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A Parametric Family of Quartic Thue Inequalities

BERNADIN IBRAHIMPAŠIĆ
Pedagogical Faculty, University of Bihać, Džanića mahala 36,
77000 Bihać, Bosnia and Herzegovina
bernadin@bih.net.ba

Abstract. In this paper we prove that the only primitive solution of the Thue inequality

$$\left|x^4 - 2cx^3y + 2x^2y^2 + 2cxy^3 + y^4\right| \leq 6c + 4,$$

where $c \geq 5$ is an integer, are $(x, y) = (\pm 1, 0), (0, \pm 1), (1, \pm 1), (-1, \pm 1)$.

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1. Introduction

Let $f \in \mathbb{Z}[X, Y]$ be a homogeneous irreducible polynomial of degree $n \geq 3$ and $\mu \neq 0$ fixed integer. Then the Diophantine equation

$$f(x, y) = \mu \tag{1.1}$$

is called Thue equation in honour of A. Thue. In 1909, Thue [21] proved that equation (1.1) has only finitely many solutions $x, y \in \mathbb{Z}$. His proof was not effective. In 1968, Baker [1] gave an upper bound for the solutions of Thue equation, based on the theory of linear forms in logarithms of algebraic numbers. Since then, algorithms for the solution of single Thue equations have been developed (see [5, 19, 23]).

Starting with Thomas [20] in 1990, parametrized families of Thue equations have been considered (see [12, 13] for references).

In this paper, we consider the family of Thue inequalities

$$\left|x^4 - 2cx^3y + 2x^2y^2 + 2cxy^3 + y^4\right| \leq 6c + 4\,. \tag{1.2}$$

We will apply the method of Tzanakis introduced in [22] and used in [7, 8, 9, 11, 15, 25]. The application of Tzanakis method for solving Thue equations of the special type has several adventages (see [22, 8, 9]). We transform the problem of solving of Thue equation to solving the system of simultaneous pellian equations.

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The theory of continued fractions can be used in order to determine values of μ for which the equation $f(x,y)=\mu$ has a solution. We will use characterization in terms of continued fractions of α of all fractions a/b satisfying the inequality

$$\left|\alpha-\frac{a}{b}\right|<\frac{k}{b^2}, \tag{1.3}$$

where k is some positive integer. We will find the sets of all values of μ for which the equation (1.4) or the equation (1.5) has a solution. Comparing these sets we find the set of all values of μ for which the system (1.4) and (1.5) has a solution.

From the comparison of a lower bound for solutions of this system, obtained using the congruence method introduced in [10], and an upper bound obtained from a theorem of Bennett [4] on simultaneous approximations of algebraic numbers, we obtained results for $c \geq 53776$. For $c \leq 53775$ we use a theorem of Baker and Wüstholz [3] and a version of the reduction procedure due to Baker and Davenport [2].

Our main result is the following theorem.

Theorem 1.1. *Let $c \geq 3$ be an integer. The only primitive solutions to Thue inequality*

$$\left|x^4-2cx^3y+2x^2y^2+2cxy^3+y^4\right| \leq 6c+4$$

are

$$\begin{array}{lll} \text{(i)} & (x,y) = & (\pm1,0),(0,\pm1),(1,\pm1),(-1,\pm1), \quad c\geq 5, \\ \text{(ii)} & (x,y) = & (\pm1,0),(0,\pm1),(1,\pm1),(-1,\pm1), \\ & & (2,1),(-2,-1),(1,-2),(-1,2), \quad c=4, \\ \text{(iii)} & (x,y) = & (\pm1,0),(0,\pm1),(1,\pm1),(-1,\pm1), \\ & & (2,1),(-2,-1),(1,-2),(-1,2), \\ & & (3,2),(-3,-2),(2,-3),(-2,3), \quad c=3. \end{array}$$

Note that Thue inequality $\left|x^4-2c^2x^3y+2x^2y^2+2c^2xy^3+y^4\right| \leq \frac{c}{2}$, where $c \geq 1$ was completely solved in [16]. It was shown that all primitive solutions of this inequality are given by $(x,y)=(0,\pm1),(\pm1,0),(1,\pm1),(-1,\pm1)$. The result due to Dujella and Ibrahimpašić [6] allows us to have a rather large right side in inequality (1.2) compared to the result in [16] cited above.

Let $f(x,y)=x^4-2cx^3y+2x^2y^2+2cxy^3+y^4$. Note that, because $f(x,y)$ is homogeneous, it is enough to consider only primitive solutions of (1.2), i.e. those with $\gcd(x,y)=1$. Furthermore, since $f(a,b)=f(-a,-b)=f(b,-a)=f(-b,a)$, it suffices to find only all nonnegative solutions of (1.2).

It is trivial to check that for $c=0$ and $c=1$ all nonnegative solutions of (1.2) are $(1,0),(0,1)$ and $(1,1)$, where $f(1,0)=f(0,1)=1$ and $f(1,1)=4$.

For $c=2$ we have $f(x,y)=\left(x^2-2xy-y^2\right)^2$. In this case inequality (1.2) has infinitely many primitive solutions corresponding to the equations $f(x,y)=1$ and $f(x,y)=4$. In the first case, all nonnegative solutions are given by $(x,y)=(a_{n+1},a_n)$, with $a_{n+2}=2a_{n+1}+a_n,\ \ a_1=1,\ a_2=0$. In the second case, all nonnegative solutions are given by $(x,y)=(b_{n+1},b_n)$, with $b_{n+2}=2b_{n+1}+b_n,\ \ b_1=1,\ b_2=1$.

From now on, we assume that $c \geq 3$.

Solving the Thue equation $f(x,y) = \mu$, where $|\mu| \leq 6c+4$, by the method of Tzanakis (for more details see [8, 9]) reduces to solving the system of pellian equations with one common unknown

$$cV^2 - (c+2)U^2 = -2\mu \tag{1.4}$$

$$cZ^2 - (c-2)U^2 = 2\mu, \tag{1.5}$$

where

$$U = x^2 + y^2, \quad V = x^2 + 2xy - y^2, \quad Z = -x^2 + 2xy + y^2.$$

Since $\gcd(x,y) = 1$, we have $\gcd(U,V) = \gcd(U,Z) = \gcd(U,V,Z) = 1$ or $\gcd(U,V) = \gcd(U,Z) = \gcd(U,V,Z) = 2$.

2. Continued fractions

In this section, we will consider the connections between solutions of the equations (1.4) and (1.5) and continued fraction expansion of the corresponding quadratic irrationals.

