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Lifting Property of the Jacobson Radical in Associative Pairs

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Abstract. An ideal I in a ring R is called a lifting ideal if idempotents can be lifted modulo every left ideal contained in I. In this paper we extend this notion to the context of associative pairs and characterize when the Jacobson radical of an associative pair is a lifting ideal.

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1. Introduction

Let $A = (A^+, A^-)$ be a pair of modules over an associative commutative unital ring K and $\langle\ \rangle^\sigma$:

$$\begin{aligned} \langle\ \rangle^\sigma : A^\sigma \times A^{-\sigma} \times A^\sigma &\quad\rightarrow\quad A^\sigma \\ (x^\sigma, y^{-\sigma}, z^\sigma) &\mapsto \quad \langle x^\sigma y^{-\sigma} z^\sigma \rangle^\sigma \end{aligned}$$

for $\sigma \in \{+, -\}$, two K-trilinear mappings called triple products. A is called an associative K-pair, if the identities

$$\begin{aligned} \langle\langle x^\sigma y^{-\sigma} z^\sigma \rangle^\sigma u^{-\sigma} v^\sigma \rangle^\sigma &= \langle x^\sigma \langle y^{-\sigma} z^\sigma u^{-\sigma} \rangle^\sigma v^\sigma \rangle^\sigma \\ &= \langle x^\sigma y^{-\sigma} \langle z^\sigma u^{-\sigma} v^\sigma \rangle^\sigma \rangle^\sigma \end{aligned}$$

are satisfied for $x^\sigma, z^\sigma, v^\sigma \in A^\sigma$, $y^{-\sigma}, u^{-\sigma} \in A^{-\sigma}$, and $\sigma \in \{+, -\}$. From now on, for the sake of simplicity, we will use $\langle\ \rangle$ instead of $\langle\ \rangle^\sigma$ if no confusion can arise. The classical example of an associative pair is $(\mathcal{M}_{p\times q}(R), \mathcal{M}_{q\times p}(R))$, where R is an associative K-algebra and p, q are natural numbers.

A K-submodule I^σ of A^σ is called a *right ideal* (resp. *left ideal*) of A^σ if $\langle I^\sigma A^{-\sigma} A^\sigma \rangle \subseteq I^\sigma$ (resp. $\langle A^\sigma A^{-\sigma} I^\sigma \rangle \subseteq I^\sigma$). A *right ideal* of A is a couple (I^+, I^-), where I^σ is a right ideal of A^σ for all $\sigma \in \{+, -\}$. Similarly, we can define left

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ideal of A, while I^σ is said to be a *two-sided ideal* of A^σ, if I^σ is both a right ideal and a left ideal. An *ideal* of A is a couple (I^+, I^-) of two-sided ideals which satisfy $\langle A^\sigma I^\sigma A^\sigma \rangle \subseteq I^\sigma$ for all $\sigma \in \{+,-\}$. The definitions of homomorphisms of associative pairs, the multiplication operators can be found in [3, 4].

An element $(e^+, e^-) \in A$ is said to be an *idempotent* if $e^\sigma = \langle e^\sigma e^{-\sigma} e^\sigma \rangle$ for all $\sigma \in \{+,-\}$. An element $x^\sigma \in A^\sigma$ is said to be *von Neumann regular* if there exists $y^{-\sigma} \in A^{-\sigma}$ such that $x^\sigma = \langle x^\sigma y^{-\sigma} x^\sigma \rangle$.

For an associative K-pair $A = (A^+, A^-)$, Loos proved that there exists a unital associative K-algebra $\mathcal{U}_A$, the standard embedding of A, with two orthogonal idempotents $e_1,\ e_2$ satisfying $1 = e_1 + e_2$, and such that

$$(A^+, A^-) \simeq (e_1 \mathcal{U}_A e_2, e_2 \mathcal{U}_A e_1). \tag{1.1}$$

We are going to sketch the construction of such a K-algebra. Let $\mathcal{U}_{11}^{(A)}$ be the subalgebra of $End_K(A^+) \times End_K(A^-)^{op}$ generated by (Id, Id) and the set $\{x^+x^- = (L(x^+, x^-), R(x^+, x^-)) | x^\sigma \in A^\sigma\}$ and let $\mathcal{U}_{22}^{(A)}$ be the subalgebra of $End_K(A^-) \times End_K(A^+)^{op}$ generated by (Id, Id) and the set $\{x^-x^+ = (L(x^-, x^+), R(x^-, x^+)) | x^\sigma \in A^\sigma\}$, where $L(x^\sigma, x^{-\sigma})(y^\sigma) = \langle x^\sigma x^{-\sigma} y^\sigma \rangle$,
$R(x^\sigma, x^{-\sigma})(y^{-\sigma}) = \langle y^{-\sigma} x^\sigma x^{-\sigma} \rangle$ for any $y^\sigma \in A^\sigma$ and $\sigma \in \{+,-\}$. For the sake of simplicity, we write $\mathcal{U}_{ii}$ instead of $\mathcal{U}_{ii}^{(A)}$ for $i \in \{1, 2\}$. Then A^+ is a $\mathcal{U}_{11}$-$\mathcal{U}_{22}$-bimodule for the actions $(x^+x^-)a^+ = \langle x^+x^-a^+ \rangle$, $a^+(x^-x^+) = \langle a^+x^-x^+ \rangle$. Similarly, A^- is a $\mathcal{U}_{22}$-$\mathcal{U}_{11}$-bimodule for the actions $(x^-x^+)a^- = \langle x^-x^+a^- \rangle$, $a^-(x^+x^-) = \langle a^-x^+x^- \rangle$. Let $\mathcal{U}_{12} = A^+$, $\mathcal{U}_{21} = A^-$, then $\mathcal{U}_{11} \oplus \mathcal{U}_{12} \oplus \mathcal{U}_{21} \oplus \mathcal{U}_{22}$ is a unital associative K-algebra endowed with the product

