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Coefficient Estimates and Landau-Bloch's Constant for Planar Harmonic Mappings

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Abstract. The aim of this paper is to study the properties of planar harmonic mappings. The main results are as follows. First, by using the subordination of analytic functions, a sharp coefficient estimate is obtained and several applications are given. Then two results about Landau-Bloch's constant are proved: one for planar harmonic mappings and the other for $L(f)$, where L represents the linear complex operator $L = z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}$ defined on the class of complex-valued C^1 functions in the plane and f is an open harmonic mapping.

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1. Preliminaries and main results

One of the long standing open problems in function theory is that of determining the precise value of the schlicht Landau-Bloch's constant for holomorphic mappings of the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Analogous problem of estimating the Landau-Bloch's constant for harmonic mappings has been one of the recent investigations by a number of authors [1, 3, 4, 6, 8, 9, 11, 13, 14, 18]. One of the main aims of this paper is to use subordination as a tool to derive a sharp coefficient estimate for harmonic mappings and as a consequence, we obtain improved estimates for Landau-Bloch's constant both for harmonic and biharmonic mappings.

A sense-preserving (planar) harmonic mapping f of $\mathbb{D}$ is a solution of the elliptic differential equation

$$\overline{f_{\overline{z}}(z)} = \omega(z) f_z(z)$$

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where ω, known as the analytic dilatation of f, is an analytic function in $\mathbb{D}$ with $\omega(\mathbb{D}) \subset \mathbb{D}$. One of the useful representations of sense-preserving harmonic mappings f in $\mathbb{D}$ is that $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$. In this case, $\omega(z) = g'(z)/h'(z)$ and the Jacobian

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2 = |h'|^2(1 - |\omega|^2)$$

is positive.

For harmonic mappings f of $\mathbb{D}$, we use the following standard notations:

$$\Lambda_f(z) = \max_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = \big| |f_z(z)| - |f_{\overline{z}}(z)| \big|.$$

Then $J_f = \lambda_f \Lambda_f$ if $J_f \geq 0$.

We say that $f \in \mathcal{H}_M(\mathbb{D})$ if f is harmonic in $\mathbb{D}$ and $|f(z)| \leq M$ for $z \in \mathbb{D}$. We use the canonical decomposition $f = h + \overline{g}$ with the analytic functions h and g having the power series

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Theorem 1.1. *Suppose $f \in \mathcal{H}_M(\mathbb{D})$. Then $|a_0| \leq M$ and for each $n \geq 1$,*

$$|a_n| + |b_n| \leq \frac{4M}{\pi}. \tag{1.1}$$

The estimate (1.1) is sharp for any $n \geq 1$. For each $n \geq 1$, the extremal function is

$$f_n(z) = \frac{2M\alpha}{\pi} \arg\left(\frac{1 + \beta z^n}{1 - \beta z^n}\right), \quad |\alpha| = |\beta| = 1$$

or $f(z) \equiv M$.

We shall prove the theorem in Section 2, and the proof depends on the principle of subordination. The inequality (1.1) for $n = 1$ can be obtained as a consequence of the harmonic version of the Schwarz's lemma due to Chen, Gauthier and Hengartner [3, Theorem 1(1)] (see also Heinz [12, Lemma]). In [18, Theorem 4] (see also [9, Lemma 3]) a weaker estimate, namely, $|a_n| + |b_n| \leq 2M$ for $n \geq 1$, was used to obtain estimates for Bloch constants for planar harmonic mappings. We recall that a four times continuously differentiable complex-valued mapping F of $\mathbb{D}$ is *biharmonic* if and only if ΔF satisfies the biharmonic equation $\Delta(\Delta F) = 0$, where $\Delta F = 4F_{z\overline{z}}$ denotes the Laplacian of F. It is easy to see that if F is biharmonic in $\mathbb{D}$ then there exist harmonic functions G and K of $\mathbb{D}$ such that $F = |z|^2 G + K$ (cf. [1, 2, 4–7])

In view of the sharp estimate from Theorem 1.1, we can obtain two Landau's theorems for planar biharmonic mappings improving the earlier results of Abdulhadi and Abu Muhanna [1] and Liu [13]. It is worth recalling that neither the normalization $f_z(0) = 1$ nor the normalization $J_f(0) = 1$ gives us a Bloch theorem for general univalent harmonic mappings. There are examples where no Bloch theorem is possible for harmonic mappings even with both of these normalizations (cf. [3]).