Dujella and Ibrahimpašić [14, 6, Propositions 2.1 and 2.2] proved several results on connection between the continued fractions and rational approximations of the form $|\alpha - a/b| < k/b^2$ for a positive integer k. They extended Worley's work [24] and gave explicit and sharp versions of [24, Theorems 1 and 2] for $k = 3, 4, 5, \ldots, 12$. They gave the pairs (r,s) which appear in the expression of solutions to (1.3) in the form $(a,b) = (rp_{m+1} \pm sp_m, rq_{m+1} \pm sq_m)$, where p_m/q_m denotes the mth convergent of the continued fraction expansion of α. Recently, Ibrahimpašić [14] extended this result to $0 \leq k \leq 13$.

Worley [24, Corollary] gave the explicit version of his result for $k = 2$. He showed, if a real number α and a rational number $\frac{a}{b}$ satisfy the inequality $\left|\alpha - \frac{a}{b}\right| < \frac{2}{b^2}$, then $\frac{a}{b} = \frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}$, where

$$(r,s) \in R_2 = \{(0,1),(1,1),(1,2),(2,1),(3,1)\},$$

or $\frac{a}{b} = \frac{sp_{m+2}-tp_{m+1}}{sq_{m+2}-tq_{m+1}}$, where

$$(s,t) \in T_2 = \{(1,1),(1,2),(1,3),(2,1)\}$$

(for an integer $m \geq -1$).

Proposition 2.1. *Let $k \in \{3,4,5,6,7,8,9,10,11,12,13\}$. If a real number α and a rational number $\frac{a}{b}$ satisfy the inequality (1.3), then $\frac{a}{b} = \frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}$, where $(r,s) \in R_k = R_{k-1} \cup R'_k$, or $\frac{a}{b} = \frac{sp_{m+2}-tp_{m+1}}{sq_{m+2}-tq_{m+1}}$, where $(s,t) \in T_k = T_{k-1} \cup T'_k$ (for an integer $m \geq -1$), where the sets R'_k and T'_k are given in the following table. Moreover, if any of the elements in sets R_k or T_k is omitted, the statement will no longer be valid.*

Table 1. Sets R'_k and T'_k

k	R'_k	T'_k
3	$\{(1,3),(4,1),(5,1)\}$	$\{(3,1),(1,4),(1,5)\}$
4	$\{(1,4),(3,2),(6,1),(7,1)\}$	$\{(4,1),(2,3),(1,6),(1,7)\}$
5	$\{(1,5),(2,3),(8,1),(9,1)\}$	$\{(5,1),(3,2),(1,8),(1,9)\}$
6	$\{(1,6),(5,2),(10,1),(11,1)\}$	$\{(6,1),(2,5),(1,10),(1,11)\}$
7	$\{(1,7),(2,5),(4,3),(12,1),(13,1)\}$	$\{(7,1),(5,2),(3,4),(1,12),(1,13)\}$
8	$\{(1,8),(3,4),(7,2),(14,1),(15,1)\}$	$\{(8,1),(4,3),(2,7),(1,14),(1,15)\}$
9	$\{(1,9),(5,3),(16,1),(17,1)\}$	$\{(9,1),(3,5),(1,16),(1,17)\}$
10	$\{(1,10),(9,2),(18,1),(19,1)\}$	$\{(10,1),(2,9),(1,18),(1,19)\}$
11	$\{(1,11),(2,7),(3,5),(20,1),(21,1)\}$	$\{(11,1),(7,2),(5,3),(1,20),(1,21)\}$
12	$\{(1,12),(5,4),(7,3),$ $(11,2),(22,1),(23,1)\}$	$\{(12,1),(4,5),(3,7),$ $(2,11),(1,22),(1,23)\}$
13	$\{(1,13),(3,7),(4,5),(24,1),(25,1)\}$	$\{(13,1),(7,3),(5,4),(1,24),(1,25)\}$

The simple continued fraction expansion of a quadratic irrational $\alpha = \frac{e+\sqrt{d}}{f}$ is periodic. This expansion can be obtained using the following algorithm [17, Chapter 7.7]. Multiplying the numerator and the denominator by f, if necessary, we may assume that $f|(d-e^2)$. Let $s_0 = e$, $t_0 = f$ and

$$a_n = \left\lfloor \frac{s_n+\sqrt{d}}{t_n} \right\rfloor, \quad s_{n+1} = a_n t_n - s_n, \quad t_{n+1} = \frac{d-s_{n+1}^2}{t_n} \quad \text{for } n \geq 0. \tag{2.1}$$

If $(s_j,t_j) = (s_k,t_k)$ for $j < k$, then

$$\alpha = [a_0;\ldots,a_{j-1},\overline{a_j,\ldots,a_{k-1}}].$$

Applying this algorithm to $\sqrt{\frac{c+2}{c}}$ and $\sqrt{\frac{c-2}{c}}$ we find that

$$\sqrt{\frac{c+2}{c}} = \left[1;\overline{c,2}\right] \quad \text{and} \quad \sqrt{\frac{c-2}{c}} = \left[0;1,\overline{c-2,2}\right].$$

Note, if $U = 0$ or $V = 0$ or $Z = 0$, then inequality (1.2) has no solution. Assume that (U,V,Z) is a positive solution of the system (1.4) and (1.5). Then $\frac{V}{U}$ is a good rational approximation of $\sqrt{\frac{c+2}{c}}$, and $\frac{Z}{U}$ is a good rational approximation of $\sqrt{\frac{c-2}{c}}$. If we assume that $\mu < 0$, then from (1.4) we have

$$\begin{aligned}\left|\sqrt{\frac{c+2}{c}} - \frac{V}{U}\right| &= \left|\frac{c+2}{c} - \frac{V^2}{U^2}\right| \cdot \left|\sqrt{\frac{c+2}{c}} + \frac{V}{U}\right|^{-1} \\ &< \frac{\left|(c+2)\,U^2 - cV^2\right|}{cU^2} \cdot \frac{1}{2} \cdot \sqrt{\frac{c}{c+2}} \\ &= \frac{2\,|\mu|}{cU^2} \cdot \frac{1}{2} \cdot \sqrt{\frac{c}{c+2}} \leq \frac{6c+4}{\sqrt{c\,(c+2)}}\,U^{-2} \leq \frac{6}{U^2},\ c \geq 1. \end{aligned} \tag{2.2}$$

