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Entire Functions That Share Fixed-Points

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Abstract. In this paper, we study the uniqueness problem on entire functions sharing fixed points with the same multiplicities. We generalize some previous results.

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1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the following standard notations of value distribution theory [9]: $T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \cdots$ We denote by $S(r, f)$ any function satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$ possibly outside a set r of finite linear measure.

We say that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If f and g have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

Let p be a positive integer and $a \in \mathbb{C}$. We denote by $N_p(r, \frac{1}{f-a})$ the counting function of the zeros of $f - a$ where an m-fold zero is counted m times if $m \leq p$ and p times if $m > p$. We say that a finite value z_0 is a fixed point of f if $f(z_0) = z_0$.

In answer to one famous question, Hayman [4], Fang and Hua [1], and Yang and Hua [8] obtained the following result.

Theorem 1.1. *Let f and g be two non-constant entire functions, and let $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In [3], Fang also got the following results.

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Theorem 1.2. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant tsuch that $t^n = 1$.*

Theorem 1.3. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM , then $f = g$.*

Recently, Zhang, Chen and Lin [11] proved the following result, which generalized some previous results.

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two entire functions; let n, m and k be three positive integers with $n \geq 3m + 2k + 5$, and let $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$ or $P(z) = C$, where $a_0 \neq 0$,$a_1 \ldots$,a_{m-1}, $a_m \neq 0$, $C \neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$share 1 CM, then the following conclusions hold:*

- (i) *If $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$, then $f(z) = tg(z)$ for a constant t that satisfies $t^d = 1$, where $d = (n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$; or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_1\omega_1 + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_1\omega_2 + a_0)$;*
- (ii) *If $P(z) = C$, then $f = tg$ for a constant t that satisfies $t^n = 1$, or $f(z) = b_1/\sqrt[n]{C}e^{bz}$, $g(z) = b_2/\sqrt[n]{C}e^{-bz}$ for three constants b_1, b_2 and b that satisfy $(-1)^k(b_1b_2)^n(nb)^{2k} = -1$.*

Corresponding to the above results, some authors considered the uniqueness problems of entire functions that have fixed points, see Fang and Qiu [2], Lin and Yi [7]. In the present paper, we consider the existence of fixed points of $(f^nP(f))^{(k)}$ and the corresponding uniqueness theorems, where n, k are positive integers and $P(z)$ is a nonzero polynomial, and we obtain the following results which generalize the above theorems.

Theorem 1.5. *Let $f(z)$ be a transcendental entire function, n, k, m be three positive integers with $n \geq k+2$, and let $P(z) = a_mz^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0$ or $P(z) = C$, where a_0 ,$a_1 \ldots$,a_{m-1}, $a_m \neq 0$, $C \neq 0$ are complex constants. Then $[f^nP(f)]^{(k)}$ has infinitely many fixed points.*

Remark 1.1. It is easy to see that a polynomial $Q(z)$ with degree $n \geq 1$ has exactly n fixed points (counting multiplicities), but a transcendental entire function may have no fixed points. For example, the function $f = e^{\alpha(z)} + z$ has no any fixed points, where $\alpha(z)$ is an entire function.

Here and forth, we define an integer m^*, according to the nonzero polynomial $P(z)$ in Theorem 1.6, by

$$m^* = \begin{cases} m, & P(z) \neq C; \\ 0, & P(z) = C. \end{cases}$$

Theorem 1.6. *Suppose that $P(z)$ is given by Theorem 1.5. Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let n, m and k be three positive integers with $n > 2k + m^* + 4$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share z CM, then the following conclusions hold:*

(i) *If $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$ is not a monomial, then $f(z) = tg(z)$ for a constant t that satisfies $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$; or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_1\omega_1 + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_1\omega_2 + a_0)$;*

(ii) *If $P(z) = C$ or $P(z) = a_mz^m$, then $f = tg$ for a constant t that satisfies $t^{n+m^*} = 1$, or $f(z) = b_1e^{bz^2}$, $g(z) = b_2e^{-bz^2}$ for three constants b_1, b_2 and b that satisfy $4a_m^2(b_1b_2)^{n+m}((n+m)b)^2 = -1$, or $4C^2(b_1b_2)^n(nb)^2 = -1$.*

Remark 1.2. The condition of $n \geq 3m + 2k + 5$ in Theorem 1.4 is replaced by $n > 2k + 4 + m^*$ in Theorem 1.6.

