. Maji *et al.* [7] and Molodtsov [8] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed

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out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji *et al.* [7] described the application of soft set theory to a decision making problem. Maji *et al.* [6] also studied several operations on the theory of soft sets. Chen *et al.* [1] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. By using the notions of fuzzy sets, Jun *et al.* [3] studied the fuzzy ideals in BE-algebras. Lee [5] discussed bipolar fuzzy subalgebras/ideals of BCK/BCI-algebras by using the notion of bipolar fuzzy sets. Zhan and Jun [10] applied the notion of fuzzy points to ideal theory of BCI-algebras, and generalized well-known fuzzy algebraic structures in BCI-algebras.

In this paper, we deal with the algebraic structure of BCK-algebras by applying soft set theory. We discussed the algebraic properties of soft sets in BCK-algebras. We introduced the notion of positive implicative soft ideals and positive implicative idealistic soft BCK-algebras, and gave several examples. We investigated relations between idealistic soft BCK-algebras and positive implicative idealistic soft BCK/BCI-algebras. We established the intersection, union, “AND” operation, and “OR” operation of positive implicative soft ideals and positive implicative idealistic soft BCK/BCI-algebras.

## 2. Basic Results on BCK-algebras

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra  $(X; *, 0)$  of type  $(2, 0)$  is called a *BCI-algebra* if it satisfies the following axioms:

- (I)  $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0)$ ,
- (II)  $(\forall x, y \in X) ((x * (x * y)) * y = 0)$ ,
- (III)  $(\forall x \in X) (x * x = 0)$ ,
- (IV)  $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)$ .

If a BCI-algebra  $X$  satisfies the following identity:

- (V)  $(\forall x \in X) (0 * x = 0)$ , then  $X$  is called a *BCK-algebra*.

Any BCK-algebra  $X$  satisfies the following conditions:

- (a1)  $(\forall x \in X) (x * 0 = x)$ ,
- (a2)  $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$ ,
- (a3)  $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y)$ ,
- (a4)  $(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y)$

where  $x \leq y$  if and only if  $x * y = 0$ . A BCK-algebra  $X$  is said to be *positive implicative* if it satisfies the following identity:

- (2.1)  $(\forall x, y, z \in X) ((x * y) * z = (x * z) * (y * z))$ .

A nonempty subset  $S$  of a BCK-algebra  $X$  is called a *subalgebra* of  $X$  if  $x * y \in S$  for all  $x, y \in S$ . A subset  $H$  of a BCK-algebra  $X$  is called an *ideal* of  $X$ , denoted by  $H \triangleleft X$ , if it satisfies the following axioms:

- (I1)  $0 \in H$ ,

$$(I2) (\forall x \in X) (\forall y \in H) (x * y \in H \Rightarrow x \in H).$$

Any ideal  $H$  of a BCK-algebra  $X$  satisfies the following implication:

$$(2.2) \quad (\forall x \in X) (\forall y \in H) (x \leq y \Rightarrow x \in H).$$

A subset  $H$  of a BCK-algebra  $X$  is called a *positive implicative ideal* of  $X$ , denoted by  $H \triangleleft_{pi} X$ , if it satisfies the following axioms:

$$(I1) \quad 0 \in H,$$

$$(I3) \quad (\forall x, y, z \in X) ((x * y) * z \in H, y * z \in H \Rightarrow x * z \in H).$$

We refer the reader to the book [9] for further information regarding BCK-algebras.

### 3. Basic results on soft sets

Molodtsov [8] defined the soft set in the following way: Let  $U$  be an initial universe set and  $E$  be a set of parameters. Let  $\mathcal{P}(U)$  denotes the power set of  $U$  and  $A \subset E$ .

**Definition 3.1.** [8] A pair  $(\mathcal{A}, A)$  is called a soft set over  $U$ , where  $\mathcal{A}$  is a mapping given by

$$\mathcal{A} : A \rightarrow \mathcal{P}(U).$$

In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ . For  $\varepsilon \in A$ ,  $\mathcal{A}(\varepsilon)$  may be considered as the set of  $\varepsilon$ -approximate elements of the soft set  $(\mathcal{A}, A)$ . Clearly, a soft set is not a classical set. For illustration, Molodtsov considered several examples in [8].

**Definition 3.2.** [6] Let  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  be two soft sets over a common universe  $U$ . The intersection of  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  is defined to be the soft set  $(\mathcal{G}, C)$  satisfying the following conditions:

$$(i) \quad C = A \cap B,$$

$$(ii) \quad (\forall e \in C) (\mathcal{G}(e) = \mathcal{A}(e) \text{ or } \mathcal{B}(e), \text{ (as both are same set)}).$$

In this case, we write  $(\mathcal{A}, A) \tilde{\cap} (\mathcal{B}, B) = (\mathcal{G}, C)$ .

