BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **34**(2) (2011), 231–246

Product Properties for Pairwise Lindelöf Spaces

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Abstract. In this paper we study finite product of pairwise Lindelöf bitopological spaces. We show that the product properties for pairwise Lindelöf spaces are not preserved. Further, we provide some necessary conditions for these spaces to be preserved under a finite product.

2010 Mathematics Subject Classification: 54E55

Keywords and phrases: Bitopological space, i-Lindelöf, Lindelöf, (i, j)-Lindelöf, B-Lindelöf, s-Lindelöf, p-Lindelöf, product bitopology.

1. Introduction

The Cartesian product of a collection of sets is one of the most important and widely used ideas in mathematics. The theory of product spaces constitutes a very interesting and complex part of set-theoretic topology. The Cartesian product of arbitrarily many topological spaces was defined by Tychonoff in 1930. Then almost 33 years later in 1963 the idea of bitopological spaces was initiated by J. C. Kelly, see [10] and thereafter a large number of papers have been produced in order to generalize the topological concepts to bitopological setting. In 1972, Datta [3] defined the Cartesian product of arbitrarily many bitopological spaces. It is also well-known that, the Tychonoff Product Theorem plays an important role for general product of compact topological spaces.

Recently, the authors studied pairwise Lindelöfness in [14], introduced and studied the notion of pairwise weakly Lindelöfness in bitopological spaces, see [13], weakly regular Lindelöf [12] and almost Lindelöf bitopological spaces, see [11] where the authors extended some results that were due to Cammaroto and Santoro [2], Kılıçman and Fawakhreh [5, 6, 7]. In [15], the authors also studied the mappings and pairwise continuity on pairwise Lindelöf bitopological spaces.

Communicated by Lee See Keong.
Received: October 12, 2008; *Revised:* November 10, 2009.

The purpose of the present paper is to study the product properties on pairwise Lindelöf bitopological spaces. In this paper we consider four kinds of pairwise Lindelöf spaces namely known as Lindelöf, B-Lindelöf, s-Lindelöf and p-Lindelöf spaces, for details we refer to [9]. So we shall study the product properties for every kinds of pairwise Lindelöf spaces which are mentioned. Although the compactness is preserved for finite products, the Lindelöfness may not be productive unless one or more factors are assumed to satisfy additional conditions.

The similar results yield for every kind of pairwise Lindelöf spaces that we investigate in this paper. We give some counter-examples to show that the product properties are negative. We also provide some necessary conditions for these spaces to be preserved under a finite product.

2. Preliminaries

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always topological spaces and bitopological spaces, respectively unless explicitly stated. In this paper, we shall use p- and s- to denote pairwise and semi-respectively. For instance, p-Lindelöf and s-Lindelöf stands for pairwise Lindelöf and semi-Lindelöf respectively (see [9]). Also B-Lindelöf stands for another type of pairwise. If $\mathcal{P}$ is a topological property, then (τ_i, τ_j)-$\mathcal{P}$ denotes an analogue of this property for τ_i has property $\mathcal{P}$ with respect to τ_j, and p-$\mathcal{P}$ denotes the conjunction (τ_1, τ_2)-$\mathcal{P} \wedge (\tau_2, \tau_1)$-$\mathcal{P}$, i.e., p-$\mathcal{P}$ denotes an absolute bitopological analogue of $\mathcal{P}$. As we shall see below, sometimes (τ_1, τ_2)-$\mathcal{P} \iff (\tau_2, \tau_1)$-$\mathcal{P}$ (and thus $\Leftrightarrow p$-$\mathcal{P}$) so that it suffices to consider one of these three bitopological analogue. Also sometimes τ_1-$\mathcal{P} \iff \tau_2$-$\mathcal{P}$ and thus $\mathcal{P} \iff \tau_1$-$\mathcal{P} \wedge \tau_2$-$\mathcal{P}$, i.e., (X, τ_i) has property $\mathcal{P}$ for each $i = 1, 2$.

We also note that (X, τ_i) has a property $\mathcal{P} \iff (X, \tau_1, \tau_2)$ has a property τ_i-$\mathcal{P}$. Sometimes the prefixes (τ_i, τ_j)- or τ_i- will be replaced by (i, j)- or i- respectively, if there is no chance for confusion. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the (i, j)-regular open cover of X, etc. In this paper always $i, j \in \{1, 2\}$ and $i \neq j$. For details notation we refer to [4].

Definition 2.1. *Let (X, τ_1, τ_2) be a bitopological space. A subset F of X is said to be*

(1) *i-open if F is open with respect to τ_i in X, F is said open in X if it is both 1-open and 2-open in X, or equivalently, $F = U$ for $U \in (\tau_1 \cap \tau_2)$ in X;*
(2) *i-closed if F is closed with respect τ_i in X, F is said closed in X if it is both 1-closed and 2-closed in X, or equivalently, $F = V$ and $U \in (\tau_1 \cap \tau_2)$ for $F \in V$ and $U \subseteq \tau_1 \cap \tau_2$, respectively.*

Definition 2.2. [3] *Let $\{(X_\alpha, \tau_\alpha, \sigma_\alpha) : \alpha \in \Delta\}$ be a family of bitopological spaces. On the product set $X = \prod_{\alpha \in \Delta} X_\alpha$, we define a bitopological structure (τ, σ) by taking τ as the product topology generated by the projections which (τ, τ_α)-continuous and σ as the product topology generated by the projections which (σ, σ_α)-continuous for every $\alpha \in \Delta$. The product set X with the product bitopology (τ, σ), i.e., (X, τ, σ) is called product bitopological space. The product bitopology (τ, σ) also can be denoted by $\left(\prod_{\alpha \in \Delta} \tau_\alpha, \prod_{\alpha \in \Delta} \sigma_\alpha\right)$.*

3. Product of Lindelöf bitopological spaces

Definition 3.1. [9, 14] *A bitopological space* (X, τ_1, τ_2) *is said to be i-Lindelöf (resp. i-compact) if the topological space* (X, τ_i) *is Lindelöf (resp. compact). X is said Lindelöf (resp. compact) if it is i-Lindelöf (resp. i-compact) for each* $i = 1, 2$. *Equivalently,* (X, τ_1, τ_2) *is Lindelöf (resp. compact) if every i -open cover of X has a countable (resp. finite) subcover for each* $i = 1, 2$.

It is well-known that the product of any two Lindelöf topological spaces need not be Lindelöf. In general the product of any two i-Lindelöf spaces need not be i-Lindelöf or the product of any two Lindelöf bitopological spaces need not be Lindelöf as the following example below shows.

Example 3.1. Let $\mathcal{B}$ be a collection of closed-open intervals in the real line $\mathbb{R}$:

$$\mathcal{B} = \{[a, b) : a, b \in \mathbb{R},\ a < b\}.$$

Hence $\mathcal{B}$ is a base for the lower limit topology (or Sorgenfrey topology) τ_1 on $\mathbb{R}$. Choose usual topology as topology τ_2 on $\mathbb{R}$. Thus $(\mathbb{R}, \tau_1, \tau_2)$ is a Lindelöf bitopological space (see [17]). So it is clear that $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is not $(\tau_1 \times \tau_1)$-Lindelöf for the $(\tau_1 \times \tau_1)$-closed subspace $L = \{(x, y) : y = -x\}$ is not $(\tau_1 \times \tau_1)$-Lindelöf for it is a discrete subspace (see [17]). Thus $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is not a Lindelöf bitopological space.