2. Some lemmas

Lemma 2.1. [9] *Let f be a non-constant meromorphic function, and a_0, a_1, a_2, $\ldots$ a_n be small functions of f such that $a_n \neq 0$. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + \cdots a_1f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. [6] *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(\frac{h'}{h}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f), \tag{2.1}$$

$$N_p\left(\frac{h'}{h}\right) \leq k\overline{N}(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.2}$$

Lemma 2.3. [10] *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.3}$$

where F and G are two non-constant meromorphic functions. If F and G share 1 CM and $H \not\equiv 0$, then

$$\begin{aligned} T(r, F) + T(r, G) \leq{}& 2(N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r, F) + N_2(r, G)) \\ &+ S(r, F) + S(r, G). \end{aligned} \tag{2.4}$$

Lemma 2.4. [9] *Let f be a non-constant meromorphic function, and $a_1(z)$, $a_2(z)$ and $a_3(z)$ be distinct small functions of f. Then*

$$T(r, f) < \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

Lemma 2.5. [5] *Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) = S\left(r, \frac{f'}{f}\right),$$

then $f = e^{az+b}$, where $a \neq 0$, b are constants.

Lemma 2.6. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, n, k be two positive integers with $n > k+2$, and let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ or $P(z) = C$, where a_0 ,$a_1 \ldots$,a_{m-1}, $a_m \neq 0$, $C \neq 0$ are complex constants. If $[f^n(z)P(f)]^{(k)}[g^n(z)P(g)]^{(k)} \equiv z^2$, then $P(z)$ is reduced to a nonzero monomial, that is, $P(z) = a_m z^m$ or $P(z) = C$.*

Proof. If $P(z)$ is not reduced to a nonzero monomial, then $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots a_i z^i$, where a_i is the last nonzero complex constant for $i = 0, 1, \ldots, m-1$. Since

$$[f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots a_i f^i)]^{(k)}[g^n(a_m g^m + a_{m-1}g^{m-1} + \cdots a_i g^i)]^{(k)} \equiv z^2. \tag{2.5}$$

Suppose that z_0 is a p-fold zero of f, we know that z_0 must be a $(np + ip - k)$-fold zero of $[f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots a_i f^i)]^{(k)}$. Noting that g is an entire function and $n > k+2$, it follows from (2.5) that z_0 is a zero of z^2 with the order at least 3, which is impossible. Thus f has no zeros. Let $f(z) = e^{\beta(z)}$, where $\beta(z)$ is a non-constant entire function. Then

$$(f^{m+n})^{(k)} = (e^{(m+n)\beta})^{(k)} = P_m(\beta', \beta'', \ldots \beta^{(k)})e^{(m+n)\beta}, \tag{2.6}$$

$$(f^{n+i})^{(k)} = (e^{(n+i)\beta})^{(k)} = P_i(\beta', \beta'', \ldots \beta^{(k)})e^{(n+i)\beta}, \tag{2.7}$$

where P_m and P_i are differential polynomials in $\beta', \beta'', \ldots \beta^{(k)}$. Obviously, $P_m \not\equiv 0$, $P_i \not\equiv 0$, $T(r, P_m) = S(r,f)$ and $T(r, P_i) = S(r,f)$. We obtain from (2.5) to (2.7) that

$$N\left(r, \frac{1}{a_m P_m e^{(m-i)\beta} + a_{m-1}P_{m-1}e^{(m-1-i)\beta} + \cdots a_i P_i}\right) = S(r,f).$$