Theorem 1.2. *Let $F = |z|^2G + K$ be a biharmonic mapping of $\mathbb{D}$ such that $F(0) = G(0) = K(0) = J_F(0) - 1 = 0$, $|G(z)| \leq M_1$ and $|K(z)| \leq M_2$. Then there is a constant $0 < \rho_2 < 1$ so that F is univalent in $|z| < \rho_2$. In specific ρ_2 satisfies*

$$\frac{\pi}{4M_2} - 2\rho_2 M_1 - \frac{4M_1\rho_2^2}{\pi(1-\rho_2)^2} - \frac{\sqrt{2(M_2^2-1)}\,(2\rho_2-\rho_2^2)}{(1-\rho_2)^2} = 0$$

and $F(\mathbb{D}_{\rho_2})$ contains a disk $\mathbb{D}_{R_2}$, where

$$R_2 = \frac{\pi}{4M_2}\rho_2 - \frac{\rho_2^2(4M_1\rho_2 + \pi\sqrt{2(M_2^2-1)})}{\pi(1-\rho_2)}.$$

In particular, if we set $M_1 = M_2 = M$, we easily obtain the following corollary which improves the results of Abdulhadi and Abu Muhanna [1, Theorem 1] and Liu [13, Corollary 2.8].

Corollary 1.1. *Let $F = |z|^2G + K$ be a biharmonic mapping of $\mathbb{D}$ such that $F(0) = G(0) = K(0) = J_F(0) - 1 = 0$, and G and K are both harmonic in $\mathbb{D}$, and bounded by $M \geq 1$. Then there is a constant $0 < \rho_2 < 1$ so that F is univalent in $|z| < \rho_2$. In specific ρ_2 satisfies*

$$\frac{\pi}{4M} - 2\rho_2 M - \frac{4M\rho_2^2}{\pi(1-\rho_2)^2} - \frac{\sqrt{2(M^2-1)}\,(2\rho_2-\rho_2^2)}{(1-\rho_2)^2} = 0$$

and $F(\mathbb{D}_{\rho_2})$ contains a disk $\mathbb{D}_{R_2}$, where

$$R_2 = \frac{\pi\rho_2}{4M} - \frac{\rho_2^2(4M\rho_2 + \pi\sqrt{2(M^2-1)})}{\pi(1-\rho_2)}.$$

Since $\pi/2 > 1$, clearly this corollary is an improvement of Liu [13, Corollary 2.8] (see Table 1).

Table 1. The left half columns refer to Corollary 1.1 and the right half columns refer to Corollary 2.8 in [13]

M	ρ_2	R_2	M	$\widetilde{r_2}$	$\widetilde{\sigma_2}$
1	0.288266781	0.18355165	1	0.224701365	0.147213046
2	0.04203247	0.0117912501	2	0.041014954	0.0115219145
3	0.018310479	0.0034036769	3	0.018119678	0.0033698409
4	0.010238145	0.0014246736	4	0.010178704	0.0014167515
5	0.006535294	0.0007269224	5	0.006511112	0.0007243414
6	0.004532132	0.0004199061	6	0.004520512	0.000418872
7	0.003327	0.0002641443	7	0.003320741	0.0002636667

Applying Theorem 1.1 and the proof of Theorem 1.2, we can easily obtain the following version of Landau's theorem for biharmonic mappings which clearly improves the recent result of Liu [13, Theorem 2.10] and so we omit its proof.

