Tenacity and Rupture Degree of Permutation Graphs of Complete Bipartite Graphs

1FENGWEI LI, 2QINGFANG YE AND 3XUELIANG LI

1, 2Department of Mathematics, Shaoxing University, Shaoxing Zhejiang 312000, P. R. China
3Center for Combinatorics and LPMC, Nankai University Tianjin 300071, P. R. China
1fengwei.li@eyou.com, 2fqy-y@163.com, 3x.li@eyou.com

Abstract. Computer or communication networks are so designed that they do not easily get disrupted under external attack and, moreover, these are easily reconstructed when they do get disrupted. These desirable properties of networks can be measured by various parameters such as connectivity, toughness, tenacity and rupture degree. Among these parameters, tenacity and rupture degree are comparatively better parameters to measure the vulnerability of networks. In this paper, the authors give the exact values for the tenacity and rupture degree of permutation graphs of complete bipartite graphs.

2010 Mathematics Subject Classification: 05C40, 05C90, 68R10, 94C15

Keywords and phrases: Rupture degree, tenacity, permutation graph.

1. Introduction

Throughout this paper, a graph $G = (V, E)$ always means a simple connected graph with vertex set $V$ and edge set $E$. We use Bondy and Murty [1] for terminology and notations not defined here. A set $S \subseteq V(G)$ is a cut set of $G$, if either $G - S$ is disconnected or $G - S$ has only one vertex.

Measures of the vulnerability of graphs are currently of growing interest among graph theorists and network designers. Among vulnerability parameters, much have been done recently on the toughness, binding number of different classes of graphs since these parameters are more sensitive to the structure of the graph than is the connectivity of the graph. In [4], Guichard et al. given the integrity, toughness, and binding number for permutation graphs of complete and complete bipartite graphs.

In the following two definitions, $m(G - S)$, and $\omega(G - S)$, respectively, denotes the order of the largest component and number of components in $G - S$. 
The tenacity of a graph $G$, $T(G)$, which is defined by Cozzens in [3], is defined as

$$T(G) = \min \left\{ \frac{|S| + m(G - S)}{\omega(G - S)} : S \subseteq V(G) \text{ is a vertex cut set of } G \right\}.$$  

The rupture degree of a noncomplete graph $G$, $r(G)$, introduced by Li, Zhang and Li in [7], is defined as

$$r(G) = \max \{ \omega(G - S) - |S| - m(G - S) : S \subseteq V(G) \text{ is a vertex cut set of } G \}.$$  

In particular, the tenacity and rupture degree of a complete graph $K_n$ is defined to be $n$ and $1 - n$ respectively.

Clearly, of all the above parameters, tenacity and rupture degree are comparatively appropriate for measuring the vulnerability of networks. Similarly to the relation between the toughness and scattering number, the rupture degree and tenacity also differ in showing the vulnerability of networks. This can be shown as follows. Consider the graphs $G_1$ and $G_2$ in Figure 1. It is not difficult to check that $T(G_1) = T(G_2) = 1/2$, but $r(G_1) = 3$ and $r(G_2) = 4$. Clearly $r(G_1) \neq r(G_2)$. On the other hand, we consider graphs $G_3 = K_1 + (K_{n-b-1} \cup E_b)$ and $G_4 = K_2 + (K_{n-b-3} \cup E_{b+1})$, it is obvious that $r(G_3) = r(G_4)$, but $T(G_3) \neq T(G_4)$ unless $n = 2b + 1$, where $b$ is a positive integer. Hence rupture degree is a better parameter for distinguishing the vulnerability of these two graphs $G_1$ and $G_2$, but the tenacity is a better parameter for distinguishing the vulnerability of these two graphs $G_3$ and $G_4$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1}
\caption{Graphs $G_1$ and $G_2$.}
\end{figure}

It is easy to see that the higher the tenacity (the less the rupture degree) of a network the more stable it is considered to be.

In [2], the authors introduced permutation graphs and proceeded to characterize those which are planar.