Theorem 3.1. *Let* (X, τ_1, τ_2) *be a* τ_i*-Lindelöf space and* (Y, σ_1, σ_2) *a* σ_i*-compact space. Then* $(X \times Y, \rho_1, \rho_2)$ *is* ρ_i*-Lindelöf where* ρ_i *is a product topology.*

Proof. The proof is similar with the well-known result, the Tychonoff Product Theorem, so we omit the details. ∎

In general, the converse of Theorem 3.1 is not true and we provide the following counter-example.

Example 3.2. Let τ_{coc} and τ_u denotes the cocountable topology and usual topology on $\mathbb{R}$ respectively. Then the bitopological space $(\mathbb{R} \times \mathbb{R}, \tau_{coc} \times \tau_{coc}, \tau_u \times \tau_u)$ is Lindelöf. However $(\mathbb{R}, \tau_{coc}, \tau_u)$ is Lindelöf space but not compact (see [17]).

The above result still holds if we consider an i-Lindelöf space and a collection of finite i-compact spaces as stated in the following corollary.

Corollary 3.1. *Let* $(X_m, \tau_1^m, \tau_2^m)$ *be a* τ_i^m*-Lindelöf space and*

$$\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \dots n, k \neq m, m \leq n\right\}$$

a collection of τ_i^k*-compact spaces. Then* $(\prod_{k=1}^{n} X_k, \rho_1, \rho_2)$ *is* ρ_i*-Lindelöf where* ρ_i *is a product topology.*

Proof. It follows immediately on using the fact that the topological product is commutative, associative (see [16], p. 132) and the well-known result, the Tychonoff Product Theorem. ∎

The result also holds if a collection of finite i-compact spaces is replaced by arbitrary collection of i-compact spaces since the Tychonoff Product Theorem is true for any collection of i-compact spaces.

Corollary 3.2. *Let* $\left(X_\beta, \tau_1^\beta, \tau_2^\beta\right)$ *be a* τ_i^β*-Lindelöf space and* $\{(X_\alpha, \tau_1^\alpha, \tau_2^\alpha) : \alpha \in \Delta,$ $\alpha \neq \beta\}$ *a collection of* τ_i^α*-compact spaces. Then* $\left(\prod_{\alpha\in\Delta} X_\alpha, \rho_1, \rho_2\right)$ *is* ρ_i*-Lindelöf where* ρ_i *is a product topology.*

Definition 3.2. *A bitopological space* X *is said to be* i*-*P*-space if countable intersection of* i*-open sets in* X *is* i*-open.* X *is said* P*-space if it is* i*-*P*-space for each* $i = 1, 2$.

Although the product of τ_i-Lindelöf and σ_i-Lindelöf spaces need not be $(\tau_i \times \sigma_i)$-Lindelöf, if we consider an additional condition such as τ_i-P-space to one of the space we will obtain that the product is $(\tau_i \times \sigma_i)$-Lindelöf.

Proposition 3.1. *Let* (X, τ_1, τ_2) *be a* τ_i*-Lindelöf* τ_i*-*P*-space and* (Y, σ_1, σ_2) *a* σ_i*-Lindelöf space. Then* $(X \times Y, \rho_1, \rho_2)$ *is* ρ_i*-Lindelöf where* ρ_i *is a product topology.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a ρ_i-open cover of $X \times Y$. Then each member of $\mathcal{U}$ is a union of ρ_i-basis elements of the form $V \times W$ with V is τ_i-open set in X and W is σ_i-open set in Y. We may restrict our attention to the cover $\{V_\alpha \times W_\alpha : \alpha \in \Delta\}$ of $X \times Y$ which consists of ρ_i-basis elements where each $V_\alpha \times W_\alpha$ is contained in some member of $\mathcal{U}$, since any subcover of this basic ρ_i-open cover will lead immediately to a subcover chosen from the original cover $\mathcal{U}$. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is i-homeomorphic to Y and hence Y_x is Lindelöf with respect to the inducted topology from ρ_i. So Y_x is ρ_i-Lindelöf relative to $X \times Y$ and since $\{V_\alpha \times W_\alpha : \alpha \in \Delta\}$ also covers Y_x, there must exists a countable subcover $\{V_{x,\alpha_n} \times W_{x,\alpha_n} : n \in \mathbb{N}\}$ of Y_x by sets which have a nonempty intersection with Y_x. Letting $H_x = \bigcap_{n\in\mathbb{N}} V_{x,\alpha_n}$, we see that H_x is a τ_i-open set of X containing x since X is a τ_i-P-space. The above countable subcover $\{V_{x,\alpha_n} \times W_{x,\alpha_n} : n \in \mathbb{N}\}$ actually covers $H_x \times Y$. Now $\{H_x : x \in X\}$ is a τ_i-open cover of X. Since X is τ_i-Lindelöf, there exists a countable subcover $\{H_{x_k} : k \in \mathbb{N}\}$. But then $\{\{V_{x_k,\alpha_n} \times W_{x_k,\alpha_n} : n \in \mathbb{N}\} : k \in \mathbb{N}\}$ covers $X \times Y$. Since $\{\{V_{x_k,\alpha_n} \times W_{x_k,\alpha_n} : n \in \mathbb{N}\} : k \in \mathbb{N}\}$ is a countable subcover, we have that $X \times Y$ is ρ_i-Lindelöf. ∎

Corollary 3.3. *Let* (X, τ_1, τ_2) *be a Lindelöf* P*-space and* (Y, σ_1, σ_2) *a Lindelöf space. Then* $(X \times Y, \rho_1, \rho_2)$ *is Lindelöf where* ρ_i *is a product topology.*

We note that the above result is also true if we take a collection of finite i-Lindelöf i-P-spaces and an i-Lindelöf space. By using the fact that the topological product is commutative and associative, the result will then follow by induction. We need the following lemma.

Lemma 3.1. *Let* (X, τ_1, τ_2) *be a* τ_i*-*P*-space and* (Y, σ_1, σ_2) *a* σ_i*-*P*-space. Then* $(X \times Y, \rho_1, \rho_2)$ *is* ρ_i*-*P*-space where* ρ_i *is a product topology.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable collection of ρ_i-open sets in $X \times Y$. Then each U_n is a union of ρ_i-basis elements of the form $V \times W$ where V and W are τ_i-open set and σ_i-open set of X and Y respectively. We may restrict our attention to the countable collection of ρ_i-basis element $\{V_n \times W_n : n \in \mathbb{N}\}$ of $X \times Y$ because any ρ_i-open set is a union of ρ_i-basis elements. Now

$$\bigcap_{n\in\mathbb{N}} (V_n \times W_n) = \left(\bigcap_{n\in\mathbb{N}} V_n\right) \times \left(\bigcap_{n\in\mathbb{N}} W_n\right)$$

is a ρ_i-basis element since X is τ_i-P-space and Y is σ_i-P-space. Therefore $X \times Y$ is ρ_i-P-space. ∎

Corollary 3.4. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n\right\}$ *be a collection of* τ_i^k-P*-spaces. Then* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is* ρ_i-P*-space where* ρ_i *is a product topology.*