**Definition 3.3.** [6] Let  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  be two soft sets over a common universe  $U$ . The union of  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  is defined to be the soft set  $(\mathcal{G}, C)$  satisfying the following conditions:

$$(i) \quad C = A \cup B,$$

$$(ii) \quad \text{for all } e \in C,$$

$$\mathcal{G}(e) = \begin{cases} \mathcal{A}(e) & \text{if } e \in A \setminus B, \\ \mathcal{B}(e) & \text{if } e \in B \setminus A, \\ \mathcal{A}(e) \cup \mathcal{B}(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write  $(\mathcal{A}, A) \tilde{\cup} (\mathcal{B}, B) = (\mathcal{G}, C)$ .

**Definition 3.4.** [6] If  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  are two soft sets over a common universe  $U$ , then “ $(\mathcal{A}, A)$  AND  $(\mathcal{B}, B)$ ” denoted by  $(\mathcal{A}, A) \tilde{\wedge} (\mathcal{B}, B)$  is defined by  $(\mathcal{A}, A) \tilde{\wedge} (\mathcal{B}, B) = (\mathcal{G}, A \times B)$ , where  $\mathcal{G}(x, y) = \mathcal{A}(x) \cap \mathcal{B}(y)$  for all  $(x, y) \in A \times B$ .

**Definition 3.5.** [6] If  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  are two soft sets over a common universe  $U$ , then “ $(\mathcal{A}, A)$  OR  $(\mathcal{B}, B)$ ” denoted by  $(\mathcal{A}, A) \tilde{\vee} (\mathcal{B}, B)$  is defined by  $(\mathcal{A}, A) \tilde{\vee} (\mathcal{B}, B) = (\mathcal{G}, A \times B)$ , where  $\mathcal{G}(x, y) = \mathcal{A}(x) \cup \mathcal{B}(y)$  for all  $(x, y) \in A \times B$ .

**Definition 3.6.** [6] For two soft sets  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  over a common universe  $U$ , we say that  $(\mathcal{A}, A)$  is a soft subset of  $(\mathcal{B}, B)$ , denoted by  $(\mathcal{A}, A) \tilde{\subset} (\mathcal{B}, B)$ , if it satisfies:

- (i)  $A \subset B$ ,
- (ii) For every  $\varepsilon \in A$ ,  $\mathcal{A}(\varepsilon)$  and  $\mathcal{B}(\varepsilon)$  are identical approximations.

**4. Positive implicative idealistic soft BCK-algebras**

In what follows let  $X$  and  $A$  be a BCK-algebra and a nonempty set, respectively, and  $R$  will refer to an arbitrary binary relation between an element of  $A$  and an element of  $X$ , that is,  $R$  is a subset of  $A \times X$  without otherwise specified. A set-valued function  $\mathcal{A} : A \rightarrow \mathcal{P}(X)$  can be defined as  $\mathcal{A}(x) = \{y \in X \mid (x, y) \in R\}$  for all  $x \in A$ . The pair  $(\mathcal{A}, A)$  is then a soft set over  $X$ .

**Definition 4.1.** [2] Let  $(\mathcal{A}, A)$  be a soft set over  $X$ . Then  $(\mathcal{A}, A)$  is called a soft BCK-algebra over  $X$  if  $\mathcal{A}(x)$  is a subalgebra of  $X$  for all  $x \in A$ .

**Example 4.1.** Let  $X = \{0, a, b, c, d\}$  be a BCK-algebra with the following Cayley table:

$*$	$0$	$a$	$b$	$c$	$d$
$0$	$0$	$0$	$0$	$0$	$0$
$a$	$a$	$0$	$a$	$a$	$a$
$b$	$b$	$b$	$0$	$b$	$b$
$c$	$c$	$c$	$c$	$0$	$c$
$d$	$d$	$d$	$d$	$d$	$0$

Let  $(\mathcal{A}, A)$  be a soft set over  $X$ , where  $A = X$  and  $\mathcal{A} : A \rightarrow \mathcal{P}(X)$  is a set-valued function defined by

$$\mathcal{A}(x) = \{y \in X \mid xRy \Leftrightarrow y \in x^{-1}\Phi\}$$

for all  $x \in A$  where  $\Phi = \{0, a\}$  and  $x^{-1}\Phi = \{y \in X \mid x \wedge y \in \Phi\}$ . Then  $(\mathcal{A}, A)$  is a soft BCK-algebra over  $X$  (see [2]).