For a graph $G$ with $n$ vertices labelled $1, 2, \cdots, n$, $n \geq 4$, and a permutation $\alpha \in S_n$, the symmetric group on the $n$ symbols $\{1, 2, \cdots, n\}$, the $\alpha$-permutation graph of $G$, $P_\alpha(G)$ consists of two disjoint copies of $G$, $G_x$ and $G_y$, along with the $n$ edges obtained by join $x_i$ in $G_x$ with $y_{\alpha(i)}$ in $G_y$, $i = 1, 2, \cdots, n$. 
It is well known that permutation graphs have high connectivity properties, as is shown in [9] and [10]. As special graphs, some vulnerability parameters of permutation graphs of complete bipartite graphs have been determined in [4]. In [6], we give the following decision problem.

**Problem. Not** \( \text{r-rupture} \)

**Instance:** An incomplete connected graph \( G \), and an integer \( r \).

**Question:** Does there exist an \( X \subset V(G) \) with \( \omega(G - X) \geq 2 \) such that \( \omega(G - X) > |X| + m(G - X) + r \)?

And by this decision problem, we proved that computing the rupture degree of a graph is \( \text{NP-hard} \) in general and so is the problem of determining the tenacity of a graph [8], so it is an interesting problem to determine these two parameters for some special graphs.

In this paper, formulas for computing the rupture degree and tenacity for permutation graphs of complete bipartite graphs are determined.

2. Tenacity and rupture degree of the permutation graphs of complete bipartite graphs

In this section, we fix our attention on permutation graph of complete bipartite graph, \( P_\alpha(K_{m,n}) \). Assume that \( m \leq n \) and that \( M \) and \( N \) are the sets of the partitions of size \( m \) and \( n \) respectively. Furthermore, assume that the vertices of \( M \) are labelled 1, 2, \( \cdots \), \( m \) and that vertices of \( N \) are labelled \( m + 1 \), \( m + 2 \), \( \cdots \), \( m + n \).

For the permutation graph \( P_\alpha(K_{m,n}) \), let \( M_x \) and \( M_y \) denote the partitions of the first copy of \( K_{m,n} \), \( N_x \) and \( N_y \) denote the partitions of the second copy of \( K_{m,n} \), and let \( q \) denote the number of vertices in \( M_x \) that are joined by permutation edges to vertices in \( M_y \). It is well known that the connectivity, toughness, integrity and the binding number of \( P_\alpha(K_{m,n}) \) can be expressed in terms of the parameters \( m \), \( n \), and/or \( q \) as follows.

**Theorem 2.1.** [4] For \( \alpha \in S_{m+n} \), and \( m \leq n \)

\[
 t(P_\alpha(K_{m,n})) = \begin{cases} 
 \frac{2m}{m+n-q} & \text{if } q < \frac{n^2+m^2}{n+3m} \\
 \frac{n+m}{n+q} & \text{if } q \geq \frac{n^2+m^2}{n+3m}. 
\end{cases}
\]

**Theorem 2.2.** [4] For \( \alpha \in S_{m+n} \), and \( m \leq n \)

\[
 I(P_\alpha(K_{m,n})) = \begin{cases} 
 2m + 1 & \text{if } m = n \text{ and } q \in \{0, m\} \\
 2m + 2 & \text{otherwise}. 
\end{cases}
\]

**Theorem 2.3.** [2] For \( \alpha \in S_{m+n} \), and \( m \leq n \)

\[
 b(P_\alpha(K_{m,n})) = \begin{cases} 
 \frac{n+q}{q} & \text{if } q < \frac{nm}{m+n-1} \\
 \frac{2m+2n-1}{m+2n-1} & \text{if } \frac{nm}{m+2n-1} \leq q < \frac{m^2+3mn-2m}{4n+2m-2} \\
 \frac{2m+3mn-2m}{4n+2m-2} & \text{if } \frac{m^2+3mn-2m}{4n+2m-2} \leq q. 
\end{cases}
\]

**Theorem 2.4.** [9] For \( \alpha \in S_{m+n} \), and \( m \leq n \), \( \kappa(P_\alpha(K_{m,n})) = m + 1 \).
In the following, we determine the rupture degree and tenacity of the permutation graph of complete bipartite graphs in terms of the parameters $m$, $n$ and/or $q$.