If we assume that $\mu > 0$, then for $U \geq 12$ we have

$$\left|\sqrt{\frac{c+2}{c}} - \frac{V}{U}\right| < \frac{\left|(c+2)\,U^2 - cV^2\right|}{cU^2} \cdot \frac{22}{43} \cdot \sqrt{\frac{c}{c+2}}$$

$$\leq \frac{44\,(6c+4)}{43\sqrt{c\,(c+2)}}\,U^{-2} \leq \begin{cases} \frac{6}{U^2} &, \quad c=1,2,\dots,12 \\ \frac{7}{U^2} &, \quad c \geq 13\,. \end{cases} \tag{2.3}$$

Now, we consider equation (1.5). If we assume that $\mu > 0$, we have

$$\left|\sqrt{\frac{c-2}{c}} - \frac{Z}{U}\right| < \frac{\left|(c-2)\,U^2 - cZ^2\right|}{cU^2} \cdot \frac{1}{2} \cdot \sqrt{\frac{c}{c-2}} = \frac{\mu}{U^2\sqrt{c\,(c-2)}}$$

$$\leq \frac{6c+4}{\sqrt{c\,(c-2)}}\,U^{-2} \leq \begin{cases} \frac{13}{U^2} &, \quad c=3 \\ \frac{10}{U^2} &, \quad c=4 \\ \frac{9}{U^2} &, \quad c=5,6 \\ \frac{8}{U^2} &, \quad c=7,8,9,10,11 \\ \frac{7}{U^2} &, \quad c \geq 12\,. \end{cases} \tag{2.4}$$

If we assume that $\mu < 0$, then for $U \geq 23$ we have

$$\left|\sqrt{\frac{c-2}{c}} - \frac{Z}{U}\right| < \frac{\left|(c-2)\,U^2 - cZ^2\right|}{cU^2} \cdot \frac{22}{43} \cdot \sqrt{\frac{c}{c-2}}$$

$$\leq \frac{44\,(6c+4)}{43\sqrt{c\,(c-2)}}\,U^{-2} \leq \begin{cases} \frac{13}{U^2} &, \quad c=3 \\ \frac{11}{U^2} &, \quad c=4 \\ \frac{9}{U^2} &, \quad c=5,6 \\ \frac{8}{U^2} &, \quad c=7,8,\dots,13 \\ \frac{7}{U^2} &, \quad c \geq 14\,. \end{cases} \tag{2.5}$$

In the case $U \leq 22$, from (1.5) we obtain $Z \leq 22$, and from (1.4) we obtain $V \leq 28$. By checking all possibilities, we obtain that, for $U \leq 22$, all solutions of the system (1.4) and (1.5) are:

$$(U,V,Z,\mu) = (1,1,1,1)\,,(2,2,2,4)\,,$$

for all $c \geq 3$, and additionally for $c = 3,4$ we find the following solutions:

$$(c,U,V,Z,\mu) = (3,5,7,1,-11)\,,(3,13,17,7,-11)\,,(4,5,7,1,-23)\,.$$

According to our results (Proposition 2.1 for corresponding k), applied to $\alpha = \sqrt{\frac{c+2}{c}}$ and $\alpha = \sqrt{\frac{c-2}{c}}$, all solutions of (1.4) have the form $V/U = (rp_{m+1} + up_m)\,/\,(rq_{m+1} + uq_m)$ for an index $m \geq -1$ and integers r and s, where p_m/q_m is the mth convergent of the continued fraction expansion of $\sqrt{\frac{c+2}{c}}$, and all solutions of (1.5) have the form $Z/U = (rp_{m+1} + up_m)\,/\,(rq_{m+1} + uq_m)$ for an index $m \geq -1$ and integers r and s, where p_m/q_m is the mth convergent of the continued fraction expansion of $\sqrt{\frac{c-2}{c}}$. For the determination of the corresponding μ's, we use the following result (see [9, Lemma 1]).

Lemma 2.1. *Let $\alpha\beta$ be a positive integer which is not a perfect square, and let p_n/q_n denotes the nth convergent of the continued fraction expansion of $\sqrt{\frac{\alpha}{\beta}}$. Let the sequences (s_n) and (t_n) be defined by (2.1) for the quadratic irrational $\frac{\sqrt{\alpha\beta}}{\beta}$. Then*

$$\alpha\,(rq_{n+1} + uq_n)^2 - \beta\,(rp_{n+1} + up_n)^2 = (-1)^n\left(u^2t_{n+1} + 2rus_{n+2} - r^2t_{n+2}\right).$$

Since the period of the continued fraction expansion of $\sqrt{\frac{c+2}{c}}$ is equal to 2, according to Lemma 2.1, we have to consider only the fractions $\frac{rp_{n+1}+up_n}{rq_{n+1}+uq_n}$ for $n=0$ and $n=1$. Since $\gcd(U,V)=1$ or $\gcd(U,V)=2$, we are checking all possibilities using pairs $(r,u)\in\{(r,s),(2r,2s),(s,-t),(2s,-2t)\}$, where $(r,s)\in R_k$ and $(s,t)\in T_k$, for corresponding k (according to (2.2) and (2.3)). By checking all possibilities, we obtain the following result.

Proposition 2.2. *Let μ be an integer such that $|\mu|\le 6c+4$, and that equation*

$$cV^2-(c+2)\,U^2=-2\mu$$

has solution in integers U and V such that $\gcd(U,V)=1$ *or* $\gcd(U,V)=2$. *Let*

$M_1^1=\{1,4,-2c,2c+4,-4c+1,4c+9,-6c+4\}$

$M_2^1=\left\{-\frac{c}{2},\frac{c}{2}+1,-\frac{3c}{2}+1,\frac{3c}{2}+4,-\frac{5c}{2}+4,\frac{5c}{2}+9,-\frac{7c}{2}+9,-\frac{9c}{2}+16,-\frac{11c}{2}+25\right\}$

$M_3^1=\left\{-\frac{13c}{2}+36,-\frac{15c}{2}+49,-\frac{17c}{2}+64,-\frac{19c}{2}+81,-\frac{21c}{2}+100,-\frac{23c}{2}+121\right\}$

$M_4^1=M_1^1\cup M_2^1$.