Proof. It follows by induction on k. ∎

Proposition 3.2. *Let* $\{(X_\alpha, \tau_1^\alpha, \tau_2^\alpha) : \alpha \in \Delta\}$ *be a collection of* τ_i^α-P*-spaces. Then*

$$\left(\prod\nolimits_{\alpha \in \Delta} X_\alpha, \rho_1, \rho_2\right)$$

is ρ_i-P*-space where* ρ_i *is a product topology.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable collection of ρ_i-open sets in $\prod_{\alpha \in \Delta} X_\alpha$. Then as in the proof of Lemma 3.1, we may restrict our attention to the countable collection of ρ_i-basis element $\left\{\prod \{X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\} \times V_{\beta_1}^n \times \cdots \times V_{\beta_m}^n \;:\; n \in \mathbb{N}\right\}$ of $\prod_{\alpha \in \Delta} X_\alpha$ where $V_{\beta_k}^n$ is a $\tau_i^{\beta_k}$-open set of $X_{\beta_k}, k = 1, \ldots, m$. It can be done because any ρ_i-open set is a union of ρ_i-basis elements. Now

$$\begin{aligned}
&\bigcap_{n \in \mathbb{N}} \left(\prod \{X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\} \times V_{\beta_1}^n \times \cdots \times V_{\beta_m}^n\right) \\
&= \bigcap_{n \in \mathbb{N}} \left(\prod \{X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\}\right) \times \left(\bigcap_{n \in \mathbb{N}} V_{\beta_1}^n\right) \times \cdots \times \left(\bigcap_{n \in \mathbb{N}} V_{\beta_m}^n\right) \\
&= \prod \left\{\bigcap_{n \in \mathbb{N}} X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\right\} \times \left(\bigcap_{n \in \mathbb{N}} V_{\beta_1}^n\right) \times \cdots \times \left(\bigcap_{n \in \mathbb{N}} V_{\beta_m}^n\right) \\
&= \prod \{X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\} \times \left(\bigcap_{n \in \mathbb{N}} V_{\beta_1}^n\right) \times \cdots \times \left(\bigcap_{n \in \mathbb{N}} V_{\beta_m}^n\right)
\end{aligned}$$

is a ρ_i-basis element since X_{β_k} is $\tau_i^{\beta_k}$-P-spaces. Therefore $\prod_{\alpha \in \Delta} X_\alpha$ is ρ_i-P-space. ∎

The Proposition 3.2 leads to the following corollaries.

Corollary 3.5. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n, k \neq m, m \leq n\right\}$ *be a collection of* τ_i^k*-Lindelöf* τ_i^k-P*-spaces and* $(X_m, \tau_1^m, \tau_2^m)$ *a* τ_i^m*-Lindelöf space. Then* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is* ρ_i*-Lindelöf where* ρ_i *is a product topology.*

Proof. It follows by induction of k, and noting the fact that the topological product is commutative, associative and using the Corollary 3.4. ∎

Corollary 3.6. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n, k \neq m, m \leq n\right\}$ *be a collection of Lindelöf* P*-spaces and* $(X_m, \tau_1^m, \tau_2^m)$ *a Lindelöf space. Then* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is Lindelöf where* ρ_i *is a product topology.*

Recall that a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called i-closed if $f(U)$ is σ_i-closed set in Y for every τ_i-closed set U in X, f is said closed if it is i-closed for each $i = 1, 2$. From elementary general topology it is well-known that, if X is a topological space and suppose a neighbourhood base has been fixed at each $x \in X$, then $F \subseteq X$ is closed if and only if each point $x \notin F$ has a basic neighbourhood disjoint from F (see [19]). Now we can prove the following proposition.

Proposition 3.3. *Let (X, τ_1, τ_2) be a τ_i-Lindelöf space and (Y, σ_1, σ_2) a σ_i-P-space. Then the projection $\pi_Y : (X \times Y, \rho_1, \rho_2) \to (Y, \sigma_1, \sigma_2)$ is i-closed where ρ_i is a product topology.*

Proof. Let U be a ρ_i-closed set in $X \times Y$ and let $y_0 \notin \pi_Y(U)$. Clearly $X \times \{y_0\} \cap U = \emptyset$, so the point $(x, y_0) \notin U$ has a ρ_i-basic neighbourhood $V_x \times W_{x,y_0}$ disjoint from U where V_x is τ_i-open set in X containing x and W_{x,y_0} is σ_i-open set in Y containing y_0. Now $\{V_x \times W_{x,y_0} : x \in X\}$ forms a ρ_i-open cover of $X \times \{y_0\}$ by ρ_i-open sets in $X \times Y$. Since $X \times \{y_0\}$ is i-homeomorphic to X, then $X \times \{y_0\}$ is ρ_i-Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So $X \times \{y_0\}$ is ρ_i-Lindelöf relative to $X \times Y$ and hence there exists a countable subfamily $\{V_{x_n} \times W_{x_n,y_0} : n \in \mathbb{N}\}$ such that $X \times \{y_0\} \subseteq \bigcup\limits_{n\in\mathbb{N}} (V_{x_n} \times W_{x_n,y_0}) = \left(\bigcup\limits_{n\in\mathbb{N}} V_{x_n}\right) \times \left(\bigcup\limits_{n\in\mathbb{N}} W_{x_n,y_0}\right)$. Set $W = \bigcap\limits_{n\in\mathbb{N}} W_{x_n,y_0}$ and since Y is a σ_i-P-space, W is a σ_i-open neighbourhood of y_0. We need to prove that $W \cap \pi_Y(U) = \emptyset$. Now suppose that $W \cap \pi_Y(U) \neq \emptyset$, then there exists a point $y_1 \in W$ and $y_1 \in \pi_Y(U)$. Hence $y_1 \in W_{x_n,y_0}$ for each $n \in \mathbb{N}$ and therefore $(x_n, y_1) \in V_{x_n} \times W_{x_n,y_0}$. On the other hand, $X \times \{y_1\} \cap U \neq \emptyset$ and this implies that $(x_n, y_1) \in U$ which is a contradiction. Thus $\pi_Y(U)$ is σ_i-closed set in Y. This implies that π_Y is i-closed. ∎

We can extend this result to arbitrary product space thus we have the following proposition.

Proposition 3.4. *Let $\{(X_\alpha, \tau_1^\alpha, \tau_2^\alpha) : \alpha \in \Delta, \alpha \neq \beta\}$ be a collection of τ_i^α-Lindelöf space and $\left(X_\beta, \tau_1^\beta, \tau_2^\beta\right)$ a τ_i^β-P-space. Then the projection $\pi_\beta : \left(\prod_{\alpha\in\Delta} X_\alpha, \rho_1, \rho_2\right) \to \left(X_\beta, \tau_1^\beta, \tau_2^\beta\right)$ is i-closed where ρ_i is a product topology.*

Proof. Let U be a ρ_i-closed set in $\prod_{\alpha\in\Delta} X_\alpha$ and let $y_0 \notin \pi_\beta(U)$. Clearly $\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\} \cap U = \emptyset$, so the point $(x_\alpha : \alpha \in \Delta)$ where $x_\beta = y_0$ does not belong to U has a ρ_i-basic neighbourhood $\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha} \times W_{x_\alpha,y_0}$ disjoint from U where V_{x_α} is τ_i^α-open set in X_α containing x_α, $\alpha \neq \beta$ and W_{x_α,y_0} is τ_i^β-open set in X_β containing y_0. Now $\left\{\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha} \times W_{x_\alpha,y_0} : x_\alpha \in X_\alpha\right\}$ form a ρ_i-open cover of $\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\}$ by ρ_i-open sets in $\prod_{\alpha\in\Delta} X_\alpha$. Since $\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\}$ is i-homeomorphic to $\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha$, then $\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\}$ is ρ_i-Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So $\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\}$ is ρ_i-Lindelöf relative to $\prod_{\alpha\in\Delta} X_\alpha$ and hence there exists a countable subfamily $\left\{\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha^n} \times W_{x_\alpha^n,y_0} : n \in \mathbb{N}\right\}$ such that

$$\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\} \subseteq \bigcup_{n\in\mathbb{N}} \left(\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha^n} \times W_{x_\alpha^n,y_0}\right)$$
$$= \left(\bigcup_{n\in\mathbb{N}} \left(\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha^n}\right)\right) \times \left(\bigcup_{n\in\mathbb{N}} W_{x_\alpha^n,y_0}\right).$$