In the proofs of the remaining theorems we will use the following definitions and observations. Let $M'_{x}$ be the set of vertices in $M_{x}$ that are joined by permutation edges to vertices in $M_{y}$ and let $M'_{y}$ be these vertices in $M_{y}$. So $|M'_{x}| = |M'_{y}| = q$. Let $M''_{x} = M_{x} - M'_{x}$ and $M''_{y} = M_{y} - M'_{y}$ and thus $|M''_{x}| = |M''_{y}| = m - q$. Now the vertices in $M''_{x}$ are adjacent to vertices in $N_{y}$ by permutation edges, we call this vertex set $N''_{x}$. Similarly define $N''_{y}$ to be the set of vertices in $N_{x}$ adjacent to the vertices in $M''_{y}$ by permutation edges. Thus $|N''_{x}| = |N''_{y}| = m - q$. Finally let $N'_{x} = N_{x} - N''_{x}$ and $N'_{y} = N_{y} - N''_{y}$, so $|N'_{x}| = |N'_{y}| = n - m + q$. Note that since $0 \leq q \leq m$, some of these sets may be empty. Let $K = \{M'_{x}, M'_{y}, M''_{x}, N'_{x}, N''_{x}, M'_{y}, N'_{y}, N''_{y}\}$. The relationship among these sets in $K$ is shown in Figure 2.

![Figure 2. Relationship among the sets in K.](image)

**Remark 2.1.** It is easy to see that, when $m = n = 1$, whether $q = 1$ or $q = 0$, the two graphs are isomorphic. So under this condition, we assume that $q = 0$.

To prove our main result we first give a lemma.

**Lemma 2.1.** For $\alpha$ in $S_{m+n}$ and $m \leq n$, there exists a vertex cut set $S$ of graph $P_{\alpha}(K_{m,n})$ with

$$T(P_{\alpha}(K_{m,n})) = \frac{|S| + m(P_{\alpha}(K_{m,n}) - S)}{\omega(P_{\alpha}(K_{m,n}) - S)},$$

such that for all $Z$ in $K$, if $Z \cap S$ is not empty, then $Z \subset S$.

**Proof.** By the symmetry of the permutation graph of complete bipartite graph, we do the case when $Z = M'_{x}$. Let $S'$ be the minimum vertex cut set of $P_{\alpha}(K_{m,n})$ with

$$T(P_{\alpha}(K_{m,n})) = \frac{|S'| + m(P_{\alpha}(K_{m,n}) - S')}{\omega(P_{\alpha}(K_{m,n}) - S')}.$$
and let $A_x = S' \cap M_x'$ and $B_x = M_x' - A_x$. We let $A_y$ be the neighborhood of $A_x$ in $M_y'$, and $B_y$ be the neighborhood of $B_x$ in $M_y'$. Suppose that $A_x$ and $B_x$ are both nonempty, i.e., $m \geq 2$. We first note that $T = M_x \cup M_y$ is a vertex cut set of $P_\alpha(K_{m,n})$. So by the definition of tenacity, we have

$$T(P_\alpha(K_{m,n})) \leq \frac{2m + 2}{m + n - q}.$$ 

The proof proceeds in four cases.

Case 1. If $N_x$ and $N_y$ are both contained in $S'$, let $T = S' - A_x$, then $|T| = |S'| - |A_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') + 1$. So,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} \leq \frac{|S'| + m(P_\alpha(K_{m,n}) - S') - (|A_x| - 1)}{\omega(P_\alpha(K_{m,n}) - S')} \leq \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')}.$$ 

Thus,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} = \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} ,$$

which contradicts the minimality of $S'$.

Case 2. If $N_x$ is contained in $S'$ but $N_y$ is not contained in $S'$, then let $x_i$ be an element of $A_x$ and so $y_{a(i)}$ is in $A_y$. Let $T = S' - \{x_i\}$, then $|T| = |S'| - 1$. If $y_{a(i)}$ is not contained in $S'$, then $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') + 1$. So,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} \leq \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} .$$

Thus,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} = \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} ,$$

but $T$ has one less vertex than that of $S'$, a contradiction. If $y_{a(i)}$ is contained in $S'$, then $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S') + 1$, $m(P_\alpha(K_{m,n}) - T) = m(P_\alpha(K_{m,n}) - S')$. So,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} = \frac{|S'| - 1 + m(P_\alpha(K_{m,n}) - S')} {\omega(P_\alpha(K_{m,n}) - S') + 1} < \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} ,$$

again a contradiction.