We have:

(i) *c odd:*

$c=3,\quad \mu\in M_1^1\cup\{-8c+9,-10c+16,-12c+25\}$

$c=5,\quad \mu\in M_1^1\cup\{-8c+9,-10c+16\}$

$c\ge 7,\quad \mu\in M_1^1$

(ii) *c even:*

$c=4,\quad \mu\in M_4^1\cup\left\{-\frac{13c}{2}+36,-\frac{15c}{2}+49,-8c+9,-10c+16,-12c+25\right\}$

$c=6,\quad \mu\in M_4^1\cup\left\{-\frac{13c}{2}+36,-\frac{15c}{2}+49,-\frac{17c}{2}+64,-\frac{19c}{2}+81,\right.$
$\left.-\frac{21c}{2}+100,\frac{7c}{2}+16,-8c+9\right\}$

$c=8,10,12,\quad \mu\in M_4^1\cup M_3^1\cup\left\{\frac{7c}{2}+16\right\}$

$14\le c\le 22,\quad \mu\in M_4^1\cup M_3^1\cup\left\{\frac{7c}{2}+16,\frac{9c}{2}+25\right\}$

$c=24,\quad \mu\in M_4^1\cup\left\{-\frac{13c}{2}+36,-\frac{15c}{2}+49,-\frac{17c}{2}+64,-\frac{19c}{2}+81,\frac{7c}{2}+16,\frac{9c}{2}+25\right\}$

$c=26,\quad \mu\in M_4^1\cup\left\{-\frac{13c}{2}+36,-\frac{15c}{2}+49,\frac{7c}{2}+16,\frac{9c}{2}+25\right\}$

$28\le c\le 34,\quad \mu\in M_4^1\cup\left\{-\frac{13c}{2}+36,-\frac{15c}{2}+49\right\}$

$36\le c\le 62,\quad \mu\in M_4^1\cup\left\{-\frac{13c}{2}+36,\frac{7c}{2}+16,\frac{9c}{2}+25\right\}$

$64\le c\le 80,\quad \mu\in M_4^1\cup\left\{-\frac{13c}{2}+36,\frac{7c}{2}+16,\frac{9c}{2}+25,\frac{11c}{2}+36\right\}$

$c \geq 82, \quad \mu \in M_4^1 \cup \left\{ \frac{7c}{2} + 16, \frac{9c}{2} + 25, \frac{11c}{2} + 36 \right\}.$

Now, we consider equation (1.5). Since the period of the continued fraction expansion of $\sqrt{\frac{c-2}{c}}$ is equal to 2, according to Lemma 2.1, we have to consider only the fractions $\frac{rp_{n+1}+up_n}{rq_{n+1}+uq_n}$ for $n = 0$ and $n = 1$. Since $\gcd(U, Z) = 1$ or $\gcd(U, V) = 2$, we check all possibilities using pairs $(r, u) \in \{(r, s), (2r, 2s), (s, -t), (2s, -2t)\}$, where $(r, s) \in R_k$ and $(s, t) \in T_k$, for corresponding k (according to (2.4) and (2.5)). By checking all possibilities, we obtain the following result.

Proposition 2.3. *Let μ be an integer such that $|\mu| \leq 6c + 4$, and that equation*

$$cZ^2 - (c-2)U^2 = 2\mu$$

has solution in integers U and Z such that $\gcd(U, Z) = 1$ or $\gcd(U, Z) = 2$. Let

$M_1^2 = \{1, 4, 2c, -2c+4, 4c+1, -4c+9, 6c+4, -6c+16\}$

$M_2^2 = \left\{ \frac{c}{2}, -\frac{c}{2} + 1, \frac{3c}{2} + 1, -\frac{3c}{2} + 4, \frac{5c}{2} + 4, -\frac{5c}{2} + 9, \frac{7c}{2} + 9, -\frac{7c}{2} + 16, -\frac{9c}{2} + 25, -\frac{11c}{2} + 36 \right\}$

$M_3^2 = \left\{ -\frac{15c}{2} + 64, -\frac{17c}{2} + 81, -\frac{19c}{2} + 100, -\frac{21c}{2} + 121, -\frac{23c}{2} + 144 \right\}$

$M_4^2 = \{-8c+25, -10c+36, -12c+49, -14c+64\}$

$M_5^2 = M_1^2 \cup M_2^2.$

We have:

(i) *c odd:*

$c = 3, \quad \mu \in M_1^2 \cup \{-8c+25, -10c+36, -12c+25, -16c+36\}$

$c = 5, 9, \quad \mu \in M_1^2 \cup \{-8c+25, -10c+36\}$

$c = 7, \quad \mu \in M_1^2 \cup \{-8c+25, -10c+36, -12c+25\}$

$c = 11, 13, \quad \mu \in M_1^2 \cup \{-8c+25\}$

$c \geq 15, \quad \mu \in M_1^2$

(ii) *c even:*

$c = 4, \quad \mu \in M_5^2 \cup \left\{ -\frac{13c}{2} + 49, -\frac{21c}{2} + 25, -\frac{15c}{2} + 16, -12c+25, -16c+36 \right\}$

$c = 6, \quad \mu \in M_5^2 \cup \left\{ -\frac{17c}{2} + 81, -\frac{15c}{2} + 64, -\frac{13c}{2} + 49, -\frac{19c}{2} + 81, -\frac{21c}{2} + 25, -\frac{15c}{2} + 16 \right\}$

$c = 8, \quad \mu \in M_5^2 \cup M_3^2 \cup M_4^2 \cup \left\{ -\frac{15c}{2} + 16, \frac{9c}{2} + 16 \right\}$

$c = 10, \quad \mu \in M_5^2 \cup M_3^2 \cup \left\{ -\frac{25c}{2} + 169, -\frac{15c}{2} + 16, \frac{9c}{2} + 16, -8c+25, -10c+36 \right\}$

$$c=12,\quad \mu\in M_5^2\cup M_3^2\cup\left\{-\tfrac{25c}{2}+169,-\tfrac{15c}{2}+16,\tfrac{9c}{2}+16,-8c+25\right\}$$

$$c=14,\quad \mu\in M_5^2\cup M_3^2\cup\left\{-\tfrac{25c}{2}+169,\tfrac{9c}{2}+16,-8c+25\right\}$$

$$16\le c\le 26,\quad \mu\in M_5^2\cup M_3^2\cup\left\{-\tfrac{25c}{2}+169,\tfrac{9c}{2}+25\right\}$$

$$c=28,\quad \mu\in M_5^2\cup\left\{-\tfrac{19c}{2}+100,-\tfrac{17c}{2}+81,-\tfrac{15c}{2}+64,-\tfrac{13c}{2}+49,\tfrac{9c}{2}+16\right\}$$

$$30\le c\le 34,\quad \mu\in M_5^2\cup\left\{-\tfrac{17c}{2}+81,-\tfrac{15c}{2}+64,-\tfrac{13c}{2}+49,\tfrac{9c}{2}+16\right\}$$

$$36\le c\le 40,\quad \mu\in M_5^2\cup\left\{-\tfrac{15c}{2}+64,-\tfrac{13c}{2}+49,\tfrac{9c}{2}+16\right\}$$

$$c=42,44,\quad \mu\in M_5^2\cup\left\{-\tfrac{15c}{2}+64,-\tfrac{13c}{2}+49,\tfrac{9c}{2}+16,\tfrac{11c}{2}+25\right\}$$

$$46\le c\le 106,\quad \mu\in M_5^2\cup\left\{-\tfrac{13c}{2}+49,\tfrac{9c}{2}+16,\tfrac{11c}{2}+25\right\}$$

$$c\ge 108,\quad \mu\in M_5^2\cup\left\{\tfrac{9c}{2}+16,\tfrac{11c}{2}+25\right\}.$$

Comparing the obtained results, we have proved the following result.