For any  $a \in X$  and a subset  $D$  of  $X$ , let

$$\frac{a}{D} := \{x \in X \mid x * a \in D\}, \quad \frac{a^2}{D} := \{x \in X \mid x * (x * a) \in D\}.$$

**Lemma 4.1.** For any  $a \in X$  and  $D \subset X$ , we have

$$(4.1) \quad D \triangleleft_{pi} X \Rightarrow \frac{a}{D} \triangleleft X.$$

*Proof.* Assume that  $D \triangleleft_{pi} X$ . Obviously  $0 \in \frac{a}{D}$ . Let  $x, y \in X$  be such that  $x * y \in \frac{a}{D}$  and  $y \in \frac{a}{D}$ . Then  $(x * y) * a \in D$  and  $y * a \in D$ . Since  $D \triangleleft_{pi} X$ , it follows from (I3) that  $x * a \in D$ , that is,  $x \in \frac{a}{D}$  so that  $\frac{a}{D} \triangleleft X$ . ■

**Definition 4.2.** [4] Let  $(\mathcal{A}, A)$  be a soft set over  $X$ . Then  $(\mathcal{A}, A)$  is called an idealistic soft BCK-algebra over  $X$  if it satisfies:

$$(4.2) \quad (\forall x \in A) (\alpha(x) \triangleleft X).$$

**Definition 4.3.** Let  $(\mathcal{A}, A)$  be a soft set over  $X$ . Then  $(\mathcal{A}, A)$  is called a positive implicative idealistic soft BCK-algebra over  $X$  if it satisfies:

$$(4.3) \quad (\forall x \in A) (\alpha(x) \triangleleft_{pi} X).$$

Let us illustrate this definition using the following examples.

**Example 4.2.** Let  $X = \{0, a, b, c, d\}$  be a BCK-algebra with the following Cayley table:

$*$	$0$	$a$	$b$	$c$	$d$
$0$	$0$	$0$	$0$	$0$	$0$
$a$	$a$	$0$	$a$	$0$	$0$
$b$	$b$	$b$	$0$	$b$	$0$
$c$	$c$	$c$	$c$	$0$	$0$
$d$	$d$	$d$	$c$	$b$	$0$

Let  $(\mathcal{A}, A)$  be a soft set over  $X$ , where  $A = \{0, b, c, d\}$  and  $\mathcal{A} : A \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{A}(x) = \frac{x^2}{\{0, b\}}$  for all  $x \in A$ . It is routine to check that  $(\mathcal{A}, A)$  is a positive implicative idealistic soft BCK-algebra over  $X$ . Now, let  $(\mathcal{A}, B)$  be a soft set over  $X$  where  $B = X$  and  $\mathcal{A} : B \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{A}(x) = \frac{x^2}{\{0, b\}}$  for all  $x \in B$ . Then  $\mathcal{A}(a) = \{0, b, c, d\}$  is not a positive implicative ideal of  $X$  since  $(a * c) * b = 0 * b = 0 \in \mathcal{A}(a)$  and  $c * b = c \in \mathcal{A}(a)$ , but  $a * b = a \notin \mathcal{A}(a)$ . Hence  $(\mathcal{A}, B)$  is not a positive implicative idealistic soft BCK-algebra over  $X$ .

Note that every positive implicative idealistic soft BCK-algebra over  $X$  is an idealistic soft BCK-algebra over  $X$ , but the converse is not true in general as seen in the following example.

**Example 4.3.** Let  $X = \{0, a, b, c, d\}$  be a BCK-algebra with the following Cayley table:

$*$	$0$	$a$	$b$	$c$	$d$
$0$	$0$	$0$	$0$	$0$	$0$
$a$	$a$	$0$	$0$	$a$	$0$
$b$	$b$	$a$	$0$	$b$	$0$
$c$	$c$	$c$	$c$	$0$	$0$
$d$	$d$	$d$	$d$	$d$	$0$

Let  $(\mathcal{A}, A)$  be a soft set over  $X$ , where  $A = X$  and  $\mathcal{A} : A \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{A}(x) = \frac{x^2}{\{0, a, c\}}$  for all  $x \in A$ . It is easy to verify that  $(\mathcal{A}, A)$  is a soft BCK-algebra over  $X$ . Now let  $(\mathcal{B}, I)$  be a soft set over  $X$ , where  $I = \{0, b, c, d\} \subset A$  and  $\mathcal{B} : I \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{B}(x) = \frac{x}{\{0, c\}}$  for all  $x \in I$ . Then  $\mathcal{B}(0) = \mathcal{B}(c) = \{0, c\} \triangleleft X$ ,  $\mathcal{B}(b) = \{0, a, b, c\} \triangleleft X$  and  $\mathcal{B}(d) = X \triangleleft X$ . Hence  $(\mathcal{B}, I)$  is an idealistic soft BCK-algebra over  $X$ . Now we have  $(b * a) * a = a * a = 0 \in \{0, c\}$  and  $b * a = a \notin \{0, c\}$ . Thus  $\mathcal{B}(0) = \mathcal{B}(c) = \{0, c\}$  is not a positive implicative ideal of  $X$ , and so  $(\mathcal{B}, I)$  is not a positive implicative idealistic soft BCK-algebra over  $X$ .