By Lemma 2.4 and Lemma 2.1, we have

$$\begin{aligned}
&(m-i)T(r,f) \\
&= T(r, a_m P_m e^{(m-i)\beta} + a_{m-1}P_{m-1}e^{(m-1-i)\beta} + \cdots a_{i+1}P_{i+1}e^{\beta}) + S(r,f) \\
&\leq \overline{N}\left(r, \frac{1}{a_m P_m e^{(m-i)\beta} + a_{m-1}P_{m-1}e^{(m-1-i)\beta} + \cdots a_{i+1}P_{i+1}e^{\beta}}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{a_m P_m e^{(m-i)\beta} + a_{m-1}P_{m-1}e^{(m-1-i)\beta} + \cdots + a_{i+1}P_{i+1}e^{\beta} + a_i P_i}\right) \\
&\quad + S(r,f) \\
&\leq \overline{N}\left(r, \frac{1}{a_m P_m e^{(m-i-1)\beta} + a_{m-1}P_{m-1}e^{(m-2-i)\beta} + \cdots a_{i+1}P_{i+1}}\right) + S(r,f) \\
&\leq (m-i-1)T(r,f) + S(r,f),
\end{aligned}$$

which is a contradiction. This completes the proof of Lemma 2.6. ∎

Lemma 2.7. *Assume that the assumptions of Lemma 2.6 hold, then* $f(z) = b_1 e^{bz^2}$*,* $g(z) = b_2 e^{-bz^2}$ *for three constants* b_1*,* b_2 *and* b *that satisfy* $4a_m^2(b_1b_2)^{n+m}((n+m)b)^2 = -1$*, or* $4C^2(b_1b_2)^n(nb)^2 = -1$.

Proof. From Lemma 2.6, we get $P(z) = a_m z^m$ or $P(z) = C$, we distinguish two cases.

Case A. $P(z) = a_m z^m$. In this case, we have $(a_m f^{m+n})^{(k)}(a_m g^{m+n})^{(k)} \equiv z^2$.
If $k = 1$, then

$$a_m^2 (f^{m+n})'(g^{m+n})' \equiv z^2. \tag{2.8}$$

Since f and g are entire functions and $n > k+2$, by using the similar arguments as in the proof of Lemma 2.6, we deduce from (2.8) that f and g have no zeros. Let $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are non-constant entire functions. Set

$$h(z) = \frac{1}{f(z)g(z)}, \tag{2.9}$$

we know that $h(z) = e^{\gamma(z)}$, where $\gamma(z)$ is an entire function. We claim that $\gamma(z)$ is a constant. In fact, suppose $\gamma(z)$ is a non-constant entire function, then $h(z)$ is a transcendental entire function. From (2.8), we get

$$(m+n)^2 a_m^2 (f^{n+m-1})f'(g^{n+m-1})g' \equiv z^2. \tag{2.10}$$

From (2.9) and (2.10), we have

$$\left(\frac{g'}{g} + \frac{1}{2}\frac{h'}{h}\right)^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2 h^{m+n}}{(m+n)^2 a_m^2}. \tag{2.11}$$

Let $\xi = \frac{g'}{g} + \frac{1}{2}\frac{h'}{h}$, then (2.11) becomes

$$\xi^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2 h^{m+n}}{(m+n)^2 a_m^2}. \tag{2.12}$$

If $\xi \equiv 0$, from (2.12), we get

$$h^{m+n} = \frac{(m+n)^2 a_m^2}{4z^2}\left(\frac{h'}{h}\right)^2. \tag{2.13}$$

Since $h(z) = e^{\gamma(z)}$, we obtain from (2.13) that

$$\begin{aligned}(m+n)T(r,h) &= (m+n)m(r,h) + S(r,h)\\ &\le m\left(r, \frac{1}{4z^2}\right) + 2m\left(r, \frac{h'}{h}\right) + S(r,h) = S(r,h).\end{aligned}$$

Hence h is a constant, which is a contradiction. Therefore $\xi \not\equiv 0$. Differentiating (2.12), we have

$$\begin{aligned}2\xi\xi' &= \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' - \frac{2z}{a_m^2(m+n)^2}h^{m+n} - \frac{1}{a_m^2(m+n)}z^2h^{m+n-1}h'\\ &= \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' - \frac{1}{a_m^2(m+n)^2}h^{m+n-1}(2zh + (m+n)z^2h'). \qquad (2.14)\end{aligned}$$

From (2.12) and (2.14), we obtain

$$\text{(2.15)}\quad \frac{1}{a_m^2(m+n)^2}h^{m+n}\left(2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}\right)=\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{h'}{h}\frac{\xi'}{\xi}\right).$$