Proposition 2.4. *Let μ be an integer such that $|\mu|\le 6c+4$. If the system* (1.4) *and* (1.5) *has solution U,V and Z such that* $\gcd(U,V)=\gcd(U,Z)=\gcd(U,V,Z)=1$ *or* $\gcd(U,V)=\gcd(U,Z)=\gcd(U,V,Z)=2$ *then $\mu=1$ or 4, for all integers $c\ge 3$, and $\mu=-12c+25$ for $c=3$ and 4, namely $\mu=-11$ for $c=3$ and $\mu=-23$ for $c=4$.*

Furthermore, with convention $(p_{-1},q_{-1})=(1,0)$, all solutions of the equation (1.4)*, where* $\gcd(U,V)=1$ *or* 2*, are given by:*

$$\begin{aligned}
(U,V)&=(q_{2n},p_{2n})\quad or\quad (2q_{2n+1}+q_{2n},2p_{2n+1}+p_{2n})\quad or\\
&\qquad (q_{2n+2}-2q_{2n+1},p_{2n+2}-2p_{2n+1})\quad if\quad \mu=1,\\
(U,V)&=(2q_{2n},2p_{2n})\quad or\quad (4q_{2n+1}+2q_{2n},4p_{2n+1}+2p_{2n})\quad or\\
&\qquad (2q_{2n+2}-4q_{2n+1},2p_{2n+2}-4p_{2n+1})\quad if\quad \mu=4,\\
(U,V)&=(5q_{2n}+2q_{2n-1},5p_{2n}+2p_{2n-1})\quad or\\
&\qquad (2q_{2n+1}-5q_{2n},2p_{2n+1}-5p_{2n})\quad if\quad \mu=-12c+25,
\end{aligned}$$

where $\frac{p_n}{q_n}$, $n\ge 0$, denotes the nth convergent of the continued fraction expansion of $\sqrt{\frac{c+2}{c}}$, and all solutions of the equation (1.5)*, where* $\gcd(U,Z)=1$ *or* 2*, are given by:*

$$\begin{aligned}
(U,Z)&=(q_{2n+1},p_{2n+1})\quad or\quad (2q_{2n+2}+q_{2n+1},2p_{2n+2}+p_{2n+1})\quad or\\
&\qquad (q_{2n+3}-2q_{2n+2},p_{2n+3}-2p_{2n+2})\quad if\quad \mu=1,\\
(U,Z)&=(2q_{2n+1},2p_{2n+1})\quad or\quad (4q_{2n+2}+2q_{2n+1},4p_{2n+2}+2p_{2n+1})\quad or\\
&\qquad (2q_{2n+3}-4q_{2n+2},2p_{2n+3}-4p_{2n+2})\quad if\quad \mu=4,\\
(U,Z)&=(q_{2n+1}+4q_{2n},p_{2n+1}+4p_{2n})\quad or
\end{aligned}$$

$$(4q_{2n+2} - q_{2n+1}, 4p_{2n+2} - p_{2n+1}) \quad if \quad \mu = -12c + 25,$$

where $\frac{p_n}{q_n}$, $n \geq 0$, *denotes the nth convergent of the continued fraction expansion of* $\sqrt{\frac{c-2}{c}}$.

In this way, solving Thue inequality (1.2) reduces to solving following Thue equations:

$$x^4 - 2cx^3y + 2x^2y^2 + 2cxy^3 + y^4 = 1 \tag{2.6}$$

and

$$x^4 - 2cx^3y + 2x^2y^2 + 2cxy^3 + y^4 = 4, \tag{2.7}$$

for $c \geq 3$, and Thue equation

$$x^4 - 2cx^3y + 2x^2y^2 + 2cxy^3 + y^4 = -12c + 25,$$

for $c = 3$ and $c = 4$.

3. Case $\mu = 1$

In this section we consider equation (2.6). Our main result, in this section, is the following theorem.

Theorem 3.1. *Let* $c \geq 3$ *be an integer. The only nonnegative solutions of (2.6) are* $(x, y) = (1, 0)$ *and* $(0, 1)$.

Since $\mu = 1$, the system (1.4) and (1.5) has the form:

$$cV^2 - (c+2)\,U^2 = -2 \tag{3.1}$$

$$cZ^2 - (c-2)\,U^2 = 2. \tag{3.2}$$

We will use following lemma [8, Lemma 1].

Lemma 3.1. *Let* $k \geq 2$ *be an integer. If* x *and* y *are positive integers satisfying the pellian equation*

$$(k-1)\,y^2 - (k+1)\,x^2 = -2\,,$$

then there exists integer $m \geq 0$ *such that* $x = x_m$ *and* $y = y_m$, *where the sequences* (x_m) *and* (y_m) *are given by*

$$\begin{array}{llllll} x_0 = 1\,, & x_1 = 2k-1\,, & x_{m+2} = 2kx_{m+1} - x_m & & \\ y_0 = 1\,, & y_1 = 2k+1\,, & y_{m+2} = 2ky_{m+1} - y_m\,, & m \geq 0. \end{array}$$

Lemma 3.1 implies:

Lemma 3.2. *Let* (U, V, Z) *be a positive integer solution to the system of pellian equations* (3.1) *and* (3.2). *Then there exist nonnegative integers* m *and* n *such that*

$$U = v_m = w_n,$$

where the sequences (v_m) *and* (w_n) *are given by*

$$v_0 = 1\,, \quad v_1 = 2c+1\,, \quad v_{m+2} = (2c+2)\,v_{m+1} - v_m\,, \quad m \geq 0 \tag{3.3}$$

$$w_0 = 1\,, \quad w_1 = 2c-1\,, \quad w_{n+2} = (2c-2)\,w_{n+1} - w_n\,, \quad n \geq 0. \tag{3.4}$$