Theorem 1.3. *Let $F = |z|^2G + K$ be a biharmonic mapping of $\mathbb{D}$ such that $F(0) = G(0) = K(0) = \lambda_F(0) - 1 = 0$, $|G(z)| \leq M_1$ and $|K(z)| \leq M_2$ in $\mathbb{D}$. Then there is a* *constant* $0<\rho_3<1$ *so that* F *is univalent in* $|z|<\rho_3$. *In specific* ρ_3 *satisfies*

$$1-2\rho_3M_1-\frac{4M_1\rho_3^2}{\pi(1-\rho_3)^2}-\frac{\sqrt{2(M_2^2-1)}\,(2\rho_3-\rho_3^2)}{(1-\rho_3)^2}=0$$

and $F(\mathbb{D}_{\rho_3})$ *contains a disk* $\mathbb{D}_{R_3}$, *where*

$$R_2=\rho_3-\frac{\rho_3^2\left(4M_1\rho_3+\pi\sqrt{2(M_2^2-1)}\right)}{\pi(1-\rho_3)}.$$

Also, similar discussions show that Theorems 1.1 and 1.2 of [4] can be improved by applying Theorem 1.1. In addition to these results, in Theorem 3.2, we obtain an estimate on Bloch's constant of the linear operator $L(f)$ for open harmonic mappings f. Here L denotes the complex-operator

$$L=z\frac{\partial}{\partial z}-\overline{z}\frac{\partial}{\partial \overline{z}}. \tag{1.2}$$

We see that it is linear and satisfies the usual product rule:

$$L(af+bg)=aL(f)+bL(g)\quad\text{and}\quad L(fg)=fL(g)+gL(f),$$

where a,b are complex constants, f and g are C^1 functions. In addition, the operator L possesses a number of interesting properties, e.g. L preserves both *harmonicity* and *biharmonicity*. Many other basic properties are stated for instance in [15] (see also [2, 4]).

2. Proofs of Theorems 1.1 and 1.2

In many cases, the subordination family associated with an individual function or a family plays a significant role. For two analytic functions f,g defined on $\mathbb{D}$, we say that f is *subordinate* to g, denoted by $f\prec g$, or $f(z)\prec g(z)$, if there exists a function $\omega\in\mathcal{B}_0$ such that $f(z)=g(\omega(z))$ in $\mathbb{D}$. Here $\mathcal{B}_0$ denotes the class of Schwarz functions, i.e. analytic maps ψ of $\mathbb{D}$ into itself with the normalization $\psi(0)=0$. When g is univalent in $\mathbb{D}$, $f\prec g$ if and only if $f(0)=g(0)$ and $f(\mathbb{D})\subset g(\mathbb{D})$.

Proof of Theorem 1.1. Without loss of generality, we assume $f(z)=h(z)+\overline{g(z)}$ and $|f(z)|<1$. For $\theta\in[0,2\pi)$, let

$$v_\theta(z)=\operatorname{Im}(e^{i\theta}f(z))$$

and observe that

$$v_\theta(z)=\operatorname{Im}(e^{i\theta}h(z)+\overline{e^{-i\theta}g(z)})=\operatorname{Im}(e^{i\theta}h(z)-e^{-i\theta}g(z)).$$