**Proposition 4.1.** Let  $(\mathcal{A}, A)$  and  $(\mathcal{A}, B)$  be soft sets over  $X$  where  $B \subseteq A \subseteq X$ . If  $(\mathcal{A}, A)$  is a positive implicative idealistic soft BCK-algebra over  $X$ , then so is  $(\mathcal{A}, B)$ .

*Proof.* Straightforward. ▀

By means of Example 4.2, we know that the converse of Proposition 4.1 is not true in general.

**Theorem 4.1.** *Let  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  be two positive implicative idealistic soft BCK-algebras over  $X$ . If  $A \cap B \neq \emptyset$ , then the intersection  $(\mathcal{A}, A) \tilde{\cap} (\mathcal{B}, B)$  is a positive implicative idealistic soft BCK-algebra over  $X$ .*

*Proof.* Using Definition 3.2, we can write  $(\mathcal{A}, A) \tilde{\cap} (\mathcal{B}, B) = (\mathcal{D}, C)$ , where  $C = A \cap B$  and  $\mathcal{D}(x) = \mathcal{A}(x)$  or  $\mathcal{B}(x)$  for all  $x \in C$ . Note that  $\mathcal{D} : C \rightarrow \mathcal{P}(X)$  is a mapping, and therefore  $(\mathcal{D}, C)$  is a soft set over  $X$ . Since  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  are positive implicative idealistic soft BCK-algebras over  $X$ , it follows that  $\mathcal{D}(x) = \mathcal{A}(x)$  is a positive implicative ideal of  $X$ , or  $\mathcal{D}(x) = \mathcal{B}(x)$  is a positive implicative ideal of  $X$  for all  $x \in C$ . Hence  $(\mathcal{D}, C) = (\mathcal{A}, A) \tilde{\cap} (\mathcal{B}, B)$  is a positive implicative idealistic soft BCK-algebra over  $X$ . ■

**Corollary 4.1.** *Let  $(\mathcal{A}, A)$  and  $(\mathcal{B}, A)$  be two positive implicative idealistic soft BCK-algebras over  $X$ . Then their intersection  $(\mathcal{A}, A) \tilde{\cap} (\mathcal{B}, A)$  is a positive implicative idealistic soft BCK-algebra over  $X$ .*

*Proof.* Straightforward. ■

**Theorem 4.2.** *Let  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  be two positive implicative idealistic soft BCK-algebras over  $X$ . If  $A$  and  $B$  are disjoint, then the union  $(\mathcal{A}, A) \tilde{\cup} (\mathcal{B}, B)$  is a positive implicative idealistic soft BCK-algebra over  $X$ .*

*Proof.* Using Definition 3.3, we can write  $(\mathcal{A}, A) \tilde{\cup} (\mathcal{B}, B) = (\mathcal{D}, C)$ , where  $C = A \cup B$  and for every  $e \in C$ ,

$$\mathcal{D}(e) = \begin{cases} \mathcal{A}(e) & \text{if } e \in A \setminus B, \\ \mathcal{B}(e) & \text{if } e \in B \setminus A, \\ \mathcal{A}(e) \cup \mathcal{B}(e) & \text{if } e \in A \cap B. \end{cases}$$

Since  $A \cap B = \emptyset$ , either  $x \in A \setminus B$  or  $x \in B \setminus A$  for all  $x \in C$ . If  $x \in A \setminus B$ , then  $\mathcal{D}(x) = \mathcal{A}(x)$  is a positive implicative ideal of  $X$  since  $(\mathcal{A}, A)$  is a positive implicative idealistic soft BCK-algebra over  $X$ . If  $x \in B \setminus A$ , then  $\mathcal{D}(x) = \mathcal{B}(x)$  is a positive implicative ideal of  $X$  since  $(\mathcal{B}, B)$  is a positive implicative idealistic soft BCK-algebra over  $X$ . Hence  $(\mathcal{D}, C) = (\mathcal{A}, A) \tilde{\cup} (\mathcal{B}, B)$  is a positive implicative idealistic soft BCK-algebra over  $X$ . ■

**Theorem 4.3.** *If  $(\mathcal{A}, A)$  and  $(\mathcal{B}, B)$  are positive implicative idealistic soft BCK-algebras over  $X$ , then  $(\mathcal{A}, A) \tilde{\wedge} (\mathcal{B}, B)$  is a positive implicative idealistic soft BCK-algebra over  $X$ .*

