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Moufang Loops of Odd Order $p_1 p_2 \cdots p_n q^3$

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Abstract. It has been proved that for distinct odd primes $p_1, p_2, \ldots, p_n$ and q, all Moufang loops of order $p_1 p_2 \cdots p_n q^3$ are associative if:
(1) $q \not\equiv 1 \pmod{p_1}$ and for each $i > 1$, $q^2 \not\equiv 1 \pmod{p_i}$; or
(2) $p_1 < p_2 < \cdots < p_n < q$, $q \not\equiv 1 \pmod{p_i}$, $p_i \not\equiv 1 \pmod{p_j}$ for all i, j, and the nucleus is not trivial.
In this paper, we extend these results by giving a complete resolution for Moufang loops of odd order $p_1 p_2 \cdots p_n q^3$.

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1. Introduction

A loop $\langle L, \cdot \rangle$ is a Moufang loop if it satisfies the Moufang identity $(x \cdot y) \cdot (z \cdot x) = [x \cdot (y \cdot z)] \cdot x$. One of the most important theorems in the study of Moufang loops would be Moufang's theorem: If there exist three (fixed) elements x, y, z in a Moufang loop that associate in some order, then these elements generate a group. As a corollary, Moufang loops are diassociative, i.e. for any two (fixed) elements x and y in a Moufang loop, they generate a group. Moufang loops need not be associative since there exists a nonassociative Moufang loop of order 12; see [3]. Hence, our interest is to study the question: "For what positive integer n does there exist a nonassociative Moufang loop of order n?"

In order to construct nonassociative Moufang loops, we need to eliminate those Moufang loops that will automatically become groups by virtue of their orders. This is particularly true because it is always possible to use any nonassociative Moufang loop of order m and any group of order n to construct a nonassociative Moufang loop of order mn. Consequently, if it is known that all Moufang loops of order mn are associative, then all Moufang loops of order m (and n) must also be associative.

For Moufang loops of even order, the problem is completely resolved by Chein and the first author in [4]: All Moufang loops of order $2m$ are associative if and

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only if all groups of order m are abelian. As for Moufang loops of odd order, the existence of nonassociative Moufang loops of order 3^4 and p^5 for every prime $p > 3$, has been proved by Bol [1] and Wright [18] respectively. The most recent class of nonassociative Moufang loops is constructed by the first author in [16]. He gives a product rule for nonassociative Moufang loops of order pq^3 where p and q are odd primes with $q \equiv 1 \pmod{p}$.

The proof that all Moufang loops of a particular order are associative has progressed gradually over the last four decades. We give below a list for which all Moufang loops of such orders have been proved to be groups:

(i) p, p^2, p^3 and pq where p and q are primes [3];
(ii) p^4 where p is a prime with $p > 3$ [7];
(iii) pqr and p^2q where p, q and r are odd primes with $p < q < r$ [14];
(iv) pq^2 where p and q are odd primes [8];
(v) $p_1^2 p_2^2 \cdots p_n^2$ where $p_1, p_2, \ldots, p_n$ are distinct odd primes [9];
(vi) $p^3 q_1 q_2 \cdots q_n$ [13] and $p^3 q_1^2 q_2^2 \cdots q_n^2$ [11] where $p, q_1, q_2, \ldots, q_n$ are distinct odd primes with $p < q_i$;
(vii) $p^4 q_1 q_2 \cdots q_n$ [10] and $p^4 q_1^2 q_2^2 \cdots q_n^2$ [11] where $p, q_1, q_2, \ldots, q_n$ are distinct odd primes with $3 < p < q_i$;
(viii) pq^3 where p and q are distinct odd primes with $q \not\equiv 1 \pmod{p}$ [16];
(ix) $p_1 p_2 \cdots p_n q^3$ where $p_1, p_2, \ldots, p_n, q$ are distinct odd primes with $q \not\equiv 1 \pmod{p_1}$ and $q^2 \not\equiv 1 \pmod{p_i}$ for each $i > 1$ [4];
(x) $p_1 p_2 \cdots p_n q^3$ where $p_1, p_2, \ldots, p_n, q$ are distinct odd primes with $p_i < q$, $q \not\equiv 1 \pmod{p_i}$, $p_i \not\equiv 1 \pmod{p_j}$ for all i, j, and the nucleus is not trivial [17].