Case 3. If $N_x$ is not contained in $S'$ but $N_y$ is contained in $S'$, then let $x_i$ be in $A_x$ and so $y_{a(i)}$ is in $A_y$. If $y_{a(i)}$ is in $S'$, let $T = S' - \{x_i\}$, then $|T| = |S'| - 1$, $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) = m(P_\alpha(K_{m,n}) - S') + 1$. So,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} = \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} .$$
but $T$ has one less vertex than that of $S'$, a contradiction. Hence $A_y \cap S' = \emptyset$. Now let $x_i$ be in $B_x$, then $y_{\alpha(i)}$ is contained in $B_y$. If $y_{\alpha(i)}$ is in $S'$, then let $T = S' - \{y_{\alpha(i)}\} \cup \{x_i\}$. Thus, $|T| = |S'|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S') + 1$, and $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') - 1$. So,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} \leq \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')}$$

a contradiction. So $B_y \cap S$ is empty. Thus $M_x' \cap S'$ is empty. Let $T = S' \cup B_x$, then $|T| = |S'| + |B_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S') + |B_x|$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') - |B_x|$. So,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} \leq \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} < \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')},$$

this contradicts the definition of tenacity.

**Case 4.** If $N_x$ and $N_y$ are not contained in $S'$, then consider $M_y$. If $M_y$ is contained in $S'$, let $T = S' \cup B_x$, then $|T| = |S'| + |B_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') - |B_x|$. So,

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} \leq \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} \leq \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')}.$$

Thus $T$ is a vertex cut set with

$$\frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)} = \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')}$$

and $M_x' \subseteq T$. If $M_y$ is not contained in $S'$, then it is easy to see that all of the vertices in $N_x - S'$ are in the same component since $B_x$ is nonempty, and all of the vertices in $N_y - S'$ are in the same component since $M_y \cap S'$ is nonempty. Thus $P_\alpha(K_{m,n}) - S'$ has exactly two components, one in each copy of $K_{m,n}$. If neither $x_i$ nor $y_{\alpha(i)}$ is not in $S'$ then $S'$ is not a cut set. Thus at least one of $x_i$ and $y_{\alpha(i)}$ is in $S'$ for all $i = 1, 2, \ldots, n + m$. Thus we know that $|S'| \geq n + m$, $\omega(P_\alpha(K_{m,n}) - S') = 2$. Let $C$ be the component of $P_\alpha(K_{m,n}) - S'$ containing $B_x$, then

$$T(P_\alpha(K_{m,n})) = \frac{|S'| + m(P_\alpha(K_{m,n}) - S')}{\omega(P_\alpha(K_{m,n}) - S')} \geq \frac{|S'| + |V(C)|}{\omega(P_\alpha(K_{m,n}) - S')} \geq \frac{|S'| + |B_x| + |N_x - S'|}{\omega(P_\alpha(K_{m,n}) - S')} \geq \frac{n + m + 2}{2} \geq \frac{n + m + 2}{m + n - q} \geq \frac{2m + 2}{m + n - q}.$$
Let $T = M_x \cup M_y$. Then $T$ is a vertex cut set with

$$T(P_\alpha(K_{m,n})) = \frac{|T| + m(P_\alpha(K_{m,n}) - T)}{\omega(P_\alpha(K_{m,n}) - T)}$$

and $M'_x$ is contained in $T$.

From above we know that the lemma is true if $Z = M'_x$. The above proof works for the other cases of $Z \in K$. The details are omitted.

By the above lemma we can obtain the tenacity of the permutation graph of complete bipartite graph.

