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Qualitative Behavior of Giving Up Smoking Models

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Abstract. Smoking is a large problem in the entire world. Despite overwhelming facts about the risks, smoking is still a bad habit widely spread and socially accepted. Many people start smoking during their gymnasium period. The main purpose of this paper is to determine the asymptotic behavior of a mathematical model using for giving up smoking. Our interest here is to derive and analysis the model taking into account the occasional smokers compartment in the giving up smoking model. Analysis of this model reveals that there are four equilibria, one of them is the smoking-free and the other three correspond to presence of smoking. We also present the global stability and parameter estimates that characterize the natural history of this disease with numerical simulations.

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1. Introduction

The smoking subject is an interesting area to study. There is strong medical evidence that smoking tobacco is related to more than two dozen diseases and conditions. It has negative effects on nearly every organ of the body and reduces overall health. Smoking tobacco remains the leading cause of preventable death and has negative health impacts on people of all ages in the world. The effects of smoking bring big problems both in personal and public matters. The gymnasium age, between 16 and 20 years old, is a time where many of our attitudes change; this includes the attitude towards smoking.

In order to understand the dynamics of this disease we use the concept of mathematical modeling. There is a lot of mathematical theory on the concept of diseases and epidemics, (see for example [1, 3, 6, 7, 14, 17–19]). The basic ideas in these theories are that all people in a community start as healthy. Healthy people may become infected by diseases. But infected people may become healthy again in a

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community. Epidemic models with both linear and nonlinear incidence have been studied by many authors and related literature of SIR disease transmission model is quite large, where S denotes the number of individuals that are susceptible to infection, I denotes the number of individuals that are infectious and R denotes the number of individuals that have been recovered. The chronic disease model (CDM) is a model that describes the effects of risk factors, including smoking and overweight, on the incidence and mortality of chronic diseases in the population [10]. In 2000, Castillo-Garsow *et al.* [5] proposed a simple mathematical model for giving up smoking. They consider a system with a total constant population which is divided into three classes: potential smokers, i.e. people who do not smoke yet but might become smokers in the future (P), smokers (S), and people (former smokers) who have quit smoking permanently (Q). Sharami and Gumel developed mathematical models by introducing mild and chain classes [16]. In their work they presented the development and public health impact of smoking related illnesses.

In this paper we develop a mathematical model for a giving up smoking and present its qualitative behavior. We assume that the birth rate is different from the death rate which insure that the total population is not constant. Our aim is to derive and analysis the model taking into account the occasional smokers compartment in the giving up smoking model. In the beginning, people start smoking occasionally for a variety of different reasons. Some think it looks cool. Others start because their family members or friends smoke. Statistics show that about 9 out of 10 tobacco users start occasionally and then gradually become chain smoker. Moreover, most adults who started smoking in their teens never expected to become addicted. That's why people say it's just so much easier to not start smoking at all. The model is developed from the previous work by [5] without occasional smokers compartment of total constant population size. First, we want to concentrate on a general epidemic model with bilinear incidence rate and use the stability analysis theory to find out the equilibria for the model. Analysis of this model reveals that there are four equilibria, one of them is the smoking-free and the other three correspond to presence of smoking. Then, we use the Lyapunov functional theory to establish the global stability of the giving up smoking model. Finally, we estimates the parameter that characterize the natural history of this disease and present numerical simulations.

The structure of this paper is organized as follows. In Section 2, the giving up smoking model is established under some assumption. In Section 3, the stability analysis of the proposed epidemic model is investigated. By these analysis we determine the equilibria of the system on the model. In Section 4, we present the parameter estimates and numerical simulations. Finally, we give conclusion.

2. Formulation of the model

There is a lot of mathematical theory on the concept of diseases and epidemics. The basic ideas in these theories are that all people in a community start as healthy. Healthy people may become infected (smoking). Some of the infected people may become healthy again after they quit smoking. To formulate our model, let $P(t)$ number of potential smokers at time t; $L(t)$ number of people in the population, that not smoke every day at time t, but sooner or later they become smoker say occasional smokers, $S(t)$ number of people in the population, that smoke every day at time t; $Q(t)$ number of former smokers in the population that have stopped smoking permanently at time t say quit smoker. The total population size at time t is denoted by $N(t)$ with $N(t) = P(t) + L(t) + S(t) + Q(t)$. The mathematical representation of the model consists of a system of non-linear differential equations with four state variables. The complete model is as follows:

$$(2.1)\qquad \begin{aligned} \frac{dP(t)}{dt} &= bN(t) - \beta_1 L(t)P(t) - (d_1+\mu)P(t),\\ \frac{dL(t)}{dt} &= \beta_1 L(t)P(t) - \beta_2 L(t)S(t) - (d_2+\mu)L(t),\\ \frac{dS(t)}{dt} &= \beta_2 L(t)S(t) - (\gamma + d_3+\mu)S(t),\\ \frac{dQ(t)}{dt} &= \gamma S(t) - (d_4+\mu)Q(t). \end{aligned}$$