Remark 1.1. The proof of result (iii) has a flaw in the case p^2q where $q < p$; see [15], but it is later resolved in [8] (result (iv)).

In this paper, we extend some of the results above (particularly those in (vi), (ix) and (x)) and prove that for distinct odd primes $p_1, p_2, \ldots, p_n$ and q, all Moufang loops of order $p_1 p_2 \cdots p_n q^3$ are associative if and only if $q \not\equiv 1 \pmod{p_i}$ for each i.

2. Definitions and notations

In order to make the contents of the paper as self contained as possible, we give some basic definitions and notations that are relevant. For those not listed, we refer the reader to [2] and [5].

Definition 2.1. *A quasigroup is a binary system $\langle L, \cdot \rangle$ in which specification of any two of the values x, y, z in the equation $x \cdot y = z$ uniquely determines the third value. If it further contains an identity element, then it is called a loop. (Often, when there is no risk of confusion, the notation for a loop $\langle L, \cdot \rangle$ is simplified to L instead.)*

Definition 2.2. *A subset K of a loop L is called a subloop of L ($K \leq L$) if K is a loop under the operation of L. K is a proper subloop of L if $K \neq L$.*

Definition 2.3. *A subloop K of a loop L is called a normal subloop of L ($K \trianglelefteq L$) if $xK = Kx$, $x(yK) = (xy)K$ and $(Kx)y = K(xy)$ for all $x, y \in L$.*

Remark 2.1. Suppose L is a loop in which every element has a two-sided inverse. We define

$$zT(x) = x^{-1} \cdot zx,$$
$$zL(x,y) = (yx)^{-1}(y \cdot xz),$$
$$zR(x,y) = (zx \cdot y)(xy)^{-1}.$$

$I(L) = \langle T(x), L(x,y), R(x,y) \mid x, y \in L\rangle$ is called the inner mapping group of L. K is a normal subloop of L if $K\theta = \{k\theta \mid k \in K\} = K$ for all $\theta \in I(L)$.

Definition 2.4. *Let K be a normal subloop of a loop $\langle L, \cdot\rangle$.*

(a) *Let L/K be the set of all cosets of K in L and $\odot$ a binary operation on L/K such that $xK \odot yK = (x \cdot y)K$. Then $\langle L/K, \odot\rangle$ is called a quotient loop of L.*
(b) *L/K is a proper quotient loop of L if K is not trivial.*
(c) *K is a minimal normal subloop of L if K is not trivial and for every non-trivial normal subloop H of L, $H \subseteq K \Rightarrow H = K$.*
(d) *K is a maximal normal subloop of L if K is a proper subloop of L and for every proper normal subloop H of L, $K \subseteq H \Rightarrow H = K$.*

Definition 2.5. *Let L be a finite loop, K a subloop of L and π a set of primes.*

(a) *A positive integer n is a π-number if every prime divisor of n lies in π.*
(b) *K is a π-loop if the order of every element of K is a π-number.*
(c) *K is a Hall π-subloop of L if K is a π-loop and $|K|$ is the largest π-number that divides $|L|$.*
(d) *K is a Sylow p-subloop of L if K is a Hall π-subloop of L and $\pi = \{p\}$.*

Definition 2.6. *The associator of three elements x, y, z in a loop L is the unique element (x, y, z) in L such that $xy \cdot z = (x \cdot yz)(x, y, z)$. The associator subloop of L, denoted by L_a, is the subloop generated by all the associators in L.*

Definition 2.7. *The commutator of two elements x, y in a loop L is the unique element $[x, y]$ in L such that $xy = (yx)[x, y]$. The commutator subloop of L, denoted by L_c, is the subloop generated by all the commutators in L.*

Definition 2.8. *The nucleus of a loop L, denoted by $N(L)$ or simply N, is the subloop consisting of all $n \in L$ such that $(n, x, y) = (x, n, y) = (x, y, n) = 1$ for all $x, y \in L$.*

Definition 2.9. *A loop L is a Moufang loop if it satisfies any one of the following four (equivalent) Moufang identities:*

$xy \cdot zx = (x \cdot yz)x$	*First Middle Moufang identity*
$xy \cdot zx = x(yz \cdot x)$	*Second Middle Moufang identity*
$x(y \cdot xz) = (xy \cdot x)z$	*Left Moufang identity*
$(zx \cdot y)x = z(x \cdot yx)$	*Right Moufang identity*

Remark 2.2. It is proved in [2, p. 115, Lemma 3.1] that Moufang loops have the inverse property.