Proof. The statement follows directly by applying Lemma 3.1 on (3.1) with $V \longleftrightarrow y, U \longleftrightarrow x, c \longleftrightarrow k-1, k \longleftrightarrow c+1$, and on (3.2), with $U \longleftrightarrow y, Z \longleftrightarrow x, k+1 \longleftrightarrow c, k \longleftrightarrow c-1$. ∎

In order to prove Theorem 3.1, it suffices to show that $v_m = w_n$ implies $m = n = 0$. Solving recurrences (3.3) and (3.4) we have

$$v_m = \frac{1}{2\sqrt{c+2}}\left[\left(\sqrt{c}+\sqrt{c+2}\right)\left(c+1+\sqrt{c\,(c+2)}\right)^m - \left(\sqrt{c}-\sqrt{c+2}\right)\left(c+1-\sqrt{c\,(c+2)}\right)^m\right], \tag{3.5}$$

$$w_n = \frac{1}{2\sqrt{c-2}}\left[\left(\sqrt{c}+\sqrt{c-2}\right)\left(c-1+\sqrt{c\,(c-2)}\right)^n - \left(\sqrt{c}-\sqrt{c-2}\right)\left(c-1-\sqrt{c\,(c-2)}\right)^n\right]. \tag{3.6}$$

The following lemma can be proved easily by induction.

Lemma 3.3. *Let* (v_m) *and* (w_n) *be defined by* (3.5) *and* (3.6)*. Then for all* $m, n \geq 0$ *we have*

$$v_m \equiv m\,(m+1)\,c+1 \pmod{4c^2},$$
$$w_n \equiv (-1)^{n+1}\left[n\,(n+1)\,c-1\right] \pmod{4c^2}.$$

If we assume that m and n are positive integers such that $v_m = w_n$, then we have $v_m \equiv w_n \pmod{4c^2}$. By Lemma 3.3, we have $1 \equiv (-1)^n \pmod{2c}$, and we obtain that n is even.

Suppose that $n\,(n+1) < 2c$. From (3.3) and (3.4) we have that $v_m \geq w_m$, therefore it follows that for $v_m = w_n$ we have $m \leq n$. Now we obtain $m\,(m+1) < 2c$. Lemma 3.3 implies

$$m\,(m+1)\,c+1 \equiv -n\,(n+1)\,c+1 \pmod{4c^2}$$

and

$$m\,(m+1) \equiv -n\,(n+1) \pmod{4c}. \tag{3.7}$$

Let $A = m\,(m+1)+n\,(n+1)$. Then we have $0 < A < 4c$, and by (3.7) we obtain $A \equiv 0 \pmod{4c}$, a contradiction. Now we conclude that $n\,(n+1) \geq 2c$ and we have $n > \sqrt{2c} - 0.5$. Therefore we proved:

Proposition 3.1. *If* $v_m = w_n$ *and* $m \neq 0$*, then* $n > \sqrt{2c} - 0.5$.

From (2.3) and (2.4) it follows that the solutions of the system (3.1) and (3.2) induce a good rational approximations to the numbers $\sqrt{\frac{c+2}{c}}$ and $\sqrt{\frac{c-2}{c}}$. More precisely, we have:

Lemma 3.4. *All positive integer solutions of the system of pellian equations* (3.1) *are* (3.2) *satisfy*

$$\left|\sqrt{\frac{c+2}{c}} - \frac{V}{U}\right| < \frac{1}{c}\,U^{-2} \tag{3.8}$$

(3.9) $$\left|\sqrt{\frac{c-2}{c}}-\frac{Z}{U}\right|<\frac{1}{\sqrt{c\,(c-2)}}\,U^{-2}\ .$$

Proof. The proof of (3.9) follows immediately from (2.4) with $\mu = 1$. If we assume that $U > V$, then from (3.1) we have

$$2=(c+2)\,U^2-cV^2>(c+2)\,V^2-cV^2=2V^2\,,$$

which implies that $V < 1$, a contradiction. Now we have $V \geq U$ and

$$\sqrt{\frac{c+2}{c}}+\frac{V}{U}>2\,,$$

and we obtain

$$\left|\sqrt{\frac{c+2}{c}}-\frac{V}{U}\right|<\frac{1}{c}\,U^{-2}.$$ ∎

We will find an upper bound for solutions using the following theorem of Bennett [4, Theorem 3.2].

Theorem 3.2. *If a_i, p_i, q and N are integers for $0 \leq i \leq 2$, with $a_0 < a_1 < a_2$ and $a_j = 0$ for some $0 \leq j \leq 2$, $q \geq 1$ and $N > M^9$, where*

$$M=\max_{0\leq i\leq 2}\{|a_i|\}\geq 3,$$

then we have

$$\max_{0\leq i\leq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\}>(130N\Upsilon)^{-1}\,q^{-\lambda},$$

where

$$\lambda=1+\frac{\log\left(32.04N\Upsilon\right)}{\log\left(1.68N^2\prod_{0\leq i<j\leq 2}(a_i-a_j)^{-2}\right)}$$

and

$$\Upsilon=\begin{cases}\frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1}\ , & a_2-a_1\geq a_1-a_0\\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0}\ , & a_2-a_1<a_1-a_0\ .\end{cases}$$

Remark 3.1. From the proof of Theorem 3.2 [4, Theorem 3.2] it is easy to check that statement of Theorem 3.2 is valid also for $M = 2$, if $N \geq 2699$.