**Theorem 2.5.** For $\alpha$ in $S_{m+n}$ and $m \leq n$.

1. If $1 = m = n$, $T(P_\alpha(K_{m,n})) = 3/2$.
2. If $1 = m < n$, when $q = 0$, $T(P_\alpha(K_{m,n})) = 4/(n + 1)$ when $q = 1$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{n+2}{n+1} & \text{if } 2 \leq n \leq 3 \\ \frac{4}{n} & \text{if } n > 3. \end{cases}$$

3. If $2 \leq m \leq n$, when $q = 0$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m+1}{m+n} & \text{if } m = n \\ \frac{2m+2}{m+n} & \text{if } m < n, \end{cases}$$

when $q = m$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m+2}{m+n-q} & \text{if } q < \frac{m^2+2n^2}{3m+n+4} \\ \frac{n+m+1}{n+m} & \text{if } m \geq \frac{m^2+2n^2}{3m+n+4}, \end{cases}$$

when $1 \leq q \leq m - 1$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m+2}{n+q} & \text{if } q \geq \frac{m^2+2n^2}{3m+n+4} \end{cases}.$$

**Proof.** By Lemma 2.1 we know that the vertex set satisfying the condition must be the union of the elements of $K$. It is easy to find 55 vertex cut sets of this type. But most of these sets are trivial, all but 4 of these sets may be discarded as giving too larger values for $(|S| + m(P_\alpha(K_{m,n}) - S))/\omega(P_\alpha(K_{m,n}) - S))$. The remaining sets are $S_1 = M_x \cup M'_y$, $S_2 = M_x \cup M_y$, $S_3 = M_x \cup N_y$, $S_4 = M_x \cup N'_y$, and the values for $(|S| + m(P_\alpha(K_{m,n}) - S))/\omega(P_\alpha(K_{m,n}) - S))$ given by these sets are

$$v_1 = \frac{m+2n+q}{m-q+1},$$

$$v_2 = \frac{2m+2}{n+m-q},$$

$$v_3 = \frac{m+n+2}{q+n} \quad \text{if } q \neq m \quad \text{or} \quad v_3 = \frac{m+n+1}{q+n} \quad \text{if } q = m,$$

$$v_4 = \frac{3m+n-q}{n-m+q+1}.$$
We distinguish three cases.

Case 1. When \( m = n = 1 \), it is easy to see that

(a) if \( q = 0 \), \( T(P_\alpha(K_{m,n})) = \min\{v_1, \cdots, v_4\} = v_2 = 3/2 \),

(b) if \( q = 1 \), \( T(P_\alpha(K_{m,n})) = \min\{v_1, \cdots, v_4\} = v_4 = 3/2 \).

So under this condition \( T(P_\alpha(K_{m,n})) = 3/2 \).

Case 2. When \( 1 = m < n \).

Subcase 2.1 If \( q = 0 \), then \( T(P_\alpha(K_{m,n})) = \min\{v_1, \cdots, v_4\} = v_2 = 4/(n+1) \).

Subcase 2.2 If \( q = 1 \), then

(a) when \( 2 \leq n \leq 3 \), \( T(P_\alpha(K_{m,n})) = \min\{v_1, \cdots, v_4\} = v_4 = (n+2)/(n+1) \),

(b) when \( n > 3 \), \( T(P_\alpha(K_{m,n})) = \min\{v_1, \cdots, v_4\} = v_2 = 4/n \).

Case 3. When \( 2 \leq m \leq n \).

Subcase 3.1 If \( q = 0 \),

(a) when \( m = n \), \( T(P_\alpha(K_{m,n})) = \min\{v_1, \cdots, v_4\} = v_2 = (m+n+1)/(m+n) \),

(b) when \( m < n \), \( T(P_\alpha(K_{m,n})) = \min\{v_1, \cdots, v_4\} = v_2 = (m+n+2)/(m+n) \).

Subcase 3.2 If \( 1 \leq q \leq m \), for fixed \( m \) and \( n \), when \( q \) increases, the following occur. The value \( v_1 \) increases, so the minimum value of \( v_1 \) is \( (m+2n+1)/m \). \( v_4 \) decreases, so the minimum value of \( v_4 \) is \( (2m+n)/(n+1) \). When \( q \neq m \) increases, the value \( v_3 \) decreases, so the maximum value of \( v_3 \) is \( (m+n+2)/(n+1) \). It is easy to check that the minimum value of \( v_1 \) is larger than the maximum value of \( v_2 \), and the minimum value of \( v_4 \) is larger than the maximum value of \( v_3 \). And it is also easily checked that when \( q = m \), the minimum value of \( v_4 \) is larger than the value of \( v_3 = (m+n+1)/(q+n) \). So \( S_1 \) and \( S_4 \) should be discarded. Now the value of \( v_3 \) decreases as \( q \) increases, and the intersection point for \( v_2 \) and \( v_3 \) occurs where \( q = (m^2+2n+n^2)(3m+n+4) \), \( v_3 = (m+n+2)/(q+n) \) and where \( q = (m^2+n^2+m-n)/(3m+n+3) \), \( v_3 = (m+n+1)/(q+n) \). Thus the theorem holds.