3. Lemmas

In this section we present some lemmas which will be needed in the proof of our main result.

Lemma 3.1. *Let L be a Moufang loop.*

(a) *Suppose $x \in L$ and $\theta \in I(L)$. Then $(x^n)\theta = (x\theta)^n$ for any integer n* [2, p. 117, Lemma 3.2 and p. 120, Lemma 4.1].

(b) *Suppose $x, y, u, v \in L$ and $\theta \in I(L)$. Then $(xy)\theta \cdot c = (x\theta) \cdot (y\theta \cdot c)$ where $c = [u^{-1}, v^{-1}]$ if $\theta = L(u, v)$, and $c = u^{-3}$ if $\theta = T(u)$* [2, p. 112, Lemma 2.1; p. 113, Lemma 2.2 and p. 117, Lemma 3.2].

(c) $xL(z, y) = x(x, y, z)^{-1}$ [2, p. 124, Lemma 5.4].

Lemma 3.2. *Let L be a Moufang loop. For any $x, y, z \in L$ and $n \in N$, $(xn, y, z) = (x, yn, z) = (x, y, zn) = (x, y, z)$* [8, p. 267, Lemma 1].

Lemma 3.3. *Let L be a Moufang loop and K a normal subloop of L. If L/K is a group, then $L_a \subseteq K$* [10, p. 563, Lemma 1(a)].

Lemma 3.4. [Lagrange's theorem] *Let L be a finite Moufang loop and K a subloop of L. Then $|K|$ divides $|L|$* [6, p. 42, Lagrange's theorem].

Lemma 3.5. *Let L be a Moufang loop of odd order. Suppose $H \trianglelefteq K \trianglelefteq L$ and H is a Hall subloop of K, then $H \trianglelefteq L$* [9, p. 879, Lemma 1].

Lemma 3.6. *Let L be a Moufang loop of odd order.*

(a) *L contains a Hall π-subloop where π is any set of odd primes* [5, p. 409, Theorem 12].

(b) *Suppose $K \trianglelefteq L$, $(K, K, L) = 1$ and $(|K|, |L/K|) = 1$. Then $K \subseteq N$* [5, p. 405, Theorem 10].

Lemma 3.7. *Let L be a Moufang loop of odd order and all proper subloops of L are groups.*

(a) *If there exists a minimal normal Sylow subloop in L, then L is a group* [8, p. 268, Lemma 2].

(b) *If N contains a Hall subloop of L, then L is a group* [10, p. 564, Lemma 2].

Lemma 3.8. *Let L be a Moufang loop of odd order and all proper quotient loops of L are groups. Then $(k_1k_2, \ell_1, \ell_2) = (k_1, \ell_1, \ell_2)(k_2, \ell_1, \ell_2)$ for each $k_i \in L_a$ and $\ell_i \in L$* [11, p. 483, Lemma 8].

Lemma 3.9. *Let L be a Moufang loop of odd order, K a minimal normal subloop of L and H a Hall subloop of L. Suppose all proper subloops and proper quotient loops of L are groups, $L_a \subseteq K$, $(|K|, |H|) = 1$ and $H \trianglelefteq KH$. Then L is a group* [10, p. 564, Lemma 3].

Lemma 3.10. *Let L be a nonassociative Moufang loop of odd order and all proper quotient loops of L are groups. Then*

(a) *L_a is a minimal normal subloop of L and is an elementary abelian group;*

(b) *L_a and L_c lie in every maximal normal subloop of L.*

[11, p. 478, Lemma 1 and 5, p. 402, Theorem 7].