Set $W = \bigcap\limits_{n\in\mathbb{N}} W_{x_\alpha^n,y_0}$ and since X_β is a τ_i^β-P-space, W is a τ_i^β-open neighbourhood of y_0. We need to prove that $W \cap \pi_\beta(U) = \emptyset$. Now suppose that $W \cap \pi_\beta(U) \neq \emptyset$, then there exists a point $y_1 \in W$ and $y_1 \in \pi_\beta(U)$. Hence $y_1 \in W_{x_\alpha^n, y_0}$ for each $n \in \mathbb{N}$ and therefore $(x_\alpha : \alpha \in \Delta)$ where $x_\beta = y_1$ belong to $\prod_{\alpha \in \Delta, \alpha \neq \beta} V_{x_\alpha^n} \times W_{x_\alpha^n, y_0}$. On the other hand, $\prod_{\alpha \in \Delta, \alpha \neq \beta} X_\alpha \times \{y_1\} \cap U \neq \emptyset$ and this implies that $(x_\alpha : \alpha \in \Delta)$ where $x_\beta = y_1$ belong to U which is a contradiction. Thus $\pi_Y(U)$ is τ_i^β-closed set in X_β. This implies that π_β is i-closed. ∎

4. Product of B-Lindelöf spaces

Definition 4.1. *A bitopological space (X, τ_1, τ_2) is called (i,j)-compact* [1] (*resp. (i,j)-Lindelöf* [9, 14]) *if for every i-open cover of X there is a finite* (*resp. countable*) *j-open subcover. Similarly, X is called B-compact* [1] (*resp. B-Lindelöf* [9, 14]) *if it is both $(1,2)$-compact* (*resp. $(1,2)$-Lindelöf*) *and $(2,1)$-compact* (*resp. $(2,1)$-Lindelöf*).

Theorem 4.1. *Let (X, τ_1, τ_2) be a (τ_j, τ_i)-compact space and (Y, σ_1, σ_2) a (σ_i, σ_j)-compact space. Then $(X \times Y, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-compact where ρ_i is a product topology.*

Proof. The proof of this theorem is similar to the Theorem 3.1, so we omit the details. ∎

It is clear that if (X, τ_1, τ_2) is B-Lindelöf, then (X, τ_i) must be a Lindelöf space for each $i = 1, 2$, i.e., (X, τ_1, τ_2) is a Lindelöf space. In general, the product of any two (i,j)-Lindelöf spaces need not be (i,j)-Lindelöf or the product of any two B-Lindelöf spaces need not be B-Lindelöf as the following example show.

Example 4.1. Let τ_s denotes the Sorgenfrey topology on $\mathbb{R}$. Then the bitopological space $(\mathbb{R}, \tau_s, \tau_s)$ is B-Lindelöf. However $(\mathbb{R} \times \mathbb{R}, \tau_s \times \tau_s, \tau_s \times \tau_s)$ is not B-Lindelöf, for the topological space $(\mathbb{R} \times \mathbb{R}, \tau_s \times \tau_s)$ is not Lindelöf (see [17]).

The following example gives further explanation for the B-Lindelöf spaces and to show that some of them satisfying the product invariant property.

Example 4.2. Let $\mathcal{B}_1 = \{\mathbb{R}, \{x\} : x \in \mathbb{R} \setminus \{0\}\}$ and $\mathcal{B}_2 = \{\mathbb{R}, \{x\} : x \in \mathbb{R} \setminus \{1\}\}$. Let τ_1 and τ_2 are the topologies on $\mathbb{R}$ which are generated by the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Then $(\mathbb{R}, \tau_1, \tau_2)$ is B-Lindelöf, for any τ_i-open cover of $\mathbb{R}$ must contain $\mathbb{R}$ as a member [9]. We obtain that $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is B-Lindelöf since for any $(\tau_i \times \tau_i)$-open cover of $\mathbb{R} \times \mathbb{R}$ must contain $\mathbb{R} \times \mathbb{R}$ as a member. So the bitopological space $(\mathbb{R}, \tau_1, \tau_2)$ is satisfying the product invariant property.

Theorem 4.2. *Let (X, τ_1, τ_2) be a (τ_j, τ_i)-Lindelöf space and (Y, σ_1, σ_2) a (σ_i, σ_j)-compact space. Then $(X \times Y, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-Lindelöf where ρ_i is a product topology.*

Proof. The proof is straightforward on following the Theorems 3.1 and 4.1, so we omit the details. ∎

Definition 4.2. *A bitopological space X is said to be (i,j)-P-space if countable intersection of i-open sets in X is j-open. X is said B-P-space if it is $(1,2)$-P-space and $(2,1)$-P-space.*

Proposition 4.1. *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-Lindelöf (τ_j, τ_i)-P-space and (Y, σ_1, σ_2) a (σ_i, σ_j)-Lindelöf space. Then $(X \times Y, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-Lindelöf where ρ_i is a product topology.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a ρ_i-open cover of $X \times Y$. Then as in proof of Proposition 3.1, we may restrict our attention to the cover $\{V_\alpha \times W_\alpha : \alpha \in \Delta\}$ of $X \times Y$ by the ρ_i-basis elements where each $V_\alpha \times W_\alpha$ is contained in some member of $\mathcal{U}$. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is i-homeomorphic to Y and hence Y_x is (ρ_i, ρ_j)-Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So Y_x is (ρ_i, ρ_j)-Lindelöf relative to $X \times Y$ and since $\{V_\alpha \times W_\alpha : \alpha \in \Delta\}$ also covers Y_x, there must exists a countable ρ_j-open subcover $\{V_{x,\alpha_n} \times W_{x,\alpha_n} : n \in \mathbb{N}\}$ of Y_x by sets which have a nonempty intersection with Y_x. Letting $H_x = \bigcap_{n \in \mathbb{N}} V_{x,\alpha_n}$, we see that H_x is a τ_i-open set of X containing x since X is a (τ_j, τ_i)-P-space. The above countable ρ_j-open subcover $\{V_{x,\alpha_n} \times W_{x,\alpha_n} : n \in \mathbb{N}\}$ actually covers $H_x \times Y$. Now $\{H_x : x \in X\}$ is a τ_i-open cover of X. Since X is (τ_i, τ_j)-Lindelöf, there exists a countable τ_j-open subcover $\{H_{x_k} : k \in \mathbb{N}\}$. But then $\{\{V_{x_k,\alpha_n} \times W_{x_k,\alpha_n} : n \in \mathbb{N}\} : k \in \mathbb{N}\}$ covers $X \times Y$. Since $\{\{V_{x_k,\alpha_n} \times W_{x_k,\alpha_n} : k \in \mathbb{N}\} : n \in \mathbb{N}\}$ is a countable ρ_j-open subcover, we have that $X \times Y$ is (ρ_i, ρ_j)-Lindelöf. ∎

Corollary 4.1. *Let (X, τ_1, τ_2) be a B-Lindelöf B-P-space and (Y, σ_1, σ_2) a B-Lindelöf space. Then $(X \times Y, \rho_1, \rho_2)$ is B-Lindelöf where ρ_i is a product topology.*

Now if we take a collection of finite (i, j)-Lindelöf (j, i)-P-spaces and an (i, j)-Lindelöf space then the above result is still true. One can easily see this on noting that the topological product is commutative and associative, the result will then follow by induction. We state the following lemma.