Here b is the birth rate, μ is the natural death rate, γ is the recover rate from infection (smoking), β_1 and β_2 are transmission coefficients, d_1, d_2, d_3 and d_4 represent the death rate of potential smoker, occasional smoker, smoker and quit smoker related to smoking disease, respectively. We use the classic mass action hypothesis for both positive transmission coefficients β_1 and β_2. Potential smokers acquire infection at per capita rate $\beta_1 L(t)$. The dynamics of the total population are governed by the following differential equation

$$(2.2)\qquad \frac{dN}{dt} = (b-\mu)N(t) - (d_1P(t) + d_2L(t) + d_3S(t) + d_4Q(t)).$$

If the death rate $d_1 = d_2 = d_3 = d_4 = d$, then the total population become

$$(2.3)\qquad \frac{dN}{dt} = (b - d - \mu)N(t)$$

The total population remain constant if $b = d + \mu$ (see for example [4, 12, 20]). In this paper we consider that the total population is not constant, so we assume that $b < d + \mu$. In order to understand the qualitative behavior of the model, let us consider the set Ω and the initial conditions for the system (2.1) given by:

$$(2.4)\qquad \begin{aligned} &\Omega = \{(P, L, S, Q) | 0 \le P, 0 \le L, 0 \le S, 0 \le Q, (\beta_1/b)LP \le N\},\\ &P(0) = P_0 \ge 0, L(0) = L_0 \ge 0, S(0) = S_0 \ge 0, Q(0) = S_0 \ge 0. \end{aligned}$$

Definition 2.1. *Consider the following system*

$$(2.5)\qquad \frac{dx}{dt} = f(t, x(t))$$

and autonomous system

$$(2.6)\qquad \frac{dx}{dt} = g(x), g : D \subset R^n \to R^n$$

where f, g are locally Lipschitz in $x \in R^n$ and solution exist for all positive time. If $f(t, x) \to g(x)$ as $t \to \infty$ uniformly for $x \in D$ then system (2.5) *is said to be asymptotically autonomous with limit system of* (2.6).

Lemma 2.1. *Suppose f and g are locally Lipschitz in $x \in D$ [8, 11]. If any solutions of* (2.5) *are bounded and the equilibria E of* (2.6) *is globally asymptotically stable, then any solution $x(t)$ of system* (2.5) *such that*

$$\lim_{t\to+\infty} x(t) = E.$$

Theorem 2.1. *The solution of the system* (2.1) *with initial condition* (2.4) *is nonnegative for all $t > 0$.*

Proof. By using fundamental theorem of calculus we have

$$S(t) = S(0)e^{\int_0^t (\beta_2 L(s)-(\gamma+d_3+\mu))ds}.$$

Since $S(0) > 0$, so we get $S(t) > 0$ for all $t > 0$.
In order to show that $P(t) > 0$, we multiplying both side of first equation of the system (2.1) by $e^{(d_1+\mu)t}$

$$e^{(d_1+\mu)t}P'(t) = e^{(d_1+\mu)t}(bN(t) - \beta_1 L(t)P(t) - (d_1+\mu)P(t)),$$

that is

$$(e^{(d_1+\mu)t}P(t))' = e^{(d_1+\mu)t}(bN(t) - \beta_1 L(t)P(t)),$$

by taking integration from 0 to t we get

$$P(t) = P(0)e^{-(d_1+\mu)t} + \int_0^t e^{(d_1+\mu)(s-t)}(bN(s) - \beta_1 L(s)P(s))ds.$$

Since $(\beta_1/b)LP \leq N$ and $P(0) > 0$, so we obtain $P(t) > 0$ for all $t > 0$.