Lemma 3.11. *Let L be a nonassociative Moufang loop of odd order and M a maximal normal subloop of L. Suppose all proper subloops and proper quotient loops of L are groups.*

(a) *For any $w \in M$ and $\ell \in L$, there exists some $k_0 \in L_a - \{1\}$ such that $(k_0, w, \ell) = 1$.*

(b) *If $(k, w, \ell) = 1$ for all $k \in L_a$, $w \in M$ and $\ell \in L$, then $L_a \subseteq N$.*

[11, p. 478, Lemma 2 and p. 479 Lemma 3]

Lemma 3.12. *Let L be a Moufang loop of order $p^\alpha m$ where p is a prime and $(p, m) = (p-1, p^\alpha m) = 1$. Suppose L has an element of order p^α. Then there exist a subloop P of order p^α and a normal subloop M of order m in L such that $L = PM$* [12, p. 39, Theorem 1].

Lemma 3.13. *Let L be a Moufang loop of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ where $p_1, p_2, \ldots, p_n$ are odd primes, $p_1 < p_2 < \cdots < p_n$ and $1 \le \alpha_i \le 2$ for all i. Then there exists a subloop of order $p_n^{\alpha_n}$ normal in L* [9, p. 879, Lemma 2 and p. 882, Theorem].

Lemma 3.14. *Let L be a Moufang loop of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ where $p_1, p_2, \ldots, p_n$ are odd primes, $p_1 < p_2 < \cdots < p_n$ and $1 \le \alpha_n \le 2$. Suppose all proper subloops and proper quotient loops of L are groups; and there exists a normal Sylow p_n-subloop in L. Then L is a group* [9, p. 879, Lemma 3].

Lemma 3.15. *Let L be a Moufang loop of order $p^\alpha q_1^{\beta_1}\cdots q_n^{\beta_n}$ where $p, q_1, \ldots, q_n$ are odd primes with $p < q_1 < \cdots < q_n$, $\alpha \le 3$ and $\beta_i \le 2$. Then L is a group* [11, p. 482, Theorem 1].

Lemma 3.16. *Let L be a Moufang loop of order $p_1p_2\cdots p_nq^3$ where $p_1, p_2, \ldots, p_n$ and q are distinct odd primes with $q \not\equiv 1 \pmod{p_1}$ and $q^2 \not\equiv 1 \pmod{p_i}$ for each $i \in \{2, 3, \ldots, n\}$. Then L is a group* [4, p. 240, Theorem 2.1].

Lemma 3.17. *Let L be a Moufang loop of odd order and K a normal Hall subloop of L. Suppose $K = \langle x\rangle L_a$ for some $x \in K - L_a$ and $L_a \subseteq N$. Then $K \subseteq N$.*

Proof. Take $u, v \in K$ and $\ell \in L$. Then $u = x^\alpha k_1$, $v = x^\beta k_2$ for some $\alpha, \beta \in \mathbb{Z}^+$ and $k_i \in L_a$.

$$\begin{aligned}(u, v, \ell) &= (x^\alpha k_1, x^\beta k_2, \ell)\\ &= (x^\alpha, x^\beta, \ell) \quad \text{by Lemma 3.2 since } k_i \in N\\ &= 1 \qquad\qquad\quad\ \text{by diassociativity.}\end{aligned}$$

Hence $(K, K, L) = 1$. Since K is a Hall subloop of L, $(|K|, |L/K|) = 1$. Thus, we are through by Lemma 3.6(b). ∎

Lemma 3.18. *Let L be a nonassociative Moufang loop of odd order and M a maximal normal subloop of L. Suppose all proper subloops and proper quotient loops of L are groups. Then for any $w \in M$ and $\ell \in L$, there exists some $k_0 \in L_a - \{1\}$ such that $(u^{-1}k_0u, w, \ell) = 1$ for all $u \in M$.*

Proof. Take any $w \in M$ and $\ell \in L$. By Lemma 3.11(a), there exists some $k_0 \in L_a - \{1\}$ such that $(k_0, w, \ell) = 1$.

Write $c = [\ell^{-1}, w^{-1}]$. Since $M \trianglelefteq L$, $c = \ell w\ell^{-1}w^{-1} = wT(\ell^{-1}) \cdot w^{-1} \in M$.