We may apply Theorem 3.2 for $a_0 = -2, a_1 = 0, a_2 = 2, N = c, M = 2$, $p_0 = Z, p_1 = U, p_2 = V$ and $q = U$. Since $a_2 - a_1 = a_1 - a_0 = 2$, we have $\Upsilon = \frac{32}{3}$. Since $M = 2$, then from Remark 3.1, we apply Theorem 3.2 with $c \geq 2699$. We have

(3.10) $$\lambda=1+\frac{\log\left(341.76c\right)}{\log\left(\frac{1.68c^2}{256}\right)}\ ,$$

and for $c \geq 2699$ we obtain

$$\left(130\cdot c\cdot\frac{32}{3}\right)^{-1}U^{-\lambda}<\frac{1}{\sqrt{c\,(c-2)}}\,U^{-2}$$

which implies

$$U^{2-\lambda}<1388\,.$$

If $c \geq 52078$ then $2-\lambda > 0$, and we have

$$\log U < \frac{7.236}{2-\lambda} \ . \tag{3.11}$$

Therefore, from (3.10) we obtain

$$\frac{1}{2-\lambda} = \frac{1}{1-\frac{\log(341.76c)}{\log\left(\frac{1.68c^2}{256}\right)}} < \frac{\log\left(0.0065625c^2\right)}{\log\left(0.0000192c\right)} \ . \tag{3.12}$$

From (3.6) we obtain $w_n > (2c-3)^n$. Proposition 3.1 implies, that if $(m,n) \neq (0,0)$ then $U > (2c-3)^{\sqrt{2c}-0.5}$, and we have

$$\log U > \left(\sqrt{2c}-0.5\right)\log\left(2c-3\right) \ . \tag{3.13}$$

From (3.11), (3.12) and (3.13) we have

$$\sqrt{2c}-0.5 < \frac{7.236\log\left(0.0065625c^2\right)}{\log\left(2c-3\right)\cdot\log\left(0.0000192c\right)} \ ,$$

which yields a contradiction for $c \geq 53776$. Now we have proved:

Proposition 3.2. *If $c \geq 53776$, then $(m,n) = (0,0)$ is the only solution of equation $v_m = w_n$.*

Now we will apply a version of the reduction procedure due to Baker and Davenport [2] in order to prove Theorem 3.1 for $3 \leq c \leq 53775$.

Lemma 3.5. *If $v_m = w_n$ and $m \neq 0$, then*

$$0 < n\log\left(c-1+\sqrt{c\left(c-2\right)}\right) - m\log\left(c+1+\sqrt{c\left(c+2\right)}\right)$$
$$+\log\frac{\sqrt{c+2}\left(\sqrt{c}+\sqrt{c-2}\right)}{\sqrt{c-2}\left(\sqrt{c}+\sqrt{c+2}\right)} < 0.8794\left(c+1+\sqrt{c\left(c+2\right)}\right)^{-2m} \ .$$

Proof. See the proof of [8, Lemma 5] or [9, Lemma 5]. ∎

Now we will apply the following theorem of Baker and Wüstholz [3] to the linear form from Lemma 3.5.

Theorem 3.3. *For a linear form $\Lambda = b_1\log\alpha_1 + \cdots + b_l\log\alpha_l \neq 0$ in logarithms of l algebraic numbers $\alpha_1,\ldots,\alpha_l$ with rational integer coefficients $b_1,\ldots,b_l$ we have*

$$\log\Lambda \geq -18\left(l+1\right)!l^{l+1}\left(32d\right)^{l+2}h'\left(\alpha_1\right)\cdots h'\left(\alpha_l\right)\log\left(2ld\right)\log B,$$

where $B = \max\{|b_j| : 1 \leq j \leq l\}$, and where d is the degree of the number field generated by $\alpha_1,\ldots,\alpha_l$.

Here

$$h'\left(\alpha\right) = \frac{1}{d}\max\left\{h\left(\alpha\right),\left|\log\alpha\right|,1\right\} \ ,$$

and $h\left(\alpha\right)$ denotes the standard logarithmic Weil height of α.

We have

$$\Lambda = n\log\left(c-1+\sqrt{c\left(c-2\right)}\right) - m\log\left(c+1+\sqrt{c\left(c+2\right)}\right) +$$

$$+\log\frac{\sqrt{c+2}\left(\sqrt{c}+\sqrt{c-2}\right)}{\sqrt{c-2}\left(\sqrt{c}+\sqrt{c+2}\right)}\ ,$$

and $l=3, d=4, B=n$,

$$\alpha_1=c-1+\sqrt{c\,(c-2)},\quad \alpha_2=c+1+\sqrt{c\,(c+2)},$$

$$\alpha_3=\frac{\sqrt{c+2}\left(\sqrt{c-2}+\sqrt{c}\right)}{\sqrt{c-2}\left(\sqrt{c+2}+\sqrt{c}\right)}\ .$$

The minimal polynomials for α_1, α_2 and α_3 are:

$$x^2-(2c-2)\,x+1=0,$$

$$x^2-(2c+2)\,x+1=0,$$

$$(c-2)^2\,x^4-2\,(c-2)\left(c^2-4\right)x^3-6\left(c^2-4\right)x^2+2\,(c+2)\left(c^2-4\right)x+(c+2)^2=0\,,$$

respectively. The conjugates are:

$$\alpha_1'=c-1-\sqrt{c\,(c-2)},\quad \alpha_2'=c+1-\sqrt{c\,(c+2)},$$

$$\alpha_3'=\frac{\sqrt{c+2}\left(\sqrt{c-2}-\sqrt{c}\right)}{\sqrt{c-2}\left(\sqrt{c+2}-\sqrt{c}\right)},\quad \alpha_3''=\frac{\sqrt{c+2}\left(\sqrt{c-2}+\sqrt{c}\right)}{\sqrt{c-2}\left(\sqrt{c+2}-\sqrt{c}\right)},$$

$$\alpha_3'''=\frac{\sqrt{c+2}\left(\sqrt{c-2}-\sqrt{c}\right)}{\sqrt{c-2}\left(\sqrt{c+2}+\sqrt{c}\right)}\ .$$

Under the assumption that $3\le c\le 53775$, we find that $h'(\alpha_1)<\frac{1}{2}\log(2c)$, $h'(\alpha_2)<5.793$ and $h'(\alpha_3)<\frac{1}{4}\cdot\log\left[(c-2)^2\cdot 1.540\cdot 3.248\cdot 107554.001\cdot 1\right]<8.746$.

By Lemma 3.5 we have

$$\log\Lambda<\log\left[0.8794\cdot\left(c+1+\sqrt{c\,(c+2)}\right)^{-2m}\right]<-2m\log(2c)\ , \tag{3.14}$$

and Theorem 3.3 implies

(3.15)

$$\log\Lambda\ge-18\cdot(3+1)!\cdot 3^{3+1}\cdot(32\cdot 4)^{3+2}\cdot h'(\alpha_1)\cdot h'(\alpha_2)\cdot h'(\alpha_3)\cdot\log(2\cdot 3\cdot 4)\cdot\log(n)\ .$$

From (3.14) and (3.15) we have

$$2m\log(2c)<18\cdot 4!\cdot 3^4\cdot 128^5\cdot\frac{1}{2}\log(2c)\cdot 5.793\cdot 8.746\cdot\log(24)\cdot\log(n)\,,$$

and finally we obtain

$$\frac{m}{\log n}<4.840\cdot 10^{16}\,. \tag{3.16}$$

From Lemma 3.5 we obtain

$$\frac{n}{m}<1.578\,. \tag{3.17}$$

Now, (3.16) and (3.17) imply $\frac{n}{\log n}<7.638\cdot 10^{16}$, and finally we obtain

$$n<3.255\cdot 10^{18}\,.$$

This large upper bound can be reduced using the following lemma, which is slight modification of [10, Lemma 5a].