In order to show that $Q(t) > 0$, we suppose that $Q(t) < 0$, at some time t. Let $\Gamma_2 = \inf\{t; Q(t) = 0\}$. There exist $\epsilon > 0$ such that $0 < t_1 - \Gamma_2 < \epsilon$ and $Q(t_1) < 0$ obviously $dQ(\Gamma_2)/dt < 0$. But from fourth equation of the system (2.1) we obtain

$$\frac{dQ(\Gamma_2)}{dt} = \gamma S(\Gamma_2) \geq 0,$$

which is contradiction for any $t > 0$. Hence $Q(t_1) > 0$ and thus $Q(t) > 0$, for all time $t > 0$. Next we have to show that $L(t) > 0$ for all $t > 0$. Since the total population size $N(t)$ at time t is positive with $N(t) = P(t) + L(t) + S(t) + Q(t)$. Also by the previous stepping, it is obvious that $P(t) > 0$, $S(t) > 0$ and $Q(t) > 0$ for all time $t > 0$ so we get $L(t) = N(t) - (L(t) + S(t) + Q(t)) \geq 0$ for positive time t. Hence, we proved that the solution of all individuals are nonnegative for all $t > 0$. ∎

Lemma 2.2. *All feasible solution of the system* (2.1) *are bounded and enter the region*

$$\Omega_\epsilon = \{(P, L, S, Q) \in R_+ : P + L + S + Q = N \leq N_0 e^{-K}\},$$

where $K = \mu + d_m - b$, $d_m = \min\{d_1, d_2, d_3, d_4\}$, and $b < \mu + d_m$.

Proof. It is shown that all the dependent variable of the model are nonnegative for the positive parameters. Continuity of the right hand side of the system (2.1) and its derivative imply that the model is well-posed for $N(t) > 0$ for positive time t. The dynamics of the total population are governed by

$$\frac{dN}{dt} = (b - \mu)N(t) - (d_1 P(t) + d_2 L(t) + d_3 S(t) + d_4 Q(t))$$

$$\frac{dN}{dt} \leq (b-\mu)N(t) - d_m(P(t)+L(t)+S(t)+Q(t))$$
$$\frac{dN}{dt} \leq (b-\mu-d_m)N(t),$$
$$\frac{dN}{dt} \leq -KN(t).$$

Thus, we have $0 \leq N(t) \leq N(0)e^{-Kt}$ as $t \to \infty$. Therefore, all the feasible solution of the giving up smoking model (2.1) are bounded and enter the region Ω_ϵ. This completes the proof of Lemma 2.2. ∎

3. Qualitative analysis

In this section, we will analyze the qualitative behavior of the giving up smoking model (2.1) with initial conditions (2.4).

Theorem 3.1. *The giving up smoking model* (2.1) *has three steady states as follows:*

(i) *The trivial equilibrium exists, and is given by* $E_0 = (0,0,0,0)$.

(ii) *The smoking-free equilibrium point* $E_s = (N,0,0,0)$ *exists for all parameter values.*

(iii) *The unique positive epidemic equilibrium of system* (2.1)
$E_+ = (\beta_2 bN^\star/(\beta_1(d_3+\mu+\gamma)+\beta_2(d_1+\mu)), (d_3+\mu+\gamma)/\beta_2), ((d_2+\mu)/\beta_2)[R-1], (\gamma(d_2+\mu)/\beta_2(d_4+\mu))[R-1]),$
where $R = b\beta_1\beta_2N^\star(t)/((d_2+\mu)(\beta_1(d_3+\mu+\gamma)+\beta_2(d_1+\mu)))$
and $N^\star = P^\star + L^\star + S^\star + Q^\star$.

Proof. The proof of these steady state conditions as follows:

(i) This is an obvious extreme condition, which is not very interesting at least from the biological point of view.

(ii) In the system (2.1) total population is $P(t)+L(t)+S(t)+Q(t) = N(t)$, hence (ii) is satisfied.