Take any $u \in M$. Since $L_a \subseteq M$ by Lemma 3.10(b) and M is a group, we can freely omit the parentheses when writing the product of elements in M in the proof below.

Now $(u^{-1}k_0u)L(\ell, w) \cdot c = u^{-1}L(\ell, w) \cdot k_0L(\ell, w) \cdot uL(\ell, w) \cdot c$ by applying Lemma 3.1(b) twice. After cancellation of c, we get

$$\begin{aligned}
& u^{-1}k_0u(u^{-1}k_0u, w, \ell)^{-1} \\
=& u^{-1}(u^{-1}, w, \ell)^{-1}k_0(k_0, w, \ell)^{-1}[u^{-1}L(\ell, w)]^{-1} && \text{by Lemmas 3.1(a) and (c)} \\
=& u^{-1}(u^{-1}, w, \ell)^{-1}k_0[u^{-1}(u^{-1}, w, \ell)^{-1}]^{-1} && \text{by Lemma 3.1(c)} \\
=& u^{-1}(u^{-1}, w, \ell)^{-1}k_0(u^{-1}, w, \ell)u \\
=& u^{-1}k_0u && \text{by Lemma 3.10(a).}
\end{aligned}$$

By cancellation, we get $(u^{-1}k_0u, w, \ell) = 1$. ∎

Lemma 3.19. *Let L be a nonassociative Moufang loop of odd order and M a maximal normal subloop of L. Suppose all proper subloops and proper quotient loops of L are groups; and $(k, w, \ell) \neq 1$ for some (fixed) elements $k \in L_a$, $w \in M$ and $\ell \in L$. Then L_a contains a proper nontrivial subloop which is normal in M.*

Proof. Although $(k, w, \ell) \neq 1$ for the fixed elements $k \in L_a$, $w \in M$, $\ell \in L$, but for these particular w and ℓ, there exists some $k_0 \in L_a - \{1\}$ such that $(u^{-1}k_0u, w, \ell) = 1$ for all $u \in M$, by Lemma 3.18. Let $H = \{u^{-1}k_0u \mid u \in M\}$ and $S = \langle H\rangle$. By Lemma 3.10(a), $L_a \trianglelefteq L$. So $u^{-1}k_0u \in L_a$ for all $u \in M$. Thus $H \subseteq L_a$. Hence $S \leq L_a$. Also, since L_a is a group by Lemma 3.10(a), the elements in H associate with one another.

Take $s \in S$. Since L is a finite loop, $s = h_1h_2 \cdots h_n$ where $h_i \in H$.

$$\begin{aligned}
(s, w, \ell) &= (h_1, w, \ell)(h_2, w, \ell) \cdots (h_n, w, \ell) && \text{by Lemma 3.8} \\
&= 1 && \text{by Lemma 3.18.}
\end{aligned}$$

Thus $(s, w, \ell) = 1$ for all $s \in S$. Since $(k, w, \ell) \neq 1$ and $k \in L_a$, it follows that $k \in L_a - S$. So S is a proper subloop of L_a. S is not trivial as $k_0 \in S$ and $k_0 \neq 1$.

Take $v \in M$.

$$\begin{aligned}
v^{-1}(u^{-1}k_0u)v &= (v^{-1}u^{-1})k_0(uv) && \text{as } k_0 \in M \text{ by Lemma 3.10(b)} \\
&= (uv)^{-1}k_0(uv) \in S && \text{as } uv \in M.
\end{aligned}$$

Hence $v^{-1}sv \in S$ for all $s \in S$ and $v \in M$. Since M is a group, by the definition of normal subgroups, $S \trianglelefteq M$. ∎

4. Main theorem

Theorem 4.1. *Let L be a Moufang loop of order $p_1 \cdots p_m q^3 r_1 \cdots r_n$ where $p_1, \ldots,$ $p_m, q, r_1, \ldots, r_n$ are odd primes with $p_1 < \cdots < p_m < q < r_1 < \cdots < r_n$ and $q \not\equiv 1$ (mod p_i) for all $i \in \{1, 2, \ldots, m\}$. Then L is a group.*