$$\begin{pmatrix} \alpha & x^+ \\ x^- & \beta \end{pmatrix} \cdot \begin{pmatrix} \delta & y^+ \\ y^- & \gamma \end{pmatrix} = \begin{pmatrix} \alpha\delta + x^+y^- & \alpha y^+ + x^+\gamma \\ x^-\delta + \beta y^- & x^-y^+ + \beta\gamma \end{pmatrix}$$

for all $\alpha,\ \delta \in \mathcal{U}_{11},\ \beta,\ \gamma \in \mathcal{U}_{22},\ x^\sigma,\ y^\sigma \in A^\sigma$.

It is easy to see that

$$e_1 = \begin{pmatrix} 1_{\mathcal{U}_{11}} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{U}_{22}} \end{pmatrix}$$

are two orthogonal idempotents verifying $1_{\mathcal{U}_A} = e_1 + e_2$ and hence (1.1) holds.

For an associative pair (A^+, A^-), we can define the concept of Jacobson radical of A in the following way. An element $(x^+, x^-) \in A$ is called *quasi-invertible* if $1 - x^+$ is invertible in the associative K-algebra $K1 \oplus A^+_{x^-}$, where $A^+_{x^-}$ is the associative K-algebra x^--homotope of A^+ with product $a^+ \cdot b^+ = \langle a^+x^-b^+ \rangle$. Thus (x^+, x^-) is quasi-invertible if and only if there exists $z^+ \in A^+$ such that $x^+ + z^+ = \langle x^+x^-z^+ \rangle = \langle z^+x^-x^+ \rangle$. An element $x^\sigma \in A^\sigma$ is called *properly quasi-invertible* if for each $x^{-\sigma} \in A^{-\sigma}$, (x^+, x^-) is quasi-invertible. Finally, $Rad^\sigma A$ coincides with the set of all properly quasi-invertible elements of A^σ, for all $\sigma \in \{+,-\}$. We define the *Jacobson radical* of A, denoted by $Rad\ A$, to be $Rad\ A = (Rad^+A, Rad^-A)$. Cuenca Mira *et al.* [1] proved that $Rad\ A$ is an ideal of A.

In 1977, Nicholson [8] developed the concept of suitable rings in which idempotents can be lifted modulo every left (right) ideal and are shown to coincide with the exchange rings of Warfield [9]. Let $I = (I^+, I^-)$ be a left ideal of an associative pair A. Following [2] we say that idempotents can be lifted modulo I, if for any

$(x^+, x^-) \in A$ such that $x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle \in I^\sigma$ for all $\sigma \in \{+,-\}$, there exists an idempotent $(e^+, e^-) \in A$ such that $e^\sigma - x^\sigma \in I^\sigma$ for all $\sigma \in \{+,-\}$. Let I^σ be a left ideal of A^σ, we say that von Neumann regular elements can be lifted modulo I^σ, if for any $x^\sigma \in A^\sigma$ and $a^{-\sigma} \in A^{-\sigma}$ such that $x^\sigma - \langle x^\sigma a^{-\sigma} x^\sigma \rangle \in I^\sigma$, there exists a von Neumann regular element $u^\sigma \in A^\sigma$ such that $u^\sigma - x^\sigma \in I^\sigma$.

Following [2] an associative pair $A = (A^+, A^-)$ is called *left idempotent-lifting* if idempotents can be lifted modulo any left ideal of A. For any $\sigma \in \{+,-\}$ we say that A^σ is *left regular-lifting* if von Neumann regular elements can be lifted modulo any left ideal of A^σ. It was shown in [2, Theorem 2] that these two definitions are equivalent. In [3] the same authors studied semiperfect pairs, that is, those in which idempotents can be lifted modulo *Rad* A as well as $A/Rad\ A$ are artinian, and it was shown that every artinian pair is semiperfect.

The notion of lifting ideal of a ring was introduced by Khurana and Lam [6]. An ideal I of a ring R is called a *lifting ideal* if idempotents can be lifted modulo every left ideal contained in I. In this paper, we focus on the property of *Rad* A being a lifting ideal. And we investigate the interplay of a pair and its standard embedding with regard to the property of *Rad* A being a lifting ideal.