*Proof.* By means of Definition 3.4, we know that

$$(\mathcal{A}, A) \tilde{\wedge} (\mathcal{B}, B) = (\mathcal{D}, A \times B),$$

where  $\mathcal{D}(x, y) = \mathcal{A}(x) \cap \mathcal{B}(y)$  for all  $(x, y) \in A \times B$ . Since  $\mathcal{A}(x)$  and  $\mathcal{B}(y)$  are positive implicative ideals of  $X$ , the intersection  $\mathcal{A}(x) \cap \mathcal{B}(y)$  is also a positive implicative ideal of  $X$ . Hence  $\mathcal{D}(x, y)$  is a positive implicative ideal of  $X$  for all  $(x, y) \in A \times B$ , and therefore  $(\mathcal{D}, A \times B) = (\mathcal{A}, A) \tilde{\wedge} (\mathcal{B}, B)$  is a positive implicative idealistic soft BCK-algebra over  $X$ . ■

**Definition 4.4.** *A positive implicative idealistic soft BCK-algebra  $(\mathcal{A}, A)$  over  $X$  is said to be trivial (resp., whole) if  $\mathcal{A}(x) = \{0\}$  (resp.,  $\mathcal{A}(x) = X$ ) for all  $x \in A$ .*

**Example 4.4.** Let  $X = \{0, a, b, c, d\}$  be the BCK-algebra which is described in Example 4.2. Consider  $A = \{c, d\} \subset X$  and a set-valued function  $\mathcal{A} : A \rightarrow \mathcal{P}(X)$  defined by  $\mathcal{A}(x) = \frac{x}{\{0, b\}}$  for all  $x \in A$ . Then  $\mathcal{A}(c) = \frac{c}{\{0, b\}} = X$  and  $\mathcal{A}(d) = \frac{d}{\{0, b\}} = X$ . Hence  $(\mathcal{A}, A)$  is a whole positive implicative idealistic soft BCK-algebra over  $X$ .

**Example 4.5.** Let  $X = \{0, a, b, c, d\}$  be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Let  $\mathcal{B} : \{0\} \rightarrow \mathcal{P}(X)$  be a set-valued function given by  $\mathcal{B}(0) = \frac{0}{\{0\}}$ . Then  $(\mathcal{B}, \{0\})$  is a zero positive implicative idealistic soft BCK-algebra over  $X$ .

The following example shows that there exists a BCK-algebra  $X$  such that a soft set  $(\mathcal{A}, \{0\})$  may not be a zero positive implicative idealistic soft BCK-algebra over  $X$ , where  $\mathcal{A} : \{0\} \rightarrow \mathcal{P}(X)$  is given by  $\mathcal{A}(0) = \frac{0}{\{0\}}$ .

**Example 4.6.** Let  $X = \{0, a, b, c, d\}$  be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Let  $\mathcal{A} : \{0\} \rightarrow \mathcal{P}(X)$  be a set-valued function given by  $\mathcal{A}(0) = \frac{0}{\{0\}}$ . Then  $\mathcal{A}(0) = \{0\}$  is not a positive implicative ideal of  $X$  since  $(c * a) * a = a * a = 0 \in \mathcal{A}(0)$  and  $c * a = a \notin \mathcal{A}(0)$ . Hence  $(\mathcal{A}, \{0\})$  is not a zero positive implicative idealistic soft BCK-algebra over  $X$ .

Since the zero ideal  $\{0\}$  in a positive implicative BCK-algebra is a positive implicative ideal, we have the following proposition.

**Proposition 4.2.** For any positive implicative BCK-algebra  $X$ , a soft set  $(\mathcal{B}, \{0\})$  over  $X$ , where  $\mathcal{B} : \{0\} \rightarrow \mathcal{P}(X)$  is given by  $\mathcal{B}(0) = \frac{0}{\{0\}}$ , is the zero positive implicative idealistic soft BCK-algebra over  $X$ .

Let  $f : X \rightarrow Y$  be a mapping of BCK-algebras. For a soft set  $(\mathcal{A}, A)$  over  $X$ ,  $(f(\mathcal{A}), A)$  is a soft set over  $Y$  where  $f(\mathcal{A}) : A \rightarrow \mathcal{P}(Y)$  is defined by  $f(\mathcal{A})(x) = f(\mathcal{A}(x))$  for all  $x \in A$ .

**Lemma 4.2.** Let  $f : X \rightarrow Y$  be an onto homomorphism of BCK-algebras. If  $(\mathcal{A}, A)$  is a positive implicative idealistic soft BCK-algebra over  $X$ , then  $(f(\mathcal{A}), A)$  is a positive implicative idealistic soft BCK-algebra over  $Y$ .