If $2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}\equiv 0$, then, we deduce from (2.15) that either $\frac{h'}{h}\equiv 0$ or $\left(\frac{h'}{h}\right)'-\frac{h'}{h}\frac{\xi'}{\xi}\equiv 0$. If $\frac{h'}{h}\equiv 0$, then h is a constant, which is a contradiction. If $\left(\frac{h'}{h}\right)'-\frac{h'}{h}\frac{\xi'}{\xi}\equiv 0$, we have

$$\text{(2.16)}\quad \frac{h'}{h}=\frac{\xi}{d},$$

where $d(\neq 0)$ is a constant. Thus we get from (2.12) and (2.16) that

$$\text{(2.17)}\quad \frac{z^2h^{m+n}}{a_m^2(m+n)^2}=\left(\frac{1}{4}-d^2\right)\left(\frac{h'}{h}\right)^2.$$

Hence, $(m+n)T(r,h)=S(r,h)$, which is also a contradiction.

Now we assume that $2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}\not\equiv 0$. Since $h=e^{\gamma(z)}$ and $\xi=\frac{g'}{g}+\frac{1}{2}\frac{h'}{h}$, from (2.12) and (2.15), we have

$$N\left(r,\frac{h'}{h}\right)=S(r,h),\quad N(r,\xi)=S(r,h),$$

and

$$\begin{aligned}(m+n)T(r,h)&=(m+n)m(r,h)\le m\left(r,\frac{1}{2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}}\right)\\&\quad+m\left(r,\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{h'}{h}\frac{\xi'}{\xi}\right)\right)+O(1)\\&\le m\left(r,\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{h'}{h}\frac{\xi'}{\xi}\right)\right)+m\left(r,2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}\right)\\&\quad+N\left(r,2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}\right)\\&\le N\left(r,\frac{\xi'}{\xi}\right)+S(r,h)+S(r,\xi)\\&\le T(r,\xi)+S(r,h)+S(r,\xi).\qquad\text{(2.18)}\end{aligned}$$

Note that $h=e^{\gamma(z)}$ is a transcendental entire function, we get from (2.12) that

$$\begin{aligned}2T(r,\xi)=T(r,\xi^2)+S(r,\xi)&=T\left(r,\frac{1}{4}\left(\frac{h'}{h}\right)^2-\frac{z^2h^{m+n}}{a_m^2(m+n)^2}\right)+S(r,\xi)\\&=N\left(r,\frac{1}{4}\left(\frac{h'}{h}\right)^2-\frac{z^2h^{m+n}}{a_m^2(m+n)^2}\right)\end{aligned}$$

$$+ m\left(r, \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2 h^{m+n}}{a_m^2 (m+n)^2}\right) + S(r,\xi)$$

$$\leq (m+n)m(r,h) + N\left(r, \left(\frac{h'}{h}\right)^2\right) + S(r,h) + S(r,\xi)$$

$$\leq (m+n)T(r,h) + S(r,h) + S(r,\xi). \tag{2.19}$$

Combining with (2.18), we have

$$\frac{(m+n)}{2} T(r,h) = S(r,h),$$

which is a contradiction. Thus, $\gamma(z)$ is a constant, and so $h(z) = e^{\gamma(z)}$ is also a constant. From (2.9), we obtain

$$f(z)g(z) = e^{\alpha(z)} e^{\beta(z)} = c_0, \tag{2.20}$$

where $c_0 (\neq 0)$ is a constant. So we have

$$\beta(z) = -\alpha(z) + c_1, \tag{2.21}$$

for a constant c_1. Substituting $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$ into (2.10), we get from (2.20) and (2.21) that

$$f(z) = b_1 e^{bz^2},\ g(z) = b_2 e^{-bz^2},$$

where b_1, b_2 and b are three constants that satisfy $4a_m^2 (b_1 b_2)^{n+m} ((m+n)b)^2 = -1$.