Lemma 4.1. *Let (X, τ_1, τ_2) be a (τ_i, τ_j)-P-space and (Y, σ_1, σ_2) a (σ_i, σ_j)-P-space. Then $(X \times Y, \rho_1, \rho_2)$ is (ρ_i, ρ_j)-P-space where ρ_i is a product topology.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable collection of ρ_i-open sets in $X \times Y$. Then as in the proof of Lemma 3.1, we may restrict our attention to the countable collection of ρ_i-basis element $\{V_n \times W_n : n \in \mathbb{N}\}$ of $X \times Y$ because any ρ_i-open set is a union of ρ_i-basis elements. Now $\bigcap_{n \in \mathbb{N}} (V_n \times W_n) = \left(\bigcap_{n \in \mathbb{N}} V_n\right) \times \left(\bigcap_{n \in \mathbb{N}} W_n\right)$ is a ρ_j-basis element since X is (τ_i, τ_j)-P-space and Y is (σ_i, σ_j)-P-space. Therefore $X \times Y$ is (ρ_i, ρ_j)-P-space. ∎

Corollary 4.2. *Let $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n\right\}$ be a collection of $\left(\tau_i^k, \tau_j^k\right)$-P-spaces. Then $\left(\prod_{k=1}^n X_k, \rho_1, \rho_2\right)$ is (ρ_i, ρ_j)-P-space where ρ_i is a product topology.*

Proof. It follows by induction of k. ∎

Proposition 4.2. *Let $\{(X_\alpha, \tau_1^\alpha, \tau_2^\alpha) : \alpha \in \Delta\}$ be a collection of $\left(\tau_i^\alpha, \tau_j^\alpha\right)$-P-spaces. Then $\left(\prod_{\alpha \in \Delta} X_\alpha, \rho_1, \rho_2\right)$ is (ρ_i, ρ_j)-P-space where ρ_i is a product topology.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable collection of ρ_i-open sets in $\prod_{\alpha \in \Delta} X_\alpha$. Then as in the proof of Lemma 3.1, we may restrict our attention to the countable collection of ρ_i-basis element $\left\{\prod \{X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\} \times V_{\beta_1}^n \times \cdots \times V_{\beta_m}^n : n \in \mathbb{N}\right\}$ of

$\prod_{\alpha\in\Delta} X_\alpha$ where $V^n_{\beta_k}$ is a $\tau_i^{\beta_k}$-open set of $X_{\beta_k}, k = 1, \ldots, m$. Now

$$\bigcap_{n\in\mathbb{N}} \left(\prod \{X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\} \times V^n_{\beta_1} \times \cdots \times V^n_{\beta_m}\right)$$
$$= \prod \{X_\alpha : \alpha \neq \beta_1, \ldots, \beta_m\} \times \left(\bigcap_{n\in\mathbb{N}} V^n_{\beta_1}\right) \times \cdots \times \left(\bigcap_{n\in\mathbb{N}} V^n_{\beta_m}\right)$$

is a ρ_j-basis element since X_{β_k} is $\left(\tau_i^{\beta_k}, \tau_j^{\beta_k}\right)$-$P$-spaces. Therefore $\prod_{\alpha\in\Delta} X_\alpha$ is (ρ_i, ρ_j)-P-space. ∎

Corollary 4.3. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n, k \neq m, m \leq n\right\}$ *be a collection of* $\left(\tau_i^k, \tau_j^k\right)$*-Lindelöf* $\left(\tau_j^k, \tau_i^k\right)$*-P-spaces and* $(X_m, \tau_1^m, \tau_2^m)$ *a* $\left(\tau_i^m, \tau_j^m\right)$*-Lindelöf space. Then*

$$\left(\prod_{k=1}^n X_k, \rho_1, \rho_2\right)$$

is (ρ_i, ρ_j)*-Lindelöf where* ρ_i *is a product topology.*

Proof. It follows by induction of k and the Corollary 4.2. ∎

Corollary 4.4. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n, k \neq m, m \leq n\right\}$ *be a collection of B-Lindelöf B-P-spaces and* $(X_m, \tau_1^m, \tau_2^m)$ *a B-Lindelöf space. Then* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is B-Lindelöf where* ρ_i *is a product topology.*

Proposition 4.3. *Let* (X, τ_1, τ_2) *be a* (τ_j, τ_i)*-Lindelöf* τ_j*-P-space and* (Y, σ_1, σ_2) *a* (σ_i, σ_j)*-Lindelöf space. Then* $(X \times Y, \rho_1, \rho_2)$ *is* (ρ_i, ρ_j)*-Lindelöf where* ρ_i *is a product topology.*

Proof. Similar with the proof of the Proposition 4.1. ∎

Corollary 4.5. *Let* (X, τ_1, τ_2) *be a B-Lindelöf P-space and* (Y, σ_1, σ_2) *a B-Lindelöf space. Then* $(X \times Y, \rho_1, \rho_2)$ *is B-Lindelöf where* ρ_i *is a product topology.*

Corollary 4.6. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n, k \neq m, m \leq n\right\}$ *be a collection of* $\left(\tau_j^k, \tau_i^k\right)$*-Lindelöf* τ_j^k*-P-spaces and* $(X_m, \tau_1^m, \tau_2^m)$ *a* $\left(\tau_i^m, \tau_j^m\right)$*-Lindelöf space. Then* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is* (ρ_i, ρ_j)*-Lindelöf where* ρ_i *is a product topology.*

Proof. It follows by induction of k and the Corollary 3.4. ∎

Corollary 4.7. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n, k \neq m, m \leq n\right\}$ *be a collection of B-Lindelöf P-spaces and* $(X_m, \tau_1^m, \tau_2^m)$ *a B-Lindelöf space. Then* $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ *is B-Lindelöf where* ρ_i *is a product topology.*

Definition 4.3. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called* (i, j)*-closed if* $f(U)$ *is* σ_j*-closed set in* Y *for every* τ_i*-closed set* U *in* X*,* f *is said pairwise closed if it is both* $(1, 2)$*-closed and* $(2, 1)$*-closed.*

Proposition 4.4. *Let* (X, τ_1, τ_2) *be a* (τ_i, τ_j)*-Lindelöf space and* (Y, σ_1, σ_2) *a* σ_j*-P-space. Then the projection* $\pi_Y : (X \times Y, \rho_1, \rho_2) \to (Y, \sigma_1, \sigma_2)$ *is* (i, j)*-closed where* ρ_i *is a product topology.*

Proof. The proof is similar with the proof of the Proposition 3.3 and thus we omit the details. ∎

We proceed this result to the arbitrary product spaces and we have the following proposition.