*Proof.* For every  $x \in A$ , we have  $f(\mathcal{A})(x) = f(\mathcal{A}(x))$  is a positive implicative ideal of  $Y$  since  $\mathcal{A}(x)$  is a positive implicative ideal of  $X$  and its onto homomorphic image is also a positive implicative ideal of  $Y$ . Hence  $(f(\mathcal{A}), A)$  is a positive implicative idealistic soft BCK-algebra over  $Y$ . ■

**Theorem 4.4.** *Let  $f : X \rightarrow Y$  be an onto homomorphism of BCK-algebras and let  $(\mathcal{A}, A)$  be a positive implicative idealistic soft BCK-algebra over  $X$ .*

- (i) *If  $Y$  is positive implicative and  $\mathcal{A}(x) = \ker(f)$  for all  $x \in A$ , then  $(f(\mathcal{A}), A)$  is the trivial positive implicative idealistic soft BCK-algebra over  $Y$ .*
- (ii) *If  $(\mathcal{A}, A)$  is whole, then  $(f(\mathcal{A}), A)$  is the whole positive implicative idealistic soft BCK-algebra over  $Y$ .*

*Proof.* (i) Assume that  $\mathcal{A}(x) = \ker(f)$  for all  $x \in A$ . Then  $f(\mathcal{A})(x) = f(\mathcal{A}(x)) = \{0\}$  for all  $x \in A$ . Hence  $(f(\mathcal{A}), A)$  is the trivial positive implicative idealistic soft BCK-algebra over  $Y$ .

(ii) Suppose that  $(\mathcal{A}, A)$  is whole. Then  $\mathcal{A}(x) = X$  for all  $x \in A$ , and so  $f(\mathcal{A})(x) = f(\mathcal{A}(x)) = f(X) = Y$  for all  $x \in A$ . It follows from Lemma 4.2 and Definition 4.4 that  $(f(\mathcal{A}), A)$  is the whole positive implicative idealistic soft BCK-algebra over  $Y$ . ■

**Definition 4.5.** *Let  $S$  be a subalgebra of  $X$ . A subset  $I$  of  $X$  is called a positive implicative ideal of  $X$  related to  $S$  (briefly, positive implicative  $S$ -ideal of  $X$ ), denoted by  $I \triangleleft_{pi} S$ , if it satisfies:*

- (i)  $0 \in I$ ,
- (ii)  $(\forall x, y, z \in S) ((x * y) * z \in I, y * z \in I \Rightarrow x * z \in I)$ .

Note that a positive implicative  $X$ -ideal means a positive implicative ideal.

**Definition 4.6.** *Let  $(\mathcal{A}, A)$  be a soft BCK-algebra over  $X$ . A soft set  $(\mathcal{B}, I)$  over  $X$  is called a positive implicative soft ideal of  $(\mathcal{A}, A)$ , denoted by  $(\mathcal{B}, I) \widetilde{\triangleleft}_{pi} (\mathcal{A}, A)$ , if it satisfies:*

- (i)  $I \subset A$ ,
- (ii)  $(\forall x \in I) (\mathcal{B}(x) \triangleleft_{pi} \mathcal{A}(x))$ .

Let us illustrate this definition using the following examples.

**Example 4.7.** Let  $X = \{0, a, b, c, d\}$  be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	a	c	0	a
d	d	d	d	d	0

Let  $(\mathcal{A}, A)$  be a soft set over  $X$ , where  $A = X$  and  $\mathcal{A} : A \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{A}(x) = \frac{x}{\{0, a\}}$  for all  $x \in A$ . Then  $(\mathcal{A}, A)$  is a soft BCK-algebra over  $X$ . Now take  $I = \{a, c\} \subset A$  and let  $\mathcal{B} : I \rightarrow \mathcal{P}(X)$  be a set-valued function defined by  $\mathcal{B}(x) = \frac{x}{\{0, a\}}$  for all  $x \in I$ . Then  $\mathcal{B}(a) = \{0, a, c\} \triangleleft_{pi} \mathcal{A}(a)$  and  $\mathcal{B}(c) = \{0, a, c\} \triangleleft_{pi} \mathcal{A}(c)$ . Hence  $(\mathcal{B}, I) \widetilde{\triangleleft}_{pi} (\mathcal{A}, A)$ . If we take  $J = \{a, b, c\} \subset A$  and



define a set-valued function  $\mathcal{B} : J \rightarrow \mathcal{P}(X)$  by  $\mathcal{B}(x) = \frac{x}{\{0,a\}}$  for all  $x \in J$ , then  $\mathcal{B}(b) = \{0, a, b\}$  is not a positive implicative ideal of  $\mathcal{A}(b)$  since  $(c * a) * b = a * b = a \in \mathcal{A}(b)$  and  $a * b = a \in \mathcal{A}(b)$ , but  $c * b = c \notin \mathcal{A}(b)$ . Hence  $(\mathcal{B}, J)$  is not a positive implicative soft ideal of  $(\mathcal{A}, A)$ .

**Theorem 4.5.** *Let  $(\mathcal{B}, I)$  and  $(\mathcal{B}, J)$  be soft sets over  $X$  such that  $I \subset J$ . If  $(\mathcal{B}, J)$  is a positive implicative soft ideal of a soft BCK-algebra  $(\mathcal{A}, A)$  over  $X$ , then so is  $(\mathcal{B}, I)$ .*

*Proof.* Straightforward. ▀

Example 4.7 shows that the converse of Theorem 4.5 is not valid in general.