(iii) For equilibria $P(t) = P^\star(t)$, $L(t) = L^\star(t)$, $S(t) = S^\star(t)$ and $Q(t) = Q^\star(t)$ we set $dP(t)/dt = 0$ and $dS(t)/dt = 0$ so from third equation of the system (2.1) we obtain $L^\star(t) = (d_3+\mu+\gamma)/\beta_2$. Substituting $L^\star(t)$ in first equation of the system (2.1), we get $P^\star(t) = \beta_2 bN^\star/(\beta_1(d_3+\mu+\gamma)+\beta_2(d_1+\mu))$. Now we set $dL(t)/dt = 0$ and using value of $P^\star(t)$ to obtain $S^\star(t) = ((d_2+\mu)/\beta_2)[R-1]$, where $R = b\beta_1\beta_2N^\star(t)/((d_2+\mu)(\beta_1(d_3+\mu+\gamma)+\beta_2(d_1+\mu)))$ represents the smoking generation number (basic reproductive number). It measure the average number of new smokers generated by single smoker in a population of potential smokers. Similarly by equating zero fourth equation of the system (2.1), we have $Q^\star(t) = (\gamma(d_2+\mu)/\beta_2(d_4+\mu))[R-1]$. By using the smoking generation number $R = b\beta_1\beta_2N^\star(t)/((d_2+\mu)(\beta_1(d_3+\mu+\gamma)+\beta_2(d_1+\mu)))$ and rearranging, we obtain

$$\begin{aligned} P^\star(t) &= \beta_2 bN^\star/(\beta_1(d_3+\mu+\gamma)+\beta_2(d_1+\mu)),\\ L^\star(t) &= (d_3+\mu+\gamma)/\beta_2,\\ S^\star(t) &= ((d_2+\mu)/\beta_2)[R-1],\\ Q^\star(t) &= (\gamma(d_2+\mu)/\beta_2(d_4+\mu))[R-1], \end{aligned} \tag{3.1}$$

which is the unique positive epidemic equilibrium of the system (2.1). ∎

Remark 3.1. The Jacobian matrix around the trivial equilibrium $E_0 = (0,0,0,0)$

$$J_0 = \begin{bmatrix} -d_1-\mu & 0 & 0 & 0 \\ 0 & -\mu-d_2 & & \\ 0 & 0 & -d_3-\gamma-\mu & 0 \\ 0 & 0 & \gamma & -d_4-\mu \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix around the trivial equilibrium $E_0 = (0,0,0,0)$ are $-d_1-\mu$, $-d_2-\mu$, $-\gamma-d_3-\mu$, $-d_4-\mu$. Thus all the roots have negative real part which show that the trivial equilibrium is locally stable.

Theorem 3.2. *The giving up smoking model* (2.1) *has* $E_1 = (1,0,0,0)$ *as a locally stable smoking free equilibrium if and only if* $\beta_1 < d_2+\mu$. *Otherwise* E_1*is an unstable smoking free equilibrium.*

Proof. The local stability of this equilibrium solution can be examined by linearizing the giving up smoking model (2.1) around $E_1 = (1,0,0,0)$. This equilibrium point gives us the Jacobian matrix:

$$J_1 = \begin{bmatrix} -d_1-\mu & -\beta_1 & 0 & 0 \\ 0 & \beta_1-\mu-d_2 & & \\ 0 & 0 & -d_3-\gamma-\mu & 0 \\ 0 & 0 & \gamma & -d_4-\mu \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix J_1 around smoking free equilibrium $E_1 = (1,0,0,0)$ are $-d_1-\mu$, $\beta_1-d_2-\mu$, $-\gamma-d_3-\mu$, $-d_4-\mu$. Thus, we deduce that all the roots have negative real part when $\beta_1 < d_2+\mu$ which shows that the smoking free equilibrium is locally asymptotically stable. ∎

Theorem 3.3. *The equilibrium* $E_2 = (P^{**}, L^{**}, 0, 0)$ *in the giving up smoking model* (2.1) *as a locally stable if and only if* $b\beta_1 N^{**} > (d_1+\mu)(d_2+\mu)$, *where* $P^{**} = \frac{d_2+\mu}{\beta_1}$, $L^{**} = \frac{bN^{**}}{d_2+\mu} - \frac{d_1+\mu}{\beta_1}$, *where* $N^{**} = P^{**} + L^{**} + S^{**} + Q^{**}$.