Proposition 4.5. *Let* $\{(X_\alpha, \tau_1^\alpha, \tau_2^\alpha) : \alpha \in \Delta, \alpha \neq \beta\}$ *be a collection of* $\left(\tau_i^\alpha, \tau_j^\alpha\right)$*-Lindelöf space and* $\left(X_\beta, \tau_1^\beta, \tau_2^\beta\right)$ *a* τ_j^β*-P-space. Then the projection* $\pi_\beta : \left(\prod_{\alpha\in\Delta} X_\alpha,\right.$ $\left.\rho_1, \rho_2\right) \to \left(X_\beta, \tau_1^\beta, \tau_2^\beta\right)$ *is* (i,j)*-closed where* ρ_i *is a product topology.*

Proof. Let U be a ρ_i-closed set in $\prod_{\alpha\in\Delta} X_\alpha$ and let $y_0 \notin \pi_\beta(U)$. Following the proof of Proposition 3.4, $\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\}$ is (ρ_i, ρ_j)-Lindelöf relative to $\prod_{\alpha\in\Delta} X_\alpha$ and hence there exists a countable ρ_j-open subfamily $\left\{\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha^n} \times W_{x_\alpha^n, y_0} : n \in \mathbb{N}\right\}$ such that

$$\prod_{\alpha\in\Delta,\alpha\neq\beta} X_\alpha \times \{y_0\} \subseteq \bigcup_{n\in\mathbb{N}} \left(\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha^n} \times W_{x_\alpha^n, y_0}\right)$$
$$= \left(\bigcup_{n\in\mathbb{N}} \left(\prod_{\alpha\in\Delta,\alpha\neq\beta} V_{x_\alpha^n}\right)\right) \times \left(\bigcup_{n\in\mathbb{N}} W_{x_\alpha^n, y_0}\right).$$

Set $W = \bigcap_{n\in\mathbb{N}} W_{x_\alpha^n, y_0}$ and since X_β is a τ_j^β-P-space, W is a τ_j^β-open neighbourhood of y_0 such that $W \cap \pi_\beta(U) = \emptyset$. Thus $\pi_\beta(U)$ is τ_j^β-closed set in X_β. This implies that π_β is (i,j)-closed. ∎

5. Product of s-Lindelöf spaces

Definition 5.1. *A cover* $\mathcal{U}$ *of a bitopological space* (X, τ_1, τ_2) *is called* $\tau_1\tau_2$*-open if* $\mathcal{U} \subseteq \tau_1 \cup \tau_2$ [18]*. If, in addition,* $\mathcal{U}$ *contains at least one nonempty member of* τ_1 *and at least one nonempty member of* τ_2*, it is called p-open* [8]*.*

Definition 5.2. *A bitopological space* (X, τ_1, τ_2) *is called s-compact* [3] (*resp. s-Lindelöf* [9]) *if every* $\tau_1\tau_2$*-open cover of* X *has a finite* (*resp. countable*) *subcover.*

Theorem 5.1. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *are s-compact spaces. Then* $(X \times Y, \rho_1, \rho_2)$ *is s-compact where* ρ_i *is a product topology.*

Proof. The proof is similar with the previous Theorems 3.1 and 4.1. ∎

The product is still invariant if we take a finite collection of s-compact spaces as stated in the following corollary. The result will then follow by induction.

Corollary 5.1. *Let* $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n\right\}$ *be a collection of s-compact spaces. Then* $\left(\prod_{k=1}^n X_k, \rho_1, \rho_2\right)$ *is s-compact where* ρ_i *is a product topology.*

The product of any two s-Lindelöf spaces also need not be s-Lindelöf as the following counter-example shows.

Example 5.1. Let τ_u and τ_s denotes the usual topology and Sorgenfrey topology on $\mathbb{R}$ respectively. Then the bitopological space $(\mathbb{R}, \tau_u, \tau_s)$ is s-Lindelöf. However

$$(\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u, \tau_s \times \tau_s)$$

is not s-Lindelöf, for it follows immediately from the observation that any $(\tau_s \times \tau_s)$-open cover of $(\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u, \tau_s \times \tau_s)$ is $(\tau_u \times \tau_u)(\tau_s \times \tau_s)$-open and the topological space $(\mathbb{R} \times \mathbb{R}, \tau_s \times \tau_s)$ is not Lindelöf (see [17]).

Theorem 5.2. *Let (X, τ_1, τ_2) be an s-Lindelöf space and (Y, σ_1, σ_2) an s-compact space. Then $(X \times Y, \rho_1, \rho_2)$ is s-Lindelöf where ρ_i is a product topology.*

Proof. The proof is straightforward thus we omit the details. ∎

The above result is still hold if we take an s-Lindelöf space and a collection of finite s-compact spaces as stated in the following corollary.

Corollary 5.2. *Let $(X_m, \tau_1^m, \tau_2^m)$ be an s-Lindelöf space and*

$$\left\{ \left(X_k, \tau_1^k, \tau_2^k \right) : k = 1, \ldots n, k \neq m, m \leq n \right\}$$

a collection of s-compact spaces. Then $(\prod_{k=1}^{n} X_k, \rho_1, \rho_2)$ is s-Lindelöf where ρ_i is a product topology.

Proof. It follows immediately by the fact that the product of topological space is commutative, associative in the Corollary 5.1. ∎

Definition 5.3. *Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $\tau_1\tau_2$-open cover of a bitopological space (X, τ_1, τ_2). Then X is said to be s-P-space if for each $x \in X$ there exists a countable subfamily $\{U_{x,\alpha_n} : x \in X\}$ where $x \in U_{x,\alpha_n}$ for all $n \in \mathbb{N}$ such that $\left\{ \bigcap_{n \in \mathbb{N}} U_{x,\alpha_n} : x \in X \right\} \subseteq \tau_1 \cup \tau_2$.*

Proposition 5.1. *Let (X, τ_1, τ_2) be an s-Lindelöf space s-P-space and (Y, σ_1, σ_2) an s-Lindelöf space. Then $(X \times Y, \rho_1, \rho_2)$ is s-Lindelöf where ρ_i is a product topology.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $\rho_1\rho_2$-open cover of $X \times Y$. Then $\mathcal{U} \subseteq \rho_1 \cup \rho_2$. We may restrict our attention to a cover $\mathcal{V} \times \mathcal{W}$ of $X \times Y$ with $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ and $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ are $\tau_1\tau_2$-open cover and $\sigma_1\sigma_2$-open cover of X and Y respectively where $\mathcal{V} \times \mathcal{W}$ is contained in $\mathcal{U}$, since any subcover of this cover will lead immediately to a subcover chosen from the original cover $\mathcal{U}$. Hence $\mathcal{V} \subseteq \tau_1 \cup \tau_2$ and $\mathcal{W} \subseteq \sigma_1 \cup \sigma_2$. Now for each $x \in X$, let $Y_x = \{x\} \times Y$ which is homeomorphic to Y and hence Y_x is s-Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So Y_x is s-Lindelöf relative to $X \times Y$ and since $\{V_\alpha : \alpha \in \Delta\} \times \{W_\alpha : \alpha \in \Delta\}$ also covers Y_x, there must exist a countable $\rho_1\rho_2$-open subcover $\{\{V_{x,\alpha_k}\} \times \{W_{x,\alpha_k}\} : k \in \mathbb{N}\}$ of Y_x by sets which have a nonempty intersection with Y_x. Letting $H_x = \bigcap_{k \in \mathbb{N}} V_{x,\alpha_k}$, we see that H_x contains x and hence $\{H_x : x \in X\} \subseteq \tau_1 \cup \tau_2$ since X is s-P-space. The above countable $\rho_1\rho_2$-open subcover $\{\{V_{x,\alpha_k}\} \times \{W_{x,\alpha_k}\} : k \in \mathbb{N}\}$ actually covers $H_x \times Y$. Now the family $\{H_x : x \in X\}$ is a $\tau_1\tau_2$-open cover of X. Since X is s-Lindelöf, there exists a countable subcover $\{H_{x_n} : n \in \mathbb{N}\}$. But then $\{\{\{V_{x_n,\alpha_k}\} \times \{W_{x_n,\alpha_k}\} : k \in \mathbb{N}\} : n \in \mathbb{N}\}$ covers $X \times Y$. Since $\{\{\{V_{x_n,\alpha_k}\} \times \{W_{x_n,\alpha_k}\} : k \in \mathbb{N}\} : n \in \mathbb{N}\}$ is a countable $\rho_1\rho_2$-open subcover, we have that $X \times Y$ is s-Lindelöf. ∎