The following theorem gives us the rupture degree of permutation graph of complete bipartite graphs. Note that \( T = M_x \cup M_y \) is a vertex cut set of \( P_\alpha(K_{m,n}) \), so by the definition of the rupture degree we have

\[ r(P_\alpha(K_{m,n})) \geq \omega(P_\alpha(K_{m,n}) - T) - |T| - m(P_\alpha(K_{m,n}) - T) \geq n - m - q - 2. \]

In order to prove this theorem we first introduce a lemma.

Lemma 2.2. For \( \alpha \) in \( S_{m+n} \) and \( m \leq n \), there exists a vertex cut set \( S \) of \( P_\alpha(K_{m,n}) \) with

\[ r(P_\alpha(K_{m,n})) = \omega(P_\alpha(K_{m,n}) - S) - |S| - m(P_\alpha(K_{m,n}) - S) \]

such that for all \( Z \) in \( K \), if \( Z \cap S \) is not empty, then \( Z \subset S \).
Proof. By the symmetry of the permutation graph of complete bipartite graph, We do the case when $Z = M'_x$. Let $S'$ be the minimum vertex cut set of $P_a(K_{m,n})$ with

$$r(P_a(K_{m,n})) = \omega(P_a(K_{m,n}) - S') - |S'| - m(P_a(K_{m,n}) - S'),$$

and let $A_x = S' \cap M'_x$ and $B_x = M'_x - A_x$. Suppose that $A_x$ and $B_x$ are both nonempty, i.e., $m \geq 2$. We distinguish four cases.

Case 1. If $N_x$ and $N_y$ are both contained in $S'$, let $T = S' - A_x$, then $|T| = |S'| - |A_x|$, $\omega(P_a(K_{m,n}) - T) \geq \omega(P_a(K_{m,n}) - S')$, $m(P_a(K_{m,n}) - T) \leq m(P_a(K_{m,n}) - S') + 1$. So,

$$\omega(P_a(K_{m,n}) - T) - |T| - m(P_a(K_{m,n}) - T) $$

$$\geq \omega(P_a(K_{m,n}) - S') - |S'| - m(P_a(K_{m,n}) - S') + |A_x| - 1$$

$$\geq \omega(P_a(K_{m,n}) - S') - |S' - m(P_a(K_{m,n}) - S'),$$

a contradiction to the minimality of $S'$.

Case 2. If $N_x$ is contained in $S'$ but $N_y$ is not contained in $S'$, let $T = S' - A_x$, then $|T| = |S'| - |A_x|$, $\omega(P_a(K_{m,n}) - T) \geq \omega(P_a(K_{m,n}) - S')$, $m(P_a(K_{m,n}) - T) \leq m(P_a(K_{m,n}) - S') + |A_x|$. So,

$$\omega(P_a(K_{m,n}) - T) - |T| - m(P_a(K_{m,n}) - T) $$

$$\geq \omega(P_a(K_{m,n}) - S') - |S'| - m(P_a(K_{m,n}) - S') + |A_x|$$

$$\geq \omega(P_a(K_{m,n}) - S') - |S' - m(P_a(K_{m,n}) - S'),$$

which contradicts the minimality of $S'$.

Case 3. If $N_x$ is not contained in $S'$ but $N_y$ is contained in $S'$, then $M'_y \cap S'$ is empty, the proof is similar to that of Case 3 in Lemma 2.1. Let $T = S' \cup B_x$. Thus $|T| = |S'| + |B_x|$, $\omega(P_a(K_{m,n}) - T) \geq \omega(P_a(K_{m,n}) - S') + |B_x|$, $m(P_a(K_{m,n}) - T) \leq m(P_a(K_{m,n}) - S') - |B_x|$. So,

$$\omega(P_a(K_{m,n}) - T) - |T| - m(P_a(K_{m,n}) - T) $$

$$\geq \omega(P_a(K_{m,n}) - S') - |S'| - m(P_a(K_{m,n}) - S') + |B_x|$$

$$\geq \omega(P_a(K_{m,n}) - S') - |S' - m(P_a(K_{m,n}) - S'),$$

which contradicts the definition of rupture degree.