Proof. The equilibrium point in the absence of smoker and quite smoker individuals is $(\frac{d_2+\mu}{\beta_1}, \frac{bN^{**}}{d_2+\mu} - \frac{d_1+\mu}{\beta_1})$. The Jacobian matrix of the giving up smoking model (2.1) around $E_2 = (P^{**}, L^{**}, 0, 0)$ is given by

$$J_2 = \begin{bmatrix} -\frac{b\beta_1 N^{**}}{d_2+\mu} & -(d_2+\mu) \\ \frac{b\beta_1 N^{**}}{d_2+\mu} - (d_1+\mu) & 0 \end{bmatrix}.$$

The roots of the characteristic polynomial $\psi(x) = x^2 + c_1 x + c_2$, where $c_1 = \frac{b\beta_1 N^{**}}{d_2+\mu}$ and $c_2 = b\beta_1 N^{**} - (d_1+\mu)(d_2+\mu)$. Therefore by Routh-Hurwits criteria we deduce that the roots of the polynomial $\psi(x)$ have negative real part when $b\beta_1 N^{**} > (d_1+\mu)(d_2+\mu)$, which shows that the system is asymptotically stable. ∎

In order to know stability of the system (2.1) we consider all positive parameters to construct the Jacobian matrix is given by
$J=$

$$\begin{bmatrix} -\beta_1 L(t) - (d_1+\mu) & -\beta_1 P(t) & 0 & 0 \\ -\beta_1 L(t) & \beta_1 P(t) - \beta_2 S(t) - (d_2+\mu) & -\beta_2 L(t) & 0 \\ 0 & \beta_2 S(t) & \beta_2 L(t) - (\gamma + d_3 + \mu) & 0 \\ 0 & 0 & \gamma & -(d_4+\mu) \end{bmatrix}.$$

We impose the restriction on the equilibrium points; $P_\infty(t) > 0$, $L_\infty(t) > 0$, $S_\infty(t) > 0$, $Q_\infty(t) > 0$ [4] and seek for parameter values to know the qualitative behavior of the system (2.1). The eigenvalues of the community matrix help to understand the stability of the system. Thus the characteristic equation becomes

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{3.2}$$

where the coefficients a_i, for $i = 1, 2, 3, 4$ are given by

$$\begin{aligned}
a_1 &= (\beta_1 - \beta_2)L(t) - \beta_1 P(t) + \beta_2 S(t) + (d_1 + d_2 + d_3 + d_4 + 4\mu + \gamma),\\
a_2 &= (\beta_1 L(t) + d_1 + \mu)(\beta_2(S(t) - L(t)) - \beta_1 P(t) + d_2 + d_3 + 2\mu + \gamma) + \beta_1\beta_2 P(t)L(t)\\
&\quad + (\beta_1(L(t) - P(t)) + \beta_2(S(t) - L(t)) + d_1 + d_2 + d_3 + d_4 + 3\mu + \gamma)(d_4 + \mu)\\
&\quad + (d_3 + \mu + \gamma)(d_2 + \mu - \beta_1 P(t) + \beta_2 S(t)) - \beta_2 L(t)(\beta_2 S(t) + d_2 + \mu)\\
&\quad + \beta_1^2 P(t)L(t) + \beta_2^2 L(t)S(t),\\
a_3 &= (\beta_2 L(t)(d_4 + \mu) - (d_4 + \mu)(d_3 + \mu + \gamma))(\beta_1(P(t) - L(t)) - (d_1 + d_2 + 2\mu)\\
&\quad - \beta_2 S(t)) + (\beta_2 L(t) - (d_3 + d_4 + 2\mu + \gamma))((d_1 + \mu)(\beta_1 P(t) - \beta_2 S(t))\\
&\quad - \beta_1^2 P(t)L(t) - \beta_1 L(t)(\beta_2 S(t) - \beta_1 P(t)) - (d_2 + \mu)(\beta_1 L(t) + d_2 + \mu))\\
&\quad + \beta_2^2 L(t)S(t)(\beta_1 L(t) + d_1 + d_4 + \mu),\\
a_4 &= \beta_1\beta_2^2(d_4 + \mu)L^2(t)S(t) + \beta_2^2(d_1 + \mu)(d_4 + \mu)L(t)S(t) + ((d_3 + \mu + \gamma)(d_4 + \mu)\\
&\quad - \beta_2(d_2 + \mu)L(t))(\beta_1 L(t)(\beta_2 S(t) - 2\beta_1 P(t)) + (d_1 + \mu)(\beta_2 S(t) - \beta_1 P(t))\\
&\quad + (d_2 + \mu)(\beta_1 L(t) + d_1 + \mu)).
\end{aligned}$$

The equilibrium state is asymptotically stable by Routh-Hurwitz criteria, if $a_1 > 0$, $a_3 > 0$, $a_4 > 0$ and $a_1a_2a_3 > a_3^2 + a_1^2 a_4$ [13]. These conditions ensure that all of the four eigenvalues found from (3.2) have negative real parts. We note that a_i depend on the value of N, β_1, β_2, γ, μ, d_1, d_2, d_3 and d_4. All the individual and parameter are nonnegative for $t \geq 0$.