Lemma 5.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) are s-P-spaces. Then $(X \times Y, \rho_1, \rho_2)$ is s-P-space where ρ_i is a product topology.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}, \mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ and $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ are $\rho_1\rho_2$-open cover of $X \times Y$, $\tau_1\tau_2$-open cover of X and $\sigma_1\sigma_2$-open cover of Y respectively. For each $(x, y) \in X \times Y$, let $\left\{ U_{(x,y),\alpha_n} : n \in \mathbb{N} \right\}$ be a countable subfamily of $\mathcal{U}$ containing (x, y). We may restrict our attention to a countable subfamily

$\{V_{x,\alpha_n} \times W_{y,\alpha_n} : n \in \mathbb{N}\}$ of $\mathcal{U}$ containing (x, y) where $\{V_{x,\alpha_n} : n \in \mathbb{N}\}$ is a countable subfamily of $\mathcal{V}$ containing x, $\left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \subseteq \tau_1 \cup \tau_2$; and $\{W_{y,\alpha_n} : n \in \mathbb{N}\}$ is a countable subfamily of $\mathcal{W}$ containing y, $\left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\} \subseteq \sigma_1 \cup \sigma_2$, since any countable subfamily of this form will lead immediately to a countable subfamily chosen from the original cover $\mathcal{U}$. Since

$$\begin{aligned}
&\left\{\bigcap_{n\in\mathbb{N}} (V_{x,\alpha_n} \times W_{y,\alpha_n}) : (x, y) \in X \times Y\right\} \\
&= \left\{\left(\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n}\right) \times \left(\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n}\right) : x \in X, y \in Y\right\} \\
&= \left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \times \left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\},
\end{aligned}$$

and

$$\left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \subseteq \tau_1 \cup \tau_2$$

and

$$\left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\} \subseteq \sigma_1 \cup \sigma_2,$$

then

$$\left\{\bigcap_{n\in\mathbb{N}} (V_{x,\alpha_n} \times W_{y,\alpha_n}) : (x, y) \in X \times Y\right\} \subseteq \rho_1 \cup \rho_2.$$

Therefore $X \times Y$ is s-P-space. ∎

Corollary 5.3. *Let $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n\right\}$ be a collection of s-P-spaces. Then $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ is s-P-space where ρ_i is a product topology.*

Proof. It follows by induction of k. ∎

The result of Proposition 5.1 can also be extended to a collection of finite s-Lindelöf s-P-space and an s-Lindelöf space as follows.

Corollary 5.4. *Let $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots n, k \neq m, m \leq n\right\}$ be a collection of s-Lindelöf s-P-spaces and $(X_m, \tau_1^m, \tau_2^m)$ an s-Lindelöf space. Then $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ is s-Lindelöf where ρ_i is a product topology.*

Proof. It follows by induction of k and Corollary 5.3. ∎

6. Product of p-Lindelöf spaces

Definition 6.1. *A bitopological space (X, τ_1, τ_2) is called p-compact [8] (resp. p-Lindelöf [9]) if every p-open cover of X has a finite (resp. countable) subcover.*

Theorem 6.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) are p-compact spaces. Then $(X \times Y, \rho_1, \rho_2)$ is p-compact where ρ_i is a product topology.*

Proof. The proof is similar with the Proposition 5.1. ∎

The product is still invariant if we take a finite collection of p-compact spaces thus we state the following corollary.

Corollary 6.1. *Let $\{(X_k, \tau_1^k, \tau_2^k) : k = 1, \dots n\}$ be a collection of p-compact spaces. Then $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ is p-compact where ρ_i is a product topology.*

Fora and Hdeib [9] stated that every s-Lindelöf space is Lindelöf and p-Lindelöf. In the same paper, see Example 2.34, Fora and Hdeib showed that the product of any two p-Lindelöf spaces need not be p-Lindelöf. But the product of a p-Lindelöf space and p-compact space is always p-Lindelöf.

Theorem 6.2. *Let (X, τ_1, τ_2) be a p-Lindelöf space and (Y, σ_1, σ_2) a p-compact space. Then $(X \times Y, \rho_1, \rho_2)$ is p-Lindelöf where ρ_i is a product topology.*

Proof. The proof follows on using the Theorem 6.1. ∎

The above result still holds if we take a p-Lindelöf space and a collection of finite p-compact spaces as in the following corollary.

Corollary 6.2. *Let $(X_m, \tau_1^m, \tau_2^m)$ be a p-Lindelöf space and*

$$\{(X_k, \tau_1^k, \tau_2^k) : k = 1, \dots n, k \neq m, m \leq n\}$$

a collection of p-compact spaces. Then $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ is p-Lindelöf where ρ_i is a product topology.

Proof. It follows immediately by the fact that the topological product is commutative, associative and the Corollary 6.1. ∎

Definition 6.2. *Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a p-open cover of a bitopological space (X, τ_1, τ_2). Then X is said to be p-P-space if for each $x \in X$ and any countable subfamily $\{U_{x,\alpha_n} : x \in X\}$ of $\mathcal{U}$ where $x \in U_{x,\alpha_n}$ for all $n \in \mathbb{N}$ satisfying the condition $\left\{\bigcap_{n\in\mathbb{N}} U_{x,\alpha_n} : x \in X\right\} \subseteq \tau_1 \cup \tau_2$, the following hold:*

$$\left\{\bigcap_{n\in\mathbb{N}} U_{x,\alpha_n} : x \in X\right\} \cap \tau_1 \neq \emptyset \quad \text{and} \quad \left\{\bigcap_{n\in\mathbb{N}} U_{x,\alpha_n} : x \in X\right\} \cap \tau_2 \neq \emptyset.$$

Proposition 6.1. *Let (X, τ_1, τ_2) be a p-Lindelöf space p-P-space and (Y, σ_1, σ_2) a p-Lindelöf space. Then $(X \times Y, \rho_1, \rho_2)$ is p-Lindelöf where ρ_i is a product topology.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a p-open cover of $X \times Y$. Then $\mathcal{U} \subseteq \rho_1 \cup \rho_2$, $\mathcal{U} \cap \rho_1$ contains a nonempty set and $\mathcal{U} \cap \rho_2$ contains a nonempty set. We may restrict our attention to a cover $\mathcal{V} \times \mathcal{W}$ of $X \times Y$ with $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ and $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ are p-open covers of X and Y respectively where $\mathcal{V} \times \mathcal{W}$ is contained in $\mathcal{U}$. Hence $\mathcal{V} \subseteq \tau_1 \cup \tau_2$, $\mathcal{V} \cap \tau_1$ contains a nonempty set and $\mathcal{V} \cap \tau_2$ contains a nonempty set, and $\mathcal{W} \subseteq \sigma_1 \cup \sigma_2$, $\mathcal{W} \cap \sigma_1$ contains a nonempty set and $\mathcal{W} \cap \sigma_2$ contains a nonempty set. Now for each $x \in X$, let $Y_x = \{x\} \times Y$ which is homeomorphic to Y and hence Y_x is p-Lindelöf with respect to the inducted bitopology from (ρ_1, ρ_2). So Y_x is p-Lindelöf relative to $X \times Y$ and since $\{V_\alpha : \alpha \in \Delta\} \times \{W_\alpha : \alpha \in \Delta\}$ also covers Y_x, there must exists a countable p-open subcover $\{\{V_{x,\alpha_n}\} \times \{W_{x,\alpha_n}\} : n \in \mathbb{N}\}$ of