If $k \geq 2$, then

$$a_m^2 (f^{n+m})^{(k)} (g^{n+m})^{(k)} = z^2. \tag{2.22}$$

Since f and g are entire functions and $n > k+2$, by using the arguments similar to the proof of Lemma 2.6, we know from (2.8) that f and g have no zeros. Let

$$f = e^{\alpha(z)}, \quad g = e^{\beta(z)}, \tag{2.23}$$

where $\alpha(z)$, $\beta(z)$ are non-constant entire functions. By (2.22), we have

$$N\left(r, \frac{1}{(f^{m+n})^{(k)}}\right) \leq N\left(r, \frac{1}{z^2}\right) = O(\log r). \tag{2.24}$$

Combining with (2.23) and (2.24), we obtain

$$N(r, f^{m+n}) + N\left(r, \frac{1}{f^{m+n}}\right) + N\left(r, \frac{1}{(f^{m+n})^{(k)}}\right) = O(\log r).$$

By (2.23), $T(r, \frac{(f^{m+n})'}{f^{m+n}}) = T(r, (m+n)\alpha')$. If α is transcendental, We know from Lemma 2.5 that $f = e^{\alpha} = e^{az+b}$ for some constants $a \neq 0$ and b, which is impossible. Hence α must be a polynomial, and so β is also a polynomial. We suppose that $\deg(\alpha) = p$ and $\deg(\beta) = q$. If $p = q = 1$, we have

$$f = e^{Az+B}, \quad g = e^{Cz+D}, \tag{2.25}$$

where A, B, C and D are constants that satisfy $AC \neq 0$. Substituting (2.25) into (2.22), we obtain

$$a_m^2 (m+n)^{2k} (AC)^k e^{(m+n)(A+C)z + (m+n)(B+D)} = z^2,$$

which is impossible. Thus $\max\{p,q\} > 1$. Without loss of generality, we suppose that $p > 1$. Then $(f^{m+n})^{(k)} = Q_1(z)e^{(m+n)\alpha}$, where $Q_1(z)$ is a polynomial of degree $kp - k \geq k \geq 2$. From (2.22), we have $p = k = 2$ and $q = 1$. Suppose that

$$f^{m+n} = e^{(m+n)(A_1z^2+B_1z+C_1)}, \quad g^{m+n} = e^{(m+n)(D_1z+E_1)},$$

where A_1, B_1, C_1, D_1, E_1 are constants such that $A_1D_1 \neq 0$. Then we have

$$\begin{aligned}(f^{m+n})'' = {} & (m+n)(4(m+n)A_1^2z^2 + 4(m+n)A_1B_1z + (m+n)B_1^2 \\ & + 2A_1)e^{(m+n)(A_1z^2+B_1z+C_1)},\end{aligned} \tag{2.26}$$

$$(g^{m+n})'' = (m+n)^2D_1^2e^{(m+n)(D_1z+E_1)}. \tag{2.27}$$

Substituting (2.26) and (2.27) into (2.22), we have

$$Q_2(z)e^{(m+n)(A_1z^2+(B_1+D_1)z+C_1+E_1)} = z^2,$$

where $Q_2(z)$ is a polynomial of degree 2. Since $A_1 \neq 0$, we get a contradiction.

Case B. $P(z) = C$. In this case, by the similar arguments mentioned in the Case A, f and g must satisfy $f(z) = b_1e^{bz^2}, g(z) = b_2e^{-bz^2}$, where b_1, b_2, b are constants that satisfy $4C^2(b_1b_2)^n(nb)^2 = -1$. Lemma 2.7 follows. ∎

Lemma 2.8. *Let f and g be two non-constant entire functions, n, m and k be three positive integers, and let $F = (f^n(z)P(f))^{(k)}$, $G = (g^n(z)P(g))^{(k)}$, where $P(z)$ is given by Theorem 1.5 and not a monomial. If there exist two nonzero constants a_1 and a_2 such that $\overline{N}(r, \frac{1}{F-a_1}) = \overline{N}\left(r, \frac{1}{G}\right)$ and $\overline{N}(r, \frac{1}{G-a_2}) = \overline{N}\left(r, \frac{1}{F}\right)$, then $n \leq 2k + 2 + m$.*

Proof. By the second fundamental theorem, we have

$$\begin{aligned}T(r,F) &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-a_1}\right) + S(r,F) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,F) \\ &\leq N_1\left(r, \frac{1}{F}\right) + N_1\left(r, \frac{1}{G}\right) + S(r,F).\end{aligned} \tag{2.28}$$