Example 3.1. We check for different value of β_1 and β_2 by considering $P = 153$, $L = 43$, $S = 78$ and $Q = 68$ with parameters values presented in Table 1 to know that the system is stable (or unstable).

(a) For $\beta_1 > \beta_2$ i.e. $\beta_1 = 0.84$, $\beta_2 = 0.45$, we obtained $a_1 = -76.1766$, $a_3 = 36.6659$, $a_4 = 641.2329$, which show that the equilibrium state is unstable by Routh-Hurwitz criteria.

(b) For small value of β_1 and β_2 i.e. $\beta_1 = \beta_2 = 0.00034$, we obtained $a_1 = 0.0209$, $a_3 = 4.4940 \times 10^{-7}$, $a_4 = 1.6542 \times 10^{-11}$ which show that the equilibrium state is asymptotically stable by Routh-Hurwitz criteria.

(c) $\beta_1 < \beta_2$ i.e. $\beta_1 = 0.2$, $\beta_2 = 0.8$, we obtained $a_1 = 6.8234$, $a_3 = 9.6900$, $a_4 = 95.3020$ which show that the equilibrium state is asymptotically stable by Routh-Hurwitz criteria.

Theorem 3.4. *The unique positive epidemic equilibrium $E_+ = (P^\star, L^\star, S^\star, Q^\star)$ of the giving up smoking model* (2.1) *is globally asymptotically stable if $\mu > b$.*

Proof. Let us consider the Lyapunov functional [11] along the path of the system (2.1)

$$V(P(t), L(t), S(t), Q(t)) = \frac{1}{2}(P(t) + L(t) + S(t) + Q(t))^2 + \frac{1}{2}w_1 P(t)^2 + \frac{1}{2}w_2 S(t)^2 + \frac{1}{b-\mu}(d_1 P^\star(t) + d_2 L^\star(t) + d_3 S^\star(t) + d_4 Q^\star(t)),$$

where w_1 and w_2 are some positive constant chosen later. Note that

$$V(P(t), L(t), S(t), Q(t)) \geq \frac{1}{2}(P(t) + L(t) + S(t) + Q(t))^2 + \frac{1}{2}w_1 P(t)^2 + \frac{1}{2}w_2 S(t)^2.$$

The time derivative of $V(P(t), L(t), S(t), Q(t))$ along the solution of (2.1) and (2.2)

$$\begin{aligned} V'(P(t), L(t), S(t), Q(t)) &= N(t)N(t)' + w_1 P(t)P(t)' + w_2 S(t)S(t)', \\ &= N(t)((b-\mu)N(t) - (d_1 P(t) + d_2 L(t) + d_3 S(t) + d_4 Q(t))) \\ &\quad + w_1 P(t)(bN(t) - \beta_1 L(t)P(t) - (d_1 + \mu)P(t)) \\ &\quad + w_2 S(t)(\beta_2 L(t)S(t) - (\gamma + d_3 + \mu)S(t)), \end{aligned}$$

where $'$ denotes the derivative with respect to time.

$$\begin{aligned} V'(P(t), L(t), S(t), Q(t)) &= -(\mu - b)N^2(t) - (d_1 - w_1 b)P(t)N(t) - d_4 Q(t)N(t) \\ &\quad - d_2 L(t)N(t) - \beta_1 w_1 L(t)P^2(t) - w_1(d_1 + \mu)P^2(t) \\ &\quad - w_2(\gamma + d_3 + \mu)S^2(t) - (d_3 N(t) - w_2 \beta_2 S(t))S(t). \end{aligned}$$

Choose $w_1 > 0$ and $w_2 > 0$ such that $w_1 = \frac{d_1}{b}$ and $w_2 = \frac{d_3}{\beta_2}$, and rewriting with some little rearrangement, we get

$$\begin{aligned} V'(P(t), L(t), S(t), Q(t)) = -\{&(\mu - b)N^2(t) + \beta_1 w_1 L(t)P^2(t) + (N(t) - S(t))d_3 S(t) \\ &+ d_2 L(t)N(t) + (d_1 + \mu)\frac{d_1}{b}P^2(t) + (\gamma + d_3 + \mu)\frac{d_3}{\beta_2}S^2(t) \\ &+ d_4 Q(t)N(t)\}. \end{aligned}$$