Because $|v_\theta(z)|<1$, it follows that

$$e^{i\theta}h(z)-e^{-i\theta}g(z)\prec K(z)=\lambda+\frac{2}{\pi}\log\left(\frac{1+z\xi}{1-z}\right),$$

where $\xi=e^{-i\pi\operatorname{Im}(\lambda)}$ and $\lambda=e^{i\theta}h(0)-e^{-i\theta}g(0)$. The superordinate function $K(z)$ maps $\mathbb{D}$ onto a convex domain with $K(0)=\lambda$ and $K'(0)=\frac{2}{\pi}(1+\xi)$, and therefore, by a theorem of Rogosinski [17, Theorem 2.3] (see also [10, Theorem 6.4]), it follows that

$$|a_n-e^{-2i\theta}b_n|\le\frac{2}{\pi}|1+\xi|\le\frac{4}{\pi}\qquad\text{for } n=1,2,\ldots$$

and the desired inequality (1.1) is a consequence of the arbitrariness of θ in $[0, 2\pi)$. For the proof of sharpness part, consider the functions

$$f_n(z) = \frac{2M\alpha}{\pi}\mathrm{Im}\left(\log\frac{1+\beta z^n}{1-\beta z^n}\right), \quad |\alpha| = |\beta| = 1,$$

whose values are confined to a diametral segment of the disk $\mathbb{D}_M = \{z\colon |z| < M\}$. Also,

$$f_n(z) = \frac{2M\alpha}{i\pi}\left(\sum_{k=1}^{\infty}\frac{1}{2k-1}(\beta z^n)^{2k-1} - \sum_{k=1}^{\infty}\frac{1}{2k-1}(\overline{\beta}\overline{z}^n)^{2k-1}\right),$$

which gives

$$|a_n| + |b_n| = \frac{4M}{\pi}.$$

The proof of the theorem is complete. ∎

Proof of Theorem 1.2. Suppose that $F = |z|^2G + K$ is biharmonic with $F(0) = G(0) = K(0) = J_F(0) - 1 = 0$, $|G(z)| \le M_1$, $|K(z)| \le M_2$, where

$$G(z) = g_1 + \overline{g_2} := \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty}\overline{b_n}\overline{z}^n$$

and

$$K(z) = k_1 + \overline{k_2} := \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty}\overline{d_n}\overline{z}^n$$

are harmonic in $\mathbb{D}$. Now, for fixed $0 < \rho < 1$, choose z_1, z_2 with $z_1 \neq z_2$, $|z_1| < \rho$ and $|z_2| < \rho$. It follows from the standard arguments (eg. see the proof of [1, Theorem 1]) that

$$|F(z_1) - F(z_2)| \ge |z_1 - z_2|\left\{\lambda_K(0) - 2\rho M_1 - \rho^2\sum_{n=1}^{\infty} n(|a_n| + |b_n|)\rho^{n-1}\right.$$
$$\left.- \sum_{n=2}^{\infty} n(|c_n| + |d_n|)\rho^{n-1}\right\}.$$

We observe that $J_K(0) = |c_1|^2 - |d_1|^2 = J_F(0) = 1$ and therefore, we have

$$\lambda_K(0) = \frac{1}{\Lambda_K(0)} = \frac{1}{|c_1| + |d_1|},$$

which, by Theorem 1.1, is bigger than or equal to $\pi/(4M_2)$. In view of Theorem 1.1 and [6, Theorem 1.5], we have

$$|a_n| + |b_n| \le \frac{4M_1}{\pi} \quad (n \ge 1)$$

and

$$|c_n| + |d_n| \le \sqrt{2(M_2^2 - 1)} \quad (n \ge 2),$$

respectively. Using these inequalities, as in the proof of [1, Theorem 1], we see that $|F(z_1) - F(z_2)| > 0$ if $0 < \rho < \rho_2$, where ρ_2 is the root of the following equation:

$$\frac{\pi}{4M_2} - 2\rho M_1 - \frac{4M_1}{\pi}\frac{\rho^2}{(1-\rho)^2} + \sqrt{2(M_2^2 - 1)}\left(\frac{1}{(1-\rho)^2} - 1\right) = 0$$

and the univalency of the biharmonic function F follows.