From (2.28), Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}T(r,F) \leq {} & T(r,F) - T(r, f^n(z)P(f)) + N_{k+1}\left(r, \frac{1}{f^n(z)P(f)}\right) \\ & + N_{k+1}\left(r, \frac{1}{g^n(z)P(g)}\right) + S(r,f) + S(r,g).\end{aligned}$$

Hence

$$\begin{aligned}(n+m)T(r,f) \leq {} & N_{k+1}\left(r, \frac{1}{f^n(z)P(f)}\right) + N_{k+1}\left(r, \frac{1}{g^n(z)P(g)}\right) \\ & + S(r,f) + S(r,g) \leq (k+1)\left(\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right)\right) \\ & + m(T(r,f) + T(r,g)) + S(r,f) + S(r,g).\end{aligned} \tag{2.29}$$

By the similar reasoning, we have

$$(n+m)T(r,g) \le (k+1)\left(\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right) + m(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.30}$$

From (2.29) and (2.30), we have

$$(n-2k-2-m)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g),$$

which implies that $n \le 2k+2+m$. Lemma 2.8 is thus proved. ∎

By the arguments much similar to the proof of Lemma 2.8, we have the following lemma.

Lemma 2.9. *Let f and g be two non-constant entire functions, n, m and k be three positive integers, and let $F=(f^n(z)P(f))^{(k)}$, $G=(g^n(z)P(g))^{(k)}$, where $P(z)$ is given by Theorem 1.5 and $P(z)=a_mz^m$ or $P(z)=C$. If there exist two nonzero constants a_1 and a_2 such that $\overline{N}(r,\frac{1}{F-a_1})=\overline{N}\left(r,\frac{1}{G}\right)$ and $\overline{N}(r,\frac{1}{G-a_2})=\overline{N}\left(r,\frac{1}{F}\right)$, then $n\le 2k+2-m^*$.*

3. Proof of theorems

Proof of Theorem 1.5. Set $F=f^n(z)P(f)$, by Lemma 2.4, we have

$$T(r,F^{(k)}) \le \overline{N}\left(r,\frac{1}{F^{(k)}}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}-z}\right)+S(r,f). \tag{3.1}$$

Case 1. $P(f)=a_mf^m+a_{m-1}f^{m-1}+\cdots a_1f+a_0$, where $a_m\ne 0$. By (3.1) and Lemma 2.2 with $p=1$, we obtain

$$\begin{aligned} T(r,F^{(k)}) &\le N_1\left(r,\frac{1}{F^{(k)}}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}-z}\right)+S(r,f)\\ &\le T(r,F^{(k)})-T(r,F)+N_{k+1}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}-z}\right)+S(r,f), \end{aligned} \tag{3.2}$$

and so

$$\begin{aligned} T(r,F) &\le N_{k+1}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}-z}\right)+S(r,f)\\ &\le N_{k+1}\left(r,\frac{1}{f^n}\right)+N_{k+1}\left(r,\frac{1}{a_mf^m+a_{m-1}f^{m-1}+\cdots a_1f+a_0}\right)\\ &\quad +\overline{N}\left(r,\frac{1}{F^{(k)}-z}\right)+S(r,f)\\ &\le (k+1+m)T(r,f)+\overline{N}\left(r,\frac{1}{F^{(k)}-z}\right)+S(r,f). \end{aligned}$$

Noting that $T(r,F)=(m+n)T(r,f)+S(r,f)$ and $n\ge k+2$, we get $[f^n(z)P(f)]^{(k)}$ has infinitely many fixed points .

Case 2. $P(f) = C$, where $C \neq 0$. By using the same arguments as mentioned above, we have

$$\begin{aligned} T(r,F) &\le N_{k+1}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F^{(k)}-z}\right) + S(r,f) \\ &\le N_{k+1}\left(r,\frac{1}{Cf^{n}}\right) + \overline{N}\left(r,\frac{1}{F^{(k)}-z}\right) + S(r,f) \\ &\le (k+1)T(r,f) + \overline{N}\left(r,\frac{1}{F^{(k)}-z}\right) + S(r,f). \end{aligned}$$

Note that $T(r,F) = nT(r,f) + S(r,f)$ and $n \ge k+2$, we obtain $[f^n(z)P(f)]^{(k)}$ has infinitely many fixed points. Theorem 1.5 follows. ∎