Y_x by sets which have a nonempty intersection with Y_x. Letting $H_x = \bigcap_{n\in\mathbb{N}} V_{x,\alpha_n}$, we see that H_x contains x and hence $\{H_x : x \in X\} \subseteq \tau_1 \cup \tau_2, \{H_x : x \in X\} \cap \tau_1$ contains a nonempty set and $\{H_x : x \in X\} \cap \tau_2$ contains a nonempty set since X is p-P-space. The above countable p-open subcover $\{\{V_{x,\alpha_n}\} \times \{W_{x,\alpha_n}\} : n \in \mathbb{N}\}$ actually covers $H_x \times Y$. Now the family $\{H_x : x \in X\}$ is a p-open cover of X. Since X is p-Lindel öf, there exists a countable subcover $\{H_{x_m} : m \in \mathbb{N}\}$. But then $\{\{\{V_{x_m,\alpha_n}\} \times \{W_{x_m,\alpha_n}\} : n \in \mathbb{N}\} : m \in \mathbb{N}\}$ covers $X \times Y$. Since

$$\{\{\{V_{x_m,\alpha_n}\} \times \{W_{x_m,\alpha_n}\} : n \in \mathbb{N}\} : m \in \mathbb{N}\}$$

is a countable p-open subcover, we have that $X \times Y$ is p-Lindelöf. ∎

Lemma 6.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be p-P-spaces. Then $(X \times Y, \rho_1, \rho_2)$ is p-P-space where ρ_i is a product topology.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}, \mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ and $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ be p-open cover of $X \times Y$, X and Y respectively. For each $(x, y) \in X \times Y$, let $\left\{U_{(x,y),\alpha_n} : n \in \mathbb{N}\right\}$ be a countable subfamily of $\mathcal{U}$ containing (x, y). We may restrict our attention to a countable subfamily $\{V_{x,\alpha_n} \times W_{y,\alpha_n} : n \in \mathbb{N}\}$ of $\mathcal{U}$ containing (x, y) where $\{V_{x,\alpha_n} : n \in \mathbb{N}\}$ is a countable subfamily of $\mathcal{V}$ containing x, thus $\left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \subseteq \tau_1 \cup \tau_2$, and $\left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \cap \tau_1$ contains a nonempty set, similarly, $\left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \cap \tau_2$ contains a nonempty set; and $\{W_{y,\alpha_n} : n \in \mathbb{N}\}$ is a countable subfamily of $\mathcal{W}$ containing y, thus $\left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\} \subseteq \sigma_1 \cup \sigma_2$, $\left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\} \cap \sigma_1$ contains a nonempty set, and $\left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\} \cap \sigma_2$ contains a nonempty set. Since

$$\left\{\bigcap_{n\in\mathbb{N}} (V_{x,\alpha_n} \times W_{y,\alpha_n}) : (x, y) \in X \times Y\right\} = \left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \times \left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\}$$

and $\left\{\bigcap_{n\in\mathbb{N}} V_{x,\alpha_n} : x \in X\right\} \subseteq \tau_1 \cup \tau_2$ and $\left\{\bigcap_{n\in\mathbb{N}} W_{y,\alpha_n} : y \in Y\right\} \subseteq \sigma_1 \cup \sigma_2$ satisfying the conditions stated above, then we have $\left\{\bigcap_{n\in\mathbb{N}} (V_{x,\alpha_n} \times W_{y,\alpha_n}) : (x, y) \in X \times Y\right\} \subseteq \rho_1 \cup \rho_2$, $\left\{\bigcap_{n\in\mathbb{N}} (V_{x,\alpha_n} \times W_{y,\alpha_n}) : (x, y) \in X \times Y\right\} \cap \rho_1$ contains also a nonempty set and thus $\left\{\bigcap_{n\in\mathbb{N}} (V_{x,\alpha_n} \times W_{y,\alpha_n}) : (x, y) \in X \times Y\right\}$ contains a nonempty set. Therefore $X \times Y$ is p-P-space. ∎

Corollary 6.3. *Let $\left\{\left(X_k, \tau_1^k, \tau_2^k\right) : k = 1, \ldots, n\right\}$ be a collection of p-P-spaces. Then $(\prod_{k=1}^n X_k, \rho_1, \rho_2)$ is p-P-space where ρ_i is a product topology.*

Proof. The proof follows easily by using the induction on k. ∎

The result in the Proposition 6.1 can be extended to a collection of a finite p-Lindelöf p-P-space and a p-Lindelöf space as we state in the following corollary.

Corollary 6.4. *Let* $\left\{\left(X_k,\tau_1^k,\tau_2^k\right):k=1,\dots n,k\neq m,m\leq n\right\}$ *be a collection of p-Lindelöf p-P-spaces and* (X_m,τ_1^m,τ_2^m) *a p-Lindelöf space. Then* $(\prod_{k=1}^n X_k,\rho_1,\rho_2)$ *is p-Lindelöf where ρ_i is a product topology.*

Proof. It follows by induction on k and using the Corollary 6.3. ∎

The converse of corresponding theorems, propositions and corollaries above are also true as we state in the following theorem.

Theorem 6.3. *Suppose that* $\{(X_\alpha,\tau_1^\alpha,\tau_2^\alpha):\alpha\in\Delta\}$ *be a collection of nonempty bitopological spaces. If* $\left(\prod_{\alpha\in\Delta}X_\alpha,\rho_1,\rho_2\right)$ *is ρ_i-Lindelöf (resp. Lindelöf, s-Lindelöf, p-Lindelöf, (ρ_i,ρ_j)-Lindelöf, B-Lindelöf, ρ_i-compact, compact, s-compact, p-compact, (ρ_i,ρ_j)-compact or B-compact), then each X_α is τ_i^α-Lindelöf (resp. Lindelöf, s-Lindelöf, p-Lindelöf, $\left(\tau_i^\alpha,\tau_j^\alpha\right)$-Lindelöf, B-Lindelöf, τ_i^α-compact, compact, s-compact, p-compact, $\left(\tau_i^\alpha,\tau_j^\alpha\right)$-compact or B-compact) where ρ_i is a product topology.*

Proof. Since each projection map $\pi_\alpha:\prod_{\alpha\in\Delta}X_\alpha\to X_\alpha$ is continuous open surjection, the theorem is clearly proved. ∎

Acknowledgement. The authors gratefully acknowledge that this research was partially supported by Ministry of Science, Technology and Innovations (MOSTI), Malaysia under the e-Science Grant 06-01-04-SF0115. The authors also wish to thank the referees for their constructive comments and suggestions.

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