For $|z|=\rho_2$, it follows that

$$\begin{aligned}|F(z)| &\geq |c_1z+\overline{d_1}\overline{z}|-\rho_2^2\sum_{n=1}^{\infty}(|a_n|+|b_n|)\rho_2^n-\sum_{n=2}^{\infty}(|c_n|+|d_n|)\rho_2^n\\ &\geq \frac{\pi}{4M_2}\rho_2-\frac{4M_1}{\pi}\frac{\rho_2^3}{1-\rho_2}-\sqrt{2(M_2^2-1)}\frac{\rho_2^2}{1-\rho_2}=R_2.\end{aligned}$$

The proof of the theorem is complete. ∎

3. Bloch's constant for planar harmonic mappings

In [14], Liu proved the following Lemma.

Lemma 3.1. ([13, Lemma 2.4] and [14, Lemma 2.1]) *Suppose that f is a harmonic mapping of $\mathbb{D}$ with $f(0)=\lambda_f(0)-1=0$. If $\Lambda_f\leq\Lambda$ for $z\in\mathbb{D}$, then*

$$|a_n|+|b_n|\leq\frac{\Lambda^2-1}{n\Lambda},\ \ n=2,3,\ldots.$$

Above estimates are sharp for all $n=2,3,\ldots,$ with the extremal functions

$$f_n(z)=\Lambda^2z-\int_0^z\frac{(\Lambda^3-\Lambda)\,dz}{\Lambda+z^{n-1}}.$$

As applications of Lemma 3.1, several estimates on Bloch's constant were obtained in [14], which are generalizations of the corresponding results in [3, 11], respectively. For example, the following was proved, which is an improvement of [3, Theorem 1].

Let $\mathbf{Har}(\mathbb{D},\mathbb{D})$ denote the class of all harmonic mappings of $\mathbb{D}$ satisfying $f(0)=0$ and $f(\mathbb{D})\subset\mathbb{D}$. Using the principle of subordination of analytic functions, we know that for any $f\in\mathbf{Har}(\mathbb{D},\mathbb{D})$,

$$\Lambda_f(z)\leq\frac{4}{\pi(1-|z|^2)}\ \ \text{for } z\in\mathbb{D}, \tag{3.1}$$

which is an improved version of Schwarz's lemma for harmonic mappings [3, 12, 18]. Moreover, the inequality (3.1) coincides with the result of Colonna [8] who proved that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\Lambda_f(z)\leq\frac{4}{\pi}.$$

By applying (3.1), we can improve [14, Theorem 2.3] as follows.

Theorem 3.1. *Let $f\in\mathcal{H}_M(\mathbb{D})$ with $f(0)=f_{\overline{z}}(0)=f_z(0)-1=0$. Then f is univalent in the disk $\mathbb{D}_{r_0}$ with $r_0=\phi(M_r)$ and $f(\mathbb{D}_{r_0})$ contains a univalent disk of radius at least*

$$R_0:=\max_{0<r<1}\psi(M_r), \tag{3.2}$$

where

$$\phi(x)=\frac{rx}{(x^2+x-1)},\ \ \psi(x)=r\left[1+\left(\frac{x^2-1}{x}\right)\log\left(\frac{x^2-1}{x^2+x-1}\right)\right]$$

and

$$M_r = \frac{4M}{\pi(1-r^2)}.$$

Proof. If we set

$$F(z) = \frac{f(rz)}{r}, \tag{3.3}$$

then F is a harmonic mapping of $\mathbb{D}$, and $\lambda_F(0) = 1$. Therefore by (3.1), we have

$$\Lambda_F = \Lambda_f(zr) \leq \frac{4M}{\pi(1-r^2)} = M_r.$$

Thus, by [14, Theorem 2.2], we obtain that F is univalent in the disk $|z| < \frac{r_0}{r}$, $r_0 = \phi(M_r)$, and $F(\{z : |z| < \frac{r_0}{r}\})$ contains a univalent disk $|w| < \frac{R_0}{r}$, $R_0 = \psi(M_r)$. Hence f is univalent in the disk $\mathbb{D}_{r_0}$ and $f(\mathbb{D}_{r_0})$ contains a univalent disk $\mathbb{D}_{R_0}$. The existence of (3.2) follows from the fact that