Proof of Theorem 1.6. We consider the following two cases.
(i) $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ is not a monomial. Let

$$F = \frac{(f^n(z)P(f))^{(k)}}{z}, \quad G = \frac{(g^n(z)P(g))^{(k)}}{z}. \tag{3.3}$$

Then F and G are transcendental meromorphic functions that share 1 CM. Let H be given by (2.3). If $H \not\equiv 0$, by Lemma 2.3, we know that (2.4) holds. From Lemma 2.2, we have

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &\le N_2\left(r,\frac{1}{(f^n(z)P(f))^{(k)}}\right) + S(r,f) \\ &\le T(r,(f^n(z)P(f))^{(k)}) - (m+n)T(r,f) \\ &\quad + N_{k+2}\left(r,\frac{1}{f^n(z)P(f)}\right) + S(r,f) \\ &= T(r,F) - (m+n)T(r,f) + N_{k+2}\left(r,\frac{1}{f^n(z)P(f)}\right) + S(r,f). \end{aligned} \tag{3.4}$$

Similarly, we have

$$N_2\left(r,\frac{1}{G}\right) \le T(r,G) - (m+n)T(r,g) + N_{k+2}\left(r,\frac{1}{g^n(z)P(g)}\right) + S(r,g). \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$N_2\left(r,\frac{1}{F}\right) \le N_{k+2}\left(r,\frac{1}{f^n(z)P(f)}\right) + S(r,f), \tag{3.6}$$

and

$$N_2\left(r,\frac{1}{G}\right) \le N_{k+2}\left(r,\frac{1}{g^n(z)P(g)}\right) + S(r,g). \tag{3.7}$$

Again, from (3.4) and (3.5), we have

$$\begin{aligned} (m+n)(T(r,f)+T(r,g)) \le T(r,F) + T(r,G) - N_2\left(r,\frac{1}{F}\right) - N_2\left(r,\frac{1}{G}\right) \\ + N_{k+2}\left(r,\frac{1}{f^n(z)P(f)}\right) \end{aligned}$$

$$+ N_{k+2}\left(r, \frac{1}{g^n(z)P(g)}\right) + S(r,f) + S(r,g).$$

Combining with (3.6), (3.7) and Lemma 2.3, we get

$$\begin{aligned}(m+n)(T(r,f)+T(r,g)) &\le 2N_{k+2}\left(r, \frac{1}{f^n(z)P(f)}\right) \\ &\quad + 2N_{k+2}\left(r, \frac{1}{g^n(z)P(g)}\right) + S(r,f) + S(r,g) \\ &\le (2k+4)\left(\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right)\right) + 2N_{k+2}\left(r, \frac{1}{P(f)}\right) \\ &\quad + 2N_{k+2}\left(r, \frac{1}{P(g)}\right) + S(r,f) + S(r,g). \qquad (3.8)\end{aligned}$$

Thus, we deduce that

$$(m+n-2k-4-2m)(T(r,f)+T(r,g)) \le S(r,f) + S(r,g),$$

which contradicts the assumption that $n > 2k+4+m$. Therefore $H \equiv 0$. Integrating twice, we get from (2.3) that

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \qquad (3.9)$$

where $A(\neq 0)$ and B are constants. From (3.9), we have

$$F = \frac{(B+1)G + (A-B-1)}{BG + (A-B)}, \quad G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)}. \qquad (3.10)$$

We consider the following three cases.

Case 1. Suppose that $B \neq 0, -1$. From (3.10) we have $\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r,G)$. From the second fundamental theorem, we have

$$\begin{aligned}T(r,F) &\le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r,F) \\ &= \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r,G) + S(r,F) \le \overline{N}\left(r, \frac{1}{F}\right) + S(r,F). \qquad (3.11)\end{aligned}$$

By (3.11) and the same reasoning as in the proof of (3.4), we obtain

$$\begin{aligned}T(r,F) &\le N_1\left(r, \frac{1}{F}\right) + S(r,f) \\ &\le T(r,F) - (m+n)T(r,f) + N_{k+1}\left(r, \frac{1}{f^n(z)P(f)}\right) + S(r,f).\end{aligned}$$

Hence

$$\begin{aligned}(m+n)T(r,f) &\le (k+1)\overline{N}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{P(f)}\right) + S(r,f) \\ &\le (k+m+1)T(r,f) + S(r,f),\end{aligned}$$

which contradicts $n > 2k+4+m$.