**Theorem 4.6.** *Let  $(\mathcal{A}, A)$  be a soft BCK-algebra over  $X$ . For any soft sets  $(\mathcal{B}_1, I_1)$  and  $(\mathcal{B}_2, I_2)$  over  $X$  where  $I_1 \cap I_2 \neq \emptyset$ , we have*

$$(\mathcal{B}_1, I_1) \tilde{\triangleleft}_{pi}(\mathcal{A}, A), (\mathcal{B}_2, I_2) \tilde{\triangleleft}_{pi}(\mathcal{A}, A) \Rightarrow (\mathcal{B}_1, I_1) \tilde{\cap}(\mathcal{B}_2, I_2) \tilde{\triangleleft}_{pi}(\mathcal{A}, A).$$

*Proof.* Using Definition 3.2, we can write

$$(\mathcal{B}_1, I_1) \tilde{\cap}(\mathcal{B}_2, I_2) = (\mathcal{B}, I),$$

where  $I = I_1 \cap I_2$  and  $\mathcal{B}(x) = \mathcal{B}_1(x)$  or  $\mathcal{B}_2(x)$  for all  $x \in I$ . Obviously,  $I \subset A$  and  $\mathcal{B} : I \rightarrow \mathcal{P}(X)$  is a mapping. Hence  $(\mathcal{B}, I)$  is a soft set over  $X$ . Since  $(\mathcal{B}_1, I_1) \tilde{\triangleleft}_{pi}(\mathcal{A}, A)$  and  $(\mathcal{B}_2, I_2) \tilde{\triangleleft}_{pi}(\mathcal{A}, A)$ , we know that  $\mathcal{B}(x) = \mathcal{B}_1(x) \triangleleft_{pi} \mathcal{A}(x)$  or  $\mathcal{B}(x) = \mathcal{B}_2(x) \triangleleft_{pi} \mathcal{A}(x)$  for all  $x \in I$ . Hence

$$(\mathcal{B}, I) \tilde{\cap}(\mathcal{B}_2, I_2) = (\mathcal{B}, I) \tilde{\triangleleft}_{pi}(\mathcal{A}, A).$$

This completes the proof. ▀

**Corollary 4.2.** *Let  $(\mathcal{A}, A)$  be a soft BCK-algebra over  $X$ . For any soft sets  $(\mathcal{B}, I)$  and  $(\mathcal{D}, I)$  over  $X$ , we have*

$$(\mathcal{B}, I) \tilde{\triangleleft}_{pi}(\mathcal{A}, A), (\mathcal{D}, I) \tilde{\triangleleft}_{pi}(\mathcal{A}, A) \Rightarrow (\mathcal{B}, I) \tilde{\cap}(\mathcal{D}, I) \tilde{\triangleleft}_{pi}(\mathcal{A}, A).$$

*Proof.* Straightforward. ▀

**Theorem 4.7.** *Let  $(\mathcal{A}, A)$  be a soft BCK-algebra over  $X$ . For any soft sets  $(\mathcal{B}, I)$  and  $(\mathcal{D}, J)$  over  $X$  in which  $I$  and  $J$  are disjoint, we have*

$$(\mathcal{B}, I) \tilde{\triangleleft}_{pi}(\mathcal{A}, A), (\mathcal{D}, J) \tilde{\triangleleft}_{pi}(\mathcal{A}, A) \Rightarrow (\mathcal{B}, I) \tilde{\cup}(\mathcal{D}, J) \tilde{\triangleleft}_{pi}(\mathcal{A}, A).$$

*Proof.* Assume that  $(\mathcal{B}, I) \tilde{\triangleleft}_{pi}(\mathcal{A}, A)$  and  $(\mathcal{D}, J) \tilde{\triangleleft}_{pi}(\mathcal{A}, A)$ . By means of Definition 3.3, we can write  $(\mathcal{B}, I) \tilde{\cup}(\mathcal{D}, J) = (\mathcal{H}, U)$  where  $U = I \cup J$  and for every  $x \in U$ ,

$$\mathcal{H}(x) = \begin{cases} \mathcal{B}(x) & \text{if } x \in I \setminus J, \\ \mathcal{D}(x) & \text{if } x \in J \setminus I, \\ \mathcal{B}(x) \cup \mathcal{D}(x) & \text{if } x \in I \cap J. \end{cases}$$

Since  $I \cap J = \emptyset$ , either  $x \in I \setminus J$  or  $x \in J \setminus I$  for all  $x \in U$ . If  $x \in I \setminus J$ , then  $\mathcal{H}(x) = \mathcal{B}(x) \triangleleft_{pi} \mathcal{A}(x)$  since  $(\mathcal{B}, I) \tilde{\triangleleft}_{pi}(\mathcal{A}, A)$ . If  $x \in J \setminus I$ , then  $\mathcal{H}(x) = \mathcal{D}(x) \triangleleft_{pi} \mathcal{A}(x)$  since  $(\mathcal{D}, J) \tilde{\triangleleft}_{pi}(\mathcal{A}, A)$ . Thus  $\mathcal{H}(x) \triangleleft_{pi} \mathcal{A}(x)$  for all  $x \in U$ , and so  $(\mathcal{B}, I) \tilde{\cup}(\mathcal{D}, J) = (\mathcal{H}, U) \tilde{\triangleleft}_{pi}(\mathcal{A}, A)$ . ▀

If  $I$  and  $J$  are not disjoint in Theorem 4.7, then Theorem 4.7 is not true in general as seen in the following example.