Proof. If $m = 0$, we are through by Lemma 3.15, and if $m = 1$, we are through by taking $r_1, r_2, \ldots, r_n$ as $p_2, p_3, \ldots, p_n$ in Lemma 3.16. So we need to consider now the case $m \geq 2$. Let m and n be the smallest positive integers such that

$$(*) \qquad L \text{ is not a group.}$$

Let H be a proper subloop of L. Lagrange's theorem (Lemma 3.4) gives $|H| = p_{i_1}\cdots p_{i_s}q^{\beta}r_{j_1}\cdots r_{j_t}$, where either $s<m$, $\beta<3$ or $t<n$. If $\beta<3$, then H is a group by Lemma 3.15. If $s<m$ or $t<n$, then H is a group by the minimality of m and n. Thus, every proper subloop of L is a group. The same applies to any proper quotient loop of L.

Now by Lemma 3.10(a), L_a is a minimal normal subloop of L and is an elementary abelian group. Since L is not a group, L_a is not a Sylow subloop of L by Lemma 3.7(a). So $|L_a| = q$ or q^2.

Suppose $n>0$. Now $|L/L_a| = p_1\cdots p_mq^2r_1\cdots r_n$ or $p_1\cdots p_mqr_1\cdots r_n$. Lemma 3.13 guarantees the existence of a normal subloop K/L_a of order r_n in L/L_a. Hence, $|K| = qr_n$ or q^2r_n and $K \trianglelefteq L$. Again by Lemma 3.13, there exists a normal subloop R of order r_n in K. Now $R \trianglelefteq K \trianglelefteq L$ and R is a Hall subloop of K. So by Lemma 3.5, $R \trianglelefteq L$. But R is also a Sylow r_n-subloop of L. Thus L is a group by Lemma 3.14. This contradicts our first assumption, $(*)$.

Hence $n=0$, and our problem has been reduced to the case $|L| = p_1p_2\cdots p_mq^3$. Recall that $|L_a| = q$ or q^2. We consider each case separately below:

Case 1. $|L_a| = q$.

By Lemma 3.6(a), there exists P_1, a Sylow p_1-subloop of L. Now $L_a \trianglelefteq L$ implies $L_aP_1 \le L$ where $|L_aP_1| = \frac{|L_a||P_1|}{|L_a\cap P_1|} = p_1q$. Since p_1 and q are distinct primes, $(q,p_1)=1$. As $q \not\equiv 1 \pmod{p_1}$ and $q \not\equiv 1 \pmod{q}$, it follows that $(q-1, qp_1) = 1$. It is clear that $L_a \subseteq L_aP_1$. Since L_a is a cyclic group of order q, L_aP_1 contains an element of order q. Now let $p=q$, $m=p_1$ and $\alpha=1$ as stated in Lemma 3.12, then there exists a normal subloop of order p_1 in L_aP_1. As P_1 is a Sylow subloop of L_aP_1, P_1 is the unique normal subloop of L_aP_1. It is also clear that $(|L_a|, |P_1|) = (q,p_1) = 1$. Hence L is a group by Lemma 3.9. This contradicts $(*)$.

Case 2. $|L_a| = q^2$.

Consider the quotient loop L/L_a. $|L/L_a| = p_1p_2\cdots p_mq$. Since $p_1, p_2, \ldots, p_m$ and q are distinct primes, $(q, p_1p_2\cdots p_m) = 1$. Also $(q-1, qp_1p_2\cdots p_m) = 1$ as $q \not\equiv 1 \pmod{p_i}$ for all i. By Lemma 3.6(a), there exists a Sylow q-subloop in L/L_a. Since this subloop is cyclic, L/L_a contains an element of order q. Now compare with Lemma 3.12, we let $p=q$, $m = p_1p_2\cdots p_m$ and $\alpha=1$. Then L/L_a contains a normal subloop M/L_a of order $p_1p_2\cdots p_m$. Hence $|M| = p_1p_2\cdots p_mq^2$ and M is a maximal normal subloop of L.