Case 2. Suppose that $B = 0$. From (3.10) we have

$$F = \frac{G + (A-1)}{A}, \quad G = AF - (A-1). \tag{3.12}$$

If $A \neq 1$, we get from (3.12) that $\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right)$ and $\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}(r, \frac{1}{G+(A-1)})$. By Lemma 2.8, we have $n \leq 2k+2+m$. This contradicts the assumption that $n > 2k+4+m$. Thus $A = 1$ and $F = G$, that is,

$$(f^n P(f))^{(k)} = (g^n P(g))^{(k)}.$$

By integration, we have

$$(f^n(z)P(f))^{(k-1)} = (g^n(z)P(g))^{(k-1)} + a_{k-1}.$$

where a_{k-1} is a constant. If $a_{k-1} \neq 0$, we get from Lemma 2.8 that $n \leq 2k+m$, which is a contradiction. Hence $a_{k-1} = 0$. Repeating the same process for $k-1$ times, we obtain $f^n(z)P(f) = g^n(z)P(g)$, that is

$$\begin{aligned} & f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0) \\ & = g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g + a_0). \end{aligned} \tag{3.13}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ into (3.13), we deduce

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \cdots + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$. If h is not a constant, then we know by (3.13) that f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_1\omega_1 + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_1\omega_2 + a_0)$.

Case 3. Suppose that $B = -1$. From (3.10) we obtain

$$F = \frac{A}{-G + (A+1)}, \quad G = \frac{(A+1)F - A}{F}. \tag{3.14}$$

If $A \neq -1$, we obtain from (3.14) that $\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right)$, $\overline{N}(r, F) = \overline{N}(r, \frac{1}{G-A-1})$. By the same reasoning mentioned in Case 1 and Case 2, we get a contradiction. Hence $A = -1$. From (3.14), we have $FG = 1$, that is

$$(f^n(z)P(f))^{(k)}(g^n(z)P(g))^{(k)} = z^2,$$

by Lemma 2.6, this is impossible .

(ii) $P(z) = C$ or $P(z) = a_m z^m$, we distinguish two cases.

Case A. $P(z) = a_m z^m$. In this case, we have $F = (a_m f^{n+m}(z))^{(k)}$ and $G = (a_m g^{n+m}(z))^{(k)}$. Let

$$F_1 = \frac{(a_m f^{n+m}(z))^{(k)}}{z}, \quad G_1 = \frac{(a_m g^{n+m}(z))^{(k)}}{z}.$$

Then F_1 and G_1 share 1 CM. By the similar arguments mentioned in the proof of (i), we have $F_1 \equiv G_1$ or $F_1 G_1 \equiv 1$.

If $F_1 G_1 = 1$, we obtain from Lemma 2.7 that $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ for three constants b_1, b_2 and b that satisfy $4a_m^2 (b_1 b_2)^{n+m}((n+m)b)^2 = -1$.

If $F_1 \equiv G_1$, we get

$$(a_m f^{n+m})^{(k)} = (a_m g^{n+m})^{(k)}.$$

By integration, we have

$$(a_m f^{n+m})^{(k-1)} = (a_m g^{n+m})^{(k-1)} + a_{k-1}.$$

where a_{k-1} is a constant. If $a_{k-1} \neq 0$, we get from Lemma 2.9 that $n \leq 2k + m$, which is a contradiction. Hence $a_{k-1} = 0$. Repeating the same process for $k - 1$ times, we obtain $a_m f^{n+m} = a_m g^{n+m}$, we get that $f \equiv tg$, where t is a constant that satisfies $t^{n+m} = 1$.

Case B. $P(z) = C$. In this case, by the similar arguments mentioned in the Case A, f and g must satisfy $f(z) = b_1 e^{bz^2}, g(z) = b_2 e^{-bz^2}$, where b_1, b_2 and b are three constants satisfying $4C^2 (b_1 b_2)^n (nb)^2 = -1$ or $f = tg$ for a constant t such that $t^n = 1$. This completes the proof of Theorem 1.6. ∎

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