Lemma 3.6. *Assume that M is a positive integer. Let $\frac{p}{q}$ be a convergent of the continued fraction expresion of κ such that $q > 10M$, and let $\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < n - m\kappa + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log\left(\frac{Aq}{\varepsilon}\right)}{\log B} \leq m \leq M\,.$$

We will apply this lemma, for $3 \leq c \leq 53775$, with

$$\kappa = \frac{\log \alpha_2}{\log \alpha_1}, \quad \mu = \frac{\log \alpha_3}{\log \alpha_1}, \quad A = \frac{0.8794}{\log \alpha_1}\,,$$

$$B = \left(c + 1 + \sqrt{c\,(c+2)}\right)^2, \quad M = 3.255 \cdot 10^{18}\,.$$

If the first convergent such that $q > 10M$ does not satisfy the condition $\varepsilon > 0$, then we use next convergent.

We find that the first convergent satisfy the condition $\varepsilon > 0$ in 51690 (96.13%) cases, the second in 2036 (3.78%) cases, the third in 41 (0.08%) cases and the forth convergent satisfy the condition $\varepsilon > 0$ in 6 (0.01%) cases ($c = 1518, 3487, 15695, 20727, 38466,\ 45613$). In all cases we obtained $m \leq 11$. More precisely, we obtained $m \leq 11$ in 1 case, for $c = 3$; $m \leq 10$ in 1 case, for $c = 4$; $m \leq 9$ in 2 cases, for $c = 5, 6$; $m \leq 8$ in 2 cases, for $c = 7, 8$; $m \leq 7$ in 4 cases, for $c = 9, 10, 11, 12$; $m \leq 6$ in 11 cases (minimal $c = 13$ and maximal $c = 37$); $m \leq 5$ in 25 cases (minimal $c = 21$ and maximal $c = 57$); $m \leq 4$ in 105 cases (minimal $c = 43$ and maximal $c = 217$); $m \leq 3$ in 896 cases (minimal $c = 126$ and maximal $c = 5505$); $m \leq 2$ in 38597 cases (minimal $c = 730$ and maximal $c = 53744$), and $m \leq 1$ in 14129 cases (minimal $c = 24972$ and maximal $c = 53775$).

Since, $n < 1.578m$, then $n \leq 17$, and from Proposition 3.1 we obtain $c \leq 153$. Hence, $(m, n) = (0, 0)$ is the only solution of the equation $v_m = w_n$ for $c \geq 154$.

We apply again Lemma 3.6 for $3 \leq c \leq 153$ with previously defined κ, μ, A and B, but with $M = 11$. We obtain $m \leq 1$ for $c = 3, 4, 5, 6, 7, 8, 9, 10,\ 11, 14, 22, 32, 44$ and 48, and $m \leq 0$ in all other cases.

Let $c = 3$. By Lemma 3.6 and (3.17) we have $m, n \leq 1$. Now $2c+1, 2c-1, 1$ are the values taken by v_m and w_n for $m, n \leq 1$, but these values are obviously distinct for $c \geq 3$. Therefore we have $m = n = 0$. Hence, we proved:

Proposition 3.3. *If c is an integer such that $3 \leq c \leq 53775$, then $(m, n) = (0, 0)$ is the only solution of the equation $v_m = w_n$.*

Proof of Theorem 3.1. The statement follows directly from Proposition 3.2 and Proposition 3.3. ∎

Finally, we conclude from the fact $U = x^2 + y^2 = 1$ that all solutions of equation (2.6) are $(x, y) = (1, 0), (-1, 0), (0, 1), (0, -1)$.

4. Case $\mu = 4$

In this section we consider equation (2.7), and our main result is the following theorem.

Theorem 4.1. *Let $c \geq 3$ be an integer. The only solutions to* (2.7) *are* $(x, y) = (\pm 1, \pm 1)$.

Proof. Since $\mu = 4$, the system (1.4) and (1.5) have the form:

$$cV^2 - (c+2)\,U^2 = -8$$
$$cZ^2 - (c-2)\,U^2 = 8.$$

Since Proposition 2.4 for $m = 4$, implies that U, V and Z are even, we can use substitution $U_1 = \frac{U}{2}$, $V_1 = \frac{V}{2}$ $Z_1 = \frac{Z}{2}$. Now, we have the system

$$cV_1^2 - (c+2)\,U_1^2 = -2$$
$$cZ_1^2 - (c-2)\,U_1^2 = 2.$$

From Section 3 we obtain $U_1 = 1$. It implies that $U = 2$ and we conclude that all solutions of equation (2.7) are $(x, y) = (1,1), (1,-1), (-1,1), (-1,-1)$. ∎

5. Cases $(c, \mu) = (3, -11)$ and $(4, -23)$

In the case $c = 3$ and $\mu = -11$, we have equation

$$x^4 - 6x^3y + 2x^2y^2 + 6xy^3 + y^4 = -11, \tag{5.1}$$

and the systems

$$3V^2 - 5U^2 = 22 \tag{5.2}$$

$$3Z^2 - U^2 = -22. \tag{5.3}$$

In the case $c = 4$ and $\mu = -23$, we have equation

$$x^4 - 8x^3y + 2x^2y^2 + 8xy^3 + y^4 = -23, \tag{5.4}$$

and the systems

$$4V^2 - 6U^2 = 46 \tag{5.5}$$

$$4Z^2 - 2U^2 = -46. \tag{5.6}$$

We can solve the systems (5.2)–(5.3) and (5.5)–(5.6) by Baker-Devenport reduction as in Section 3. Also, we can use the Thue equation solver in PARI/GP [18] to solve directly the equations (5.1) and (5.4). In both ways we find that all solutions of the equation (5.1) are

$$(x, y) = (2,1), (-2,-1), (1,-2), (-1,2), (3,2), (-3,-2), (2,-3), (-2,3),$$

and all solutions of the equation (5.4) are

$$(x, y) = (2,1), (-2,-1), (1,-2), (-1,2)\ .$$

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