$$\lim_{r\to 0+} \psi(M_r) = \lim_{r\to 1-} \psi(M_r) = 0.$$

The proof is complete. ∎

Let $r = \frac{\sqrt{2}}{2}$ in (3.3). Then f is univalent in the disk $\mathbb{D}_{r_0}$ with $r_0 = \phi(8M/\pi)$ and $f(\mathbb{D}_{r_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with $R_0 := \psi(8M/\pi)$, where

$$\phi(x) = \frac{x}{\sqrt{2}(x^2+x-1)} \quad \text{and} \quad \psi(x) = \frac{1}{\sqrt{2}}\left[1 + \left(\frac{x^2-1}{x}\right)\log\left(\frac{x^2-1}{x^2+x-1}\right)\right].$$

Liu [14, Theorem 2.3] obtained the above result with r_0 and R_0 by using $r_2 = \phi(4.55M)$ and $\sigma_2 = \psi(4.55M)$, respectively (see Table 2). We remark that r_0 in Theorem 3.1 is positive only when $M > \frac{\pi(\sqrt{5}-1)}{16} \approx 0.242701$. It is worth pointing out that r_0 in [14, Theorem 2.3] is positive for $M > \frac{\sqrt{5}-1}{9.1} \approx 0.135832$. By the normalization $f_{\overline{z}}(0) = f_z(0) - 1 = 0$, we easily observe that the corresponding bound M in each of [14, Theorem 2.3], [3, Theorem 3] and Theorem 3.1 satisfies the condition $M \geq \frac{\pi}{4}$. Thus, as demonstrated for example in Table 2, Theorem 3.1 improves result of Liu [14, Theorem 2.3] and hence, the result of Chen *et al.* [3, Theorem 3].

Table 2. The left half columns refer to Theorem 3.1 and the right half columns refer to Theorem 2.3 in [14]

M	$r_0 = \phi(8M/\pi)$	$R_0 = \psi(8M/\pi)$	M	$r_2 = \phi(4.55M)$	$\sigma_2 = \psi(4.55M)$
1	0.22421	0.12629	1	0.13266	0.07092
2	0.11992	0.06367	2	0.07078	0.03663
3	0.08311	0.04328	3	0.04851	0.02483

It is well-known that f is an open map (i.e. it maps every open subset of $\mathbb{D}$ to an open set in $\mathbb{C}$) which is locally one-to-one in $\mathbb{D}$ except possibly at isolated points where it behaves locally like analytic functions near zeros of derivatives. To consider an open harmonic mapping f, we call f *univalent* or locally univalent in $\mathbb{D}$ if it is *one-to-one* or locally one-to-one in $\mathbb{D}$, respectively.

Liu [14, Theorem 2.6] proved that for open harmonic mappings f of $\mathbb{D}$ normalized by $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$, $f(\mathbb{D})$ contains a univalent disk of radius at least $R \approx 0.027735$ which is an improvement of earlier known results [3, Theorem 7] and [11, Theorem 2.5]. Next we aim to obtain a similar result but for $L(f)$ defined by (1.2).

In our next result, we determine an estimate for the Bloch constant of $L(f)$ when f runs on the class of open harmonic mappings. It is worth pointing out that (see [2, Corollary 1(3)]) the operator $L(f)$ for biharmonic functions behaves much like zf' for analytic functions, for example in the sense that for f univalent and biharmonic, f is starlike in $\mathbb{D}$ if and only if $\mathrm{Re}\,(L(f)(z)/f(z)) \geq 0$ in $\mathbb{D}$.