**Example 4.8.** Let  $X = \{0, a, b, c, d\}$  be a BCK-algebra with the following Cayley table:

$*$	$0$	$a$	$b$	$c$	$d$
$0$	$0$	$0$	$0$	$0$	$0$
$a$	$a$	$0$	$0$	$0$	$0$
$b$	$b$	$b$	$0$	$b$	$0$
$c$	$c$	$c$	$c$	$0$	$0$
$d$	$d$	$d$	$c$	$b$	$0$

Let  $(\mathcal{A}, A)$  be a soft set over  $X$ , where  $A = X$  and  $\mathcal{A} : A \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{A}(x) = \frac{x^2}{\{0, b\}}$  for all  $x \in A$ . Then  $\mathcal{A}(0) = X$ ,  $\mathcal{A}(a) = \mathcal{A}(b) = \{0, b, c, d\}$ , and  $\mathcal{A}(c) = \mathcal{A}(d) = \{0, b\}$  which are subalgebras of  $X$ . Hence  $(\mathcal{A}, A)$  is a soft BCK-algebra over  $X$ . Let  $(\mathcal{B}, I)$  be a soft set over  $X$ , where  $I = \{b, c, d\}$  and  $\mathcal{B} : I \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{B}(x) = \frac{x}{\{0\}}$  for all  $x \in I$ . Then  $\mathcal{B}(b) = \{0, a, b\} \triangleleft_{pi} \{0, b, c, d\} = \mathcal{A}(b)$ ,  $\mathcal{B}(c) = \{0, a, c\} \triangleleft_{pi} \{0, b\} = \mathcal{A}(c)$ , and  $\mathcal{B}(d) = X \triangleleft_{pi} \mathcal{A}(d)$ , and so  $(\mathcal{B}, I)$  is a positive implicative soft ideal of  $(\mathcal{A}, A)$ . Let  $(\mathcal{G}, J)$  be a soft set over  $X$ , where  $J = \{b\}$  and  $\mathcal{G} : J \rightarrow \mathcal{P}(X)$  is a set-valued function defined by  $\mathcal{G}(x) = \frac{x^2}{\{0\}}$  for all  $x \in J$ . Then  $\mathcal{G}(b) = \{0, c\} \triangleleft_{pi} \{0, b, c, d\} = \mathcal{A}(b)$ , and so  $(\mathcal{G}, J)$  is a positive implicative soft ideal of  $(\mathcal{A}, A)$ . Then  $(\mathcal{S}, U) = (\mathcal{B}, I) \widetilde{\cup} (\mathcal{G}, J)$  is not a positive implicative soft ideal of  $(\mathcal{A}, A)$  since  $\mathcal{S}(b) = \mathcal{B}(b) \cup \mathcal{G}(b) = \{0, a, b, c\}$  is not an  $\mathcal{A}(b)$ -ideal because  $d * c = b \in \{0, a, b, c\}$  and  $d \notin \{0, a, b, c\}$ , and hence it is not a positive implicative  $\mathcal{A}(b)$ -ideal of  $X$ .

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## References

- [1] D. Chen, E. C. C. Tsang, D. S. Yeung and X. Wang, The parameterization reduction of soft sets and its applications, *Comput. Math. Appl.* **49** (2005), no. 5-6, 757–763.
- [2] Y. B. Jun, Soft BCK/BCI-algebras, *Comput. Math. Appl.* **56** (2008), no. 5, 1408–1413.
- [3] S. Z. Song, Y. B. Jun and K. J. Lee, Fuzzy ideals in BE-algebras, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 1, 147–153.
- [4] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, *Inform. Sci.* **178** (2008), no. 11, 2466–2475.
- [5] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 361–373.
- [6] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, *Comput. Math. Appl.* **45** (2003), no. 4-5, 555–562.
- [7] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.* **44** (2002), no. 8-9, 1077–1083.
- [8] D. Molodtsov, Soft set theory—first results, *Comput. Math. Appl.* **37** (1999), no. 4-5, 19–31.
- [9] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa, Seoul, 1994.
- [10] J. Zhan and Y. B. Jun, Generalized fuzzy ideals of BCI-algebras, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 2, 119–130.