Case 4. If neither $N_x$ nor $N_y$ is contained in $S'$, then consider $M_y$. If $M_y$ is contained in $S'$, let $T = S' \cup B_x$, then $|T| = |S'| + |B_x|$, $\omega(P_a(K_{m,n}) - T) \geq \omega(P_a(K_{m,n}) - S')$, $m(P_a(K_{m,n}) - T) \leq m(P_a(K_{m,n}) - S') - |B_x|$. So,

$$\omega(P_a(K_{m,n}) - T) - |T| - m(P_a(K_{m,n}) - T) $$

$$\geq \omega(P_a(K_{m,n}) - S') - |S'| - m(P_a(K_{m,n}) - S'),$$

Thus $T$ is a vertex cut set with $\omega(P_a(K_{m,n}) - T) - |T| - m(P_a(K_{m,n}) - T) = \omega(P_a(K_{m,n}) - S') - |S'| - m(P_a(K_{m,n}) - S')$, and $M'_x \subseteq T$. If $M_y$ is not contained in $S'$, then it is easy to see that all of the vertices in $N_x - S'$ are in the same component since $B_y$ is nonempty, and all of the vertices in $N_y - S'$ are in the same component since $M_y \cap S'$ is nonempty. Thus $P_a(K_{m,n}) - S'$ has exactly two components, one in each copy of $K_{m,n}$. If neither $x_i$ nor $y_{a(i)}$ is not in $S'$, then $S'$ is not a cut set. Thus
at least one of \(x_i\) and \(y_{\alpha(i)}\) is in \(S'\) for all \(i = 1, 2, \cdots, n + m\). Thus we know that \(|S'| \geq n + m\), \(\omega(P_{\alpha}(K_{m,n}) - S') = 2\). Let \(C\) be the component of \(P_{\alpha}(K_{m,n}) - S'\) containing \(B_x\), then
\[ r(P_{\alpha}(K_{m,n})) = \omega(P_{\alpha}(K_{m,n}) - S') - |S'| - m(P_{\alpha}(K_{m,n}) - S') \]
\[ \leq \omega(P_{\alpha}(K_{m,n}) - S') - |S'| - |V(C)| \]
\[ \leq \omega(P_{\alpha}(K_{m,n}) - S') - |S'| - |B_x| - |N_x - S'| \]
\[ \leq 2 - n - m - 2 \leq n - m - q - 2. \]

On the other hand, by the previous remark we know that \(r(P_{\alpha}(K_{m,n})) \geq n - m - q - 2\). Hence, in this case \(r(P_{\alpha}(K_{m,n})) = n - m - q - 2\). Let \(T = M_x \cup M_y\). Then \(T\) is a cut set with \(r(P_{\alpha}(K_{m,n})) = \omega(P_{\alpha}(K_{m,n}) - T) - |T| - m(P_{\alpha}(K_{m,n}) - T)\) and \(M_x' \subset T\).

From above we know that the lemma is true if \(Z = M_x'\). The above proof works for the other cases of \(Z \in K\). The details are omitted. \(\square\)

**Theorem 2.6.** For \(\alpha\) in \(S_{m+n}\) and \(m \leq n\).

1. If \(m = n = 1\), \(r(P_{\alpha}(K_{m,n})) = -1\).
2. If \(1 = m < n\), when \(q = 0\), \(r(P_{\alpha}(K_{m,n})) = n - 3\). When \(q = 1\),
\[ r(P_{\alpha}(K_{m,n})) = \begin{cases} -1 & \text{if } 2 \leq n \leq 3 \\ n - 4 & \text{if } n > 3. \end{cases} \]
3. If \(2 \leq m \leq n\), when \(q = 0\),
\[ r(P_{\alpha}(K_{m,n})) = \begin{cases} n - m - 2 & \text{if } n - m \geq 2 \\ -1 & \text{if } n - m < 2, \end{cases} \]
when \(q = m\),
\[ r(P_{\alpha}(K_{m,n})) = \begin{cases} n - 2m - 2 & \text{if } q \leq \frac{n-1}{2} \\ -1 & \text{if } q > \frac{n-1}{2}, \end{cases} \]
when \(1 \leq q \leq m - 1\),
\[ r(P_{\alpha}(K_{m,n})) = \begin{cases} n - m - q - 2 & \text{if } q \leq \frac{n}{2} \\ q - m - 2 & \text{if } q > \frac{n}{2}. \end{cases} \]