Theorem 3.2. *Let f be an open harmonic mapping of $\mathbb{D}$ normalized by $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$. Then $L(f)(\mathbb{D})$ contains a univalent disk of radius at least*

$$R = \max_{0<r<1} \varphi(r) \tag{3.4}$$

where

$$\varphi(r) = \frac{r}{\sqrt{2}} \frac{1 - \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}}{1 + \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}}, \quad M_r = \frac{2(1+r)}{1-r}.$$

Moreover, $L(f)(\mathbb{D})$ contains a univalent disk of radius at least $R \approx 0.0143328$.

Proof. It is known that for any $r \in (0,1)$, f is K_r-quasiregular on $\mathbb{D}_r$ (cf. [16]), where $K_r = \frac{1+r}{1-r}$. This implies that

$$\frac{\Lambda_f}{\lambda_f} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \leq \frac{1+r}{1-r} = K_r.$$

Let $G(z) = r^{-1} f(rz)$ for $z \in \mathbb{D}$. Then there exists a point $z_0 \in \mathbb{D}$ such that for $z \in \mathbb{D}$,

$$(1 - |z|^2)\lambda_G(z) \leq (1 - |z_0|^2)\lambda_G(z_0) = M,$$

where $M \geq 1$.

Let ϕ be a Möbius transformation of $\mathbb{D}$ onto itself with $\phi(0) = z_0$. Define F by

$$F(\xi) = G(\phi(\xi))/M \text{ for } \xi \in \mathbb{D}.$$

Then, we see that

$$(1 - |\xi|^2)\lambda_F(\xi) = \frac{(1 - |\phi(\xi)|^2)\lambda_G(\phi(\xi))}{M},$$

which gives $\lambda_F(0) = 1$ and for $\xi \in \mathbb{D}$,

$$(1 - |\xi|^2)\lambda_F(\xi) \leq 1.$$

Let $P(w) = \sqrt{2} F(w/\sqrt{2})$ for $w \in \mathbb{D}$. Then P is also K_r-quasiregular. Moreover, $\lambda_P(0) = \lambda_F(0) = 1$ and for $w \in \mathbb{D}$,

$$\Lambda_P(w) \leq K_r \lambda_P(w) = K_r \lambda_F(w/\sqrt{2}) < 2K_r = M_r.$$

Finally, we let

$$T(\zeta) = P(\zeta) - P(0) = \sum_{n=1}^{\infty} a_n \zeta^n + \sum_{n=1}^{\infty} b_n \overline{\zeta}^n \quad \text{for} \quad \zeta \in \mathbb{D}.$$

Using Lemma 3.1, we have

$$|a_n| + |b_n| \leq \frac{M_r^2 - 1}{nM_r},\ n = 2, 3, \ldots.$$

Now, to prove the univalence of $L(T)$, we adopt the standard procedure. For $\zeta_1 \neq \zeta_2$ in $\mathbb{D}_\rho$ $(0 < \rho < 1)$, by Lemma 3.1, we have

$$\begin{aligned}
|L(T)(\zeta_1) - L(T)(\zeta_2)| &= \left| \int_{[\zeta_1,\zeta_2]} L(T)_\zeta \, d\zeta + L(T)_{\overline{\zeta}} \, d\overline{\zeta} \right| \\
&\geq \left| \int_{[\zeta_1,\zeta_2]} T_\zeta(0) \, d\zeta - T_{\overline{\zeta}}(0) \, d\overline{\zeta} \right| \\
&\quad - \left| \int_{[\zeta_1,\zeta_2]} \zeta T_{\zeta\zeta}(\zeta) \, d\zeta - \overline{\zeta} T_{\overline{\zeta}\overline{\zeta}}(\zeta) \, d\overline{\zeta} \right| \\
&\quad - \left| \int_{[\zeta_1,\zeta_2]} (T_\zeta(\zeta) - T_\zeta(0)) \, d\zeta - (T_{\overline{\zeta}}(\zeta) - T_{\overline{\zeta}}(0)) \, d\overline{\zeta} \right| \\
&\geq |\zeta_1 - \zeta_2| \left\{ 1 - \frac{M_r^2 - 1}{M_r} \sum_{n=2}^{\infty} \rho^{n-1} \right. \\
&\quad \left. - \frac{M_r^2 - 1}{M_r} \sum_{n=2}^{\infty} (n-1)\rho^{n-1} \right\} \\
&\geq |\zeta_1 - \zeta_2| \left[1 - \frac{M_r^2 - 1}{M_r} \frac{\rho}{1 - \rho} - \frac{M_r^2 - 1}{M_r} \frac{\rho}{(1 - \rho)^2} \right].
\end{aligned}$$