Since all the parameters are nonnegative with the total population $N(t) \geq S(t)$, and by Theorem 2.1, we obtain that the Lyapunov functional $V'(P(t), L(t), S(t), Q(t)) < 0$, if $\mu > b$. Also, $V'(P(t), L(t), S(t), Q(t)) = 0$, if $N(t) = 0$. Therefore, by the Lasall's Invariance Principle [8] every solution of the giving up smoking model (2.1) and with the initial conditions approach to E_s as $t \to \infty$. Thus the unique positive epidemic equilibrium $E_+ = (P^*, L^*, S^*, Q^*)$ of the giving up smoking model (2.1) is globally asymptotically stable. This completes the proof. ∎

4. Parameter estimation and numerical simulation

4.1. Parameter estimation

If we find reasonable values for the parameter, then we can conclude that the model can be used to represent the dynamics of tobacco use in real life. When a person first becomes a smoker it is not likely that she/he quits for several years since tobacco contains nicotine, which is shown to be an addictive drug [2]. We assume $1/\gamma$ to be a value between 15 and 25 years. So we consider the mean value of 15 and 25, i.e. 20. Hence, the value of $1/\gamma$ is set to 7300 days. $1/b$ is an average time for a person in the system. There are strong evidence that the attitude towards smoking is starting from high (junior) school time. Therefore $1/b$ set to be 1095 days. Death from lung cancer was the leading cause of smoking, with a rate of 37 per 100,000 individuals [15]. The death rate of each individual is different from each others and depend on the real life situation. Here, we assume that $d_4 \leq d_1 \leq d_2 \leq d_3$. The natural death rate μ is per 1000 per year (currently eight in the U.S.). It is a bit harder to find a realistic value for the parameter β_1 and β_2. No statistics has been found for the subject and as a result another way was taken to solve the problem. An estimate of the β_2 value can be obtain from the unique positive epidemic equilibrium of the system (2.1) is $\beta_2 = (d_3+\mu+\gamma)/L^\star$. To investigate the parameter β_1 value we substitute the value of β_2 in the unique positive epidemic equilibrium of the system (2.1) and use the same techniques use in [20].

$$
\begin{aligned}
P^\star(t) &= \beta_2 bN/(\beta_{11}(d_3+\mu+\gamma)+\beta_2(d_1+\mu)),\\
S^\star(t) &= ((d_2+\mu)/\beta_2)[R_1^*-1],\\
Q^\star(t) &= (\gamma(d_2+\mu)/\beta_2(d_4+\mu))[R_2^*-1],
\end{aligned}
\tag{4.1}
$$

where $R_1^* = b\beta_{12}\beta_2 N^\star(t)/((d_2+\mu)(\beta_{12}(d_3+\mu+\gamma)+\beta_2(d_1+\mu)))$ and $R_2^* = b\beta_{13}\beta_2 N^\star(t)/((d_2+\mu)(\beta_{13}(d_3+\mu+\gamma)+\beta_2(d_1+\mu)))$. First we solve (4.1) and then consider $\beta_1 = (\beta_{11}+\beta_{12}+\beta_{13})/3$ to obtain reasonable value. In our numerical simulation we use different value of positive transmission coefficients β_1 and β_2 and present the impact of smoking.

After we have shown the stability of the model at the equilibrium point theoretically, we will verify this result by doing some simulations. We carried out numerical simulations using MATLAB to illustrate dynamics of the system. We consider the real data used in [9, 20] for the three individuals $P(0) = 153$, $S(0) = 79$, and $Q(0) = 68$, and assume that $S(0) \geq L(0)$. For numerical simulations of the system (2.1) we use a set of parameter values represents in Table 1.