Suppose $(k,w,\ell)=1$ for all $k\in L_a$, $w\in M$ and $\ell \in L$. Then Lemma 3.11(b) gives $L_a \subseteq N$. By Lemma 3.6(a), L contains a Sylow p_1-subloop in L. Hence, L has an element of order p_1. It is also clear that $(p_1, p_2\cdots p_mq^3) = (p_1-1, p_1p_2\cdots p_mq^3) = 1$. So by Lemma 3.12, there exists a normal subloop H_1 of order $p_2\cdots p_mq^3$ in L. By repeating the same process, we get a normal series $Q \trianglelefteq H_{m-1} \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq L$ where $|H_i| = p_{i+1}\cdots p_mq^3$ and $|Q| = q^3$. Since $Q \trianglelefteq H_{m-1} \trianglelefteq H_{m-2}$ and Q is a Hall subloop of H_{m-1}, it follows from Lemma 3.5 that $Q \trianglelefteq H_{m-2}$. By using Lemma 3.5 several times, we finally get a normal subloop Q of order q^3 in L.

Now by Lemma 3.3, L/Q is a group implies $L_a \subseteq Q$. Since $|L_a| = q^2$, there exists $x \in Q - L_a$ where $Q = \langle x\rangle L_a$. It is also clear that $(|Q|, |L/Q|) = 1$. Hence Lemma

3.17 gives $Q \subseteq N$. But Q is also a Hall subloop of L. Thus by Lemma 3.7(b), L is a group. This contradicts $(*)$.

Hence $(k, w, \ell) \neq 1$ for some fixed elements $k \in L_a$, $w \in M$ and $\ell \in L$. Then by Lemma 3.19, L_a contains a proper nontrivial subloop S which is normal in M. Clearly $|S| = q$. Thus $|M/S| = p_1 p_2 \cdots p_m q$. By using Lemma 3.12 and Lemma 3.5 repeatedly, we get a quotient loop K_m/S of order $p_m q$ normal in M/S. Hence, $|K_m| = p_m q^2$ and $K_m \trianglelefteq M$. Since $q \not\equiv 1 \pmod{p_m}$, by Lemma 3.12, $\exists\ \hat{P}/S \trianglelefteq K_m/S$ such that $|\hat{P}/S| = p_m$. Thus, $|\hat{P}| = p_m q$ where $\hat{P} \trianglelefteq K_m$. By the same argument as before, $\exists\ P \trianglelefteq \hat{P}$ such that $|P| = p_m$. Since $P \trianglelefteq \hat{P} \trianglelefteq K_m$ and P is a Hall subloop of $\hat{P}$, $P \trianglelefteq K_m$ by Lemma 3.5.

Note that K_m is also a normal Hall subloop of M, and hence $K_m \trianglelefteq L$ by Lemma 3.5. Thus L/K_m is a group and $L_a \subseteq K_m$ by Lemma 3.3. Therefore $K_m = L_a P$ and we have $P \trianglelefteq L_a P$. Now P is also a Hall subloop of L and $(|L_a|, |P|) = (q^2, p_m) = 1$. Then L is a group by Lemma 3.9. This again contradicts our first assumption, $(*)$.

Therefore, nevertheless, L is a group. ∎

Corollary 4.1. *Let $p_1, p_2, \ldots, p_n$ and q be distinct odd primes. All Moufang loops of order $p_1 p_2 \cdots p_n q^3$ are associative if and only if $q \not\equiv 1 \pmod{p_i}$ for each i.*

Proof. For $p_i > q$, it is clear that $q \not\equiv 1 \pmod{p_i}$; and if $p_i < q$, $q \not\equiv 1 \pmod{p_i}$ is a sufficient condition as assured by the main theorem. Suppose $q \equiv 1 \pmod{p_i}$ for some $i \in \{1, 2, \ldots, n\}$. Then by [16], there exists a nonassociative Moufang loop of order $p_i q^3$. Hence by using the direct product of this nonassociative Moufang loop with any group of order $(p_1 p_2 \cdots p_n)/p_i$, we get a nonassociative Moufang loop of order $p_1 p_2 \cdots p_n q^3$. Thus the condition $q \not\equiv 1 \pmod{p_i}$ for each i, is a necessary one as well. ∎

5. Open problems

Let p and q be odd primes. Are all Moufang loops of order $p^2 q^3$ and pq^4 associative if $p < q$ and $q \not\equiv 1 \pmod{p}$? The smallest unsolved case is for $p = 3$ and $q = 5$.

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