**Proof.** By Lemma 2.2 we know that the vertex set satisfying the condition must be the union of the elements of \(K\). It is easy to find 55 vertex cut sets of this type. But most of these sets are trivial, all but 5 of these sets may be discarded as giving too less values for \(\omega(P_{\alpha}(K_{m,n}) - S) - |S| - m(P_{\alpha}(K_{m,n}) - S)\). The remaining sets are \(S_1 = M_x \cup M_y, S_2 = M_x \cup N_y, S_3 = M_x \cup M_y \cup N'_x, S_4 = M_x \cup N'_y, S_5 = M_x \cup M'_y \cup N'_x\) and the values for \(\omega(P_{\alpha}(K_{m,n}) - S) - |S| - m(P_{\alpha}(K_{m,n}) - S)\) given by these sets are \(v_1 = n - m - q - 2, v_2 = q - m - 2\) if \(q \neq m\) or \(v_2 = -1\) if \(q = m, v_3 = -2q - 1, v_4 = -4m + 2q + 1, v_5 = -2m + 1\). So we distinguish three cases.
Case 1. When $m = n = 1$, it is easy to see that $r(P_\alpha(K_{m,n})) = \max\{v_1, v_2, \ldots, v_5\} = v_5 = -1$.

Case 2. When $1 = m < n$,

Subcase 2.1 If $q = 0$, it is easy to see that $r(P_\alpha(K_{m,n})) = \max\{v_1, v_2, \ldots, v_5\} = v_1 = n - 3$.

Subcase 2.2 If $q = 1$, it is easy to see that
(a) when $n = 2$, then $r(P_\alpha(K_{m,n})) = \max\{v_1, v_2, \ldots, v_5\} = v_5 = -1$,
(b) when $n > 2$, then $r(P_\alpha(K_{m,n})) = \max\{v_1, v_2, \ldots, v_5\} = v_1 = n - 4$.

Case 3. When $2 \leq m \leq n$,

Subcase 3.1 If $q = 0$, it is easy to see that
(a) when $n - m \geq 2$, then $r(P_\alpha(K_{m,n})) = \max\{v_1, v_2, \ldots, v_5\} = v_1 = n - m - 2$,
(b) when $n - m < 2$, then $r(P_\alpha(K_{m,n})) = \max\{v_1, v_2, \ldots, v_5\} = v_3 = -1$.

Subcase 3.2 If $q = m$, then, when $q \leq (n - 1)/2$, $\max\{v_1, v_2, \ldots, v_5\} = v_1 = n - 2m - 2$; when $q \geq (n - 1)/2$, $\max\{v_1, v_2, \ldots, v_5\} = v_2 = -1$.

Subcase 3.3 If $1 \leq q \leq m - 1$, under this condition, for fixed $m$ and $n$, when $q \leq n/2$, then $\max\{v_1, v_2, \ldots, v_5\} = v_1 = n - m - q - 2$; when $q \geq n/2$, $\max\{v_1, v_2, \ldots, v_5\} = v_2 = q - m - 2$. The proof is thus completed.

3. Conclusion

The rupture degree and tenacity of a graph, to some extent, represents a trade-off between the amount of work done to damage the network and how badly the network is damaged. Hence, the rupture degree and tenacity can be used to measure the vulnerability of networks. So clearly, it is of prime importance to determine this parameter for a graph. In this paper, we have obtained the exact values for the rupture degree and tenacity of permutation graphs of complete bipartite graphs. To make further progress in this direction, one could try to characterize the graphs with given rupture degree or tenacity.

Acknowledgement. This work was supported by the NSFC (No.10871226), PC-SIRT and the 973 program. The authors are thankful to the anonymous referees for their useful comments, which led to this improved version.

References