4.2. Numerical simulation

Figure 1 shows that the number of potential smoker sharply decreases during the first five days. The occasional smoker represents in Figure 2 increases sharply in the first five days and then rapidly decreases. From the potential class the potential smoker moved to the occasional smoker class at transmission rate β_1. Similarly from occasional class, the occasional smoker interact with smoker and moved to the smoker class at transmission rate β_2. Therefore, the decrease in occasional smoker individuals represented in Figure 2 bring an increase in the smoker individuals. In the first few days there are about 177 occasional smokers while at the same time there are about 84 smokers, but at the end of simulation there are about 3 occasional smokers while at the same time there about 184 smokers. In Figure 3 the smoker population increases slightly as compare to occasional smoker, while the population of quit smoker decreases represents in Figure 4. The graph of quit smoker also shows that the quitting rate of smoking is decreasing slowly than the rate of potential smoker during the first few days. In this work we use one set of parameter for the equilibria

Table 1. Parameters used for numerical simulation

Parameter	Description	Value
b	Birth rate	0.00091
μ	Natural death rate	0.0031
γ	Recovery rate	0.0031
β_1	Infection rate of smoking	0.0380
β_2	Infection rate of smoking	0.45
d_1	Disease death rate of P	0.0019
d_2	Disease death rate of L	0.00021
d_3	Disease death rate of S	0.0037
d_4	Disease death rate of Q	0.0012

of the dynamical system (2.1). Simulations with different sets of parameter values can be used in the future to obtain a sampling of possible behaviors of a dynamical system.

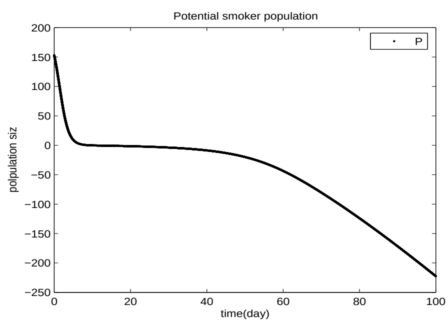

Figure 1. The plot represents the potential smoker individuals in the giving up smoking model.

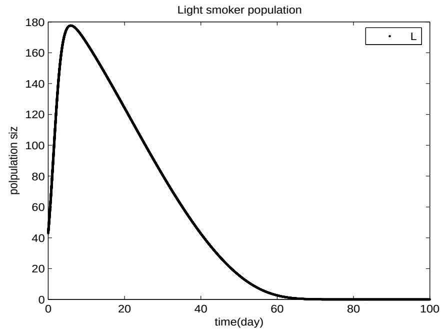

Figure 2. The plot shows the light smoker individuals in the giving up smoking model.

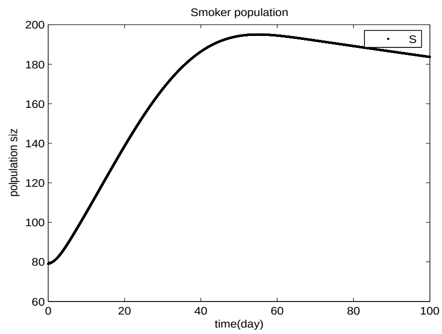

Figure 3. The plot represents the smoker individuals in the giving up smoking model.

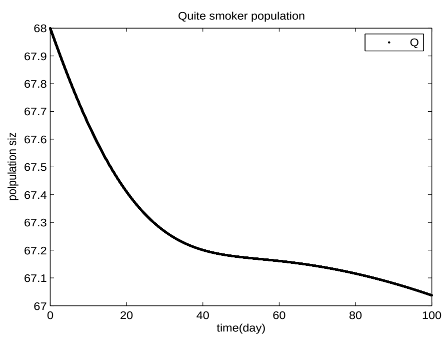

Figure 4. The plot shows the quit smoker individuals in the giving up smoking model.

5. Conclusion

We presented a non-linear model which describes the overall smoking population dynamics when the population is not assumed to remain constant and which incorporates both the natural death rate and the death rate from the disease. We derived the model taking into account the occasional smokers compartment in the giving up smoking model with bilinear incidence rate. Analysis of this model revealed that there are four equilibria, one of them is the disease-free (without smoking) and the other three correspond to presence of smoking. The stability analysis was adopted in this work to formulate the dynamics of the giving up smoking model. The previous work dealing with these models usually provide results on the qualitative properties for solutions. However, in this work we introduced the stability analysis theory in the nonlinear system and developed mathematical epidemic models which represent both the local and global behavior of smoking dynamics. Finally, we estimated the parameter that characterize the natural history of this disease and present numerical simulations. In fact, we believe that the approach introduced in this paper will be applicable in other epidemic models beyond the giving up smoking model.

One future work is the introduction of control programs in the giving up smoking model to see how this would affect the evolution of the spread of smoking and make numerical simulations to analyze the optimal parameter values for the control programs.

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