Elementary calculations show that

$$\rho_1(r) = 1 - \sqrt{1 - \frac{1}{1 + M_r - \frac{1}{M_r}}}$$

is the unique root of the equation

$$1 - \frac{M_r^2 - 1}{M_r} \frac{\rho}{1 - \rho} - \frac{M_r^2 - 1}{M_r} \frac{\rho}{(1 - \rho)^2} = 0$$

and hence, $L(T)$ is univalent in $\mathbb{D}_{\rho_1(r)}$.

Since for any ζ with $|\zeta| = \rho_1(r)$,

$$\begin{aligned}
|L(T)(\zeta)| &= |\zeta T_\zeta - \overline{\zeta} T_{\overline{\zeta}}| \\
&\geq |\zeta T_\zeta(0) - \overline{\zeta} T_{\overline{\zeta}}(0)| - |\zeta(T_\zeta - T_\zeta(0)) - \overline{\zeta}(T_{\overline{\zeta}} - T_{\overline{\zeta}}(0))| \\
&\geq \rho_1(r) \left(1 - \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \rho_1(r)^{n-1} \right) \\
&\geq \rho_1(r) \left(1 - \frac{M_r^2 - 1}{M_r} \frac{\rho_1(r)}{1 - \rho_1(r)} \right)
\end{aligned}$$

$$= \frac{1 - \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}}{1 + \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}},$$

we see that the existence of R in (3.4) follows from $L(T)(0) = 0$ and

$$\lim_{r\to 0+} \frac{r}{\sqrt{2}} \frac{1 - \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}}{1 + \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}} = \lim_{r\to 1-} \frac{r}{\sqrt{2}} \frac{1 - \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}}{1 + \sqrt{1 - \frac{1}{1+M_r - \frac{1}{M_r}}}} = 0.$$

We see that $R = \max_{0<r<1} \varphi(r) = \varphi(r_0) \approx 0.0143328$, where $r_0 \approx 0.41796$ (see Figure 1). ∎

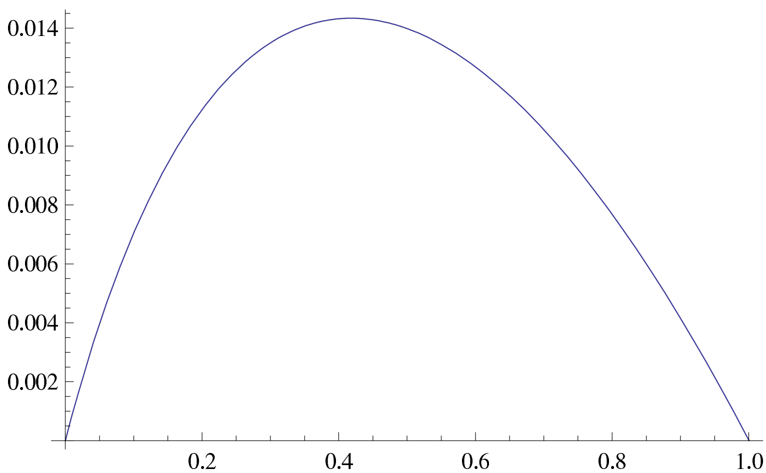

Figure 1. Graph of $\varphi(r)$ on $(0, 1)$

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