2. Main results

Firstly, we have a basic property on the Jacobson radical of A^σ, $\sigma \in \{+,-\}$, that is a direct consequence of the relation between the radical of the pair and the radical of its standard embedding $\mathcal{U}$, that is $rad\,\mathcal{U} = rad\,\mathcal{U}_{11} \oplus rad^+ A \oplus rad^- A \oplus rad\,\mathcal{U}_{22}$, see [1].

Proposition 2.1. *Let $A = (A^+, A^-)$ be an associative pair with*

$$Rad\ A = (Rad^+ A, Rad^- A).$$

For any $x^+ \in Rad^+ A$, $x^- \in A^-$, we have $x^+x^- \in Rad\,\mathcal{U}_{11}$ and $x^-x^+ \in Rad\,\mathcal{U}_{22}$. Something similar happens for any two elements $x^- \in Rad^- A$, $x^+ \in A^+$.

Lemma 2.1. *Let $A = (A^+, A^-)$ be an associative pair and $\sigma \in \{+,-\}$. For every $x^\sigma \in A^\sigma, x^{-\sigma} \in A^{-\sigma}$, the following statements are equivalent:*

(1) *There exists a von Neumann regular element $u^\sigma \in A^\sigma$ such that*
$$u^\sigma - x^\sigma \in \mathcal{U}_{ii}(x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle).$$

(2) *There exist a von Neumann regular element $u^\sigma \in A^\sigma$ and an element $\beta \in \mathcal{U}_{ii}$ such that $u^\sigma \in \mathcal{U}_{ii}x^\sigma$ and*
$$(Id - u^\sigma x^{-\sigma}) - \beta(Id - x^\sigma x^{-\sigma}) \in Rad\,\mathcal{U}_{ii}.$$

(3) *There exists a von Neumann regular element $u^\sigma \in A^\sigma$ such that $u^\sigma \in \mathcal{U}_{ii}x^\sigma$ and*
$$Id - u^\sigma x^{-\sigma} \in \mathcal{U}_{ii}(Id - x^\sigma x^{-\sigma})I.$$

Proof. The proof is a particular case of [2, Proposition 1], so we omit it here. ∎

Proposition 2.2. *For an associative pair A with $Rad\ A = (Rad^+ A, Rad^- A)$ and $\sigma \in \{+,-\}$, the following are equivalent:*

(1) *von Neumann regular elements can be lifted modulo every left ideal contained in $Rad^\sigma A$.*

(2) *Every* $x^\sigma \in A^\sigma$ *and* $x^{-\sigma} \in A^{-\sigma}$ *such that* $x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle \in Rad^\sigma A$ *satisfy the following equivalent conditions:*

(a) *There exists a von Neumann regular element* $u^\sigma \in A^\sigma$ *such that*

$$u^\sigma - x^\sigma \in \mathcal{U}_{ii}(x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle).$$

(b) *There exist a von Neumann regular element* $u^\sigma \in A^\sigma$ *and an element* $\beta \in \mathcal{U}_{ii}$ *such that* $u^\sigma \in \mathcal{U}_{ii} x^\sigma$ *and*

$$(Id - u^\sigma x^{-\sigma}) - \beta(Id - x^\sigma x^{-\sigma}) \in Rad\, \mathcal{U}_{ii}.$$

(c) *There exists a von Neumann regular element* $u^\sigma \in A^\sigma$ *such that* $u^\sigma \in \mathcal{U}_{ii} x^\sigma$ *and*

$$Id - u^\sigma x^{-\sigma} \in \mathcal{U}_{ii}(Id - x^\sigma x^{-\sigma}).$$

Proof. (1) $\Longrightarrow$ (2)(a) Let $x^\sigma \in A^\sigma$ and $x^{-\sigma} \in A^{-\sigma}$ be such that $x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle \in Rad^\sigma A$. Since $\mathcal{U}_{ii}(x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle) \subseteq Rad^\sigma A$ is a left ideal of A^σ, there exists a von Neumann regular element $u^\sigma \in A^\sigma$ such that $u^\sigma - x^\sigma \in \mathcal{U}_{ii}(x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle)$.

Conversely, let L be a left ideal of A^σ contained in $Rad^\sigma A$ and $x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle \in L \subseteq Rad^\sigma A$. There exists a von Neumann regular element $u^\sigma \in A^\sigma$ such that $u^\sigma - x^\sigma \in \mathcal{U}_{ii}(x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle) \subseteq L$ by assumption, as desired.

(2)(a), (2)(b) and (2)(c) are equivalent by Lemma 2.1. ∎

Proposition 2.3. *Let* A *be an associative pair with* $Rad\ A = (Rad^+ A, Rad^- A)$. *The following are equivalent:*

(1) *Idempotents can be lifted modulo each left ideal contained in* $Rad\ A$.

(2) *For every* $(x^+, x^-) \in A$ *such that* $x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle \in Rad^\sigma A$, *there exists an idempotent* $(e^+, e^-) \in A$ *such that* $e^\sigma - x^\sigma \in \mathcal{U}_{ii}(x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle)$ *for all* $\sigma \in \{+, -\}$.

(3) *von Neumann regular elements can be lifted modulo every left ideal contained in* $Rad^+ A$.

(4) *For every* $(x^+, x^-) \in A$ *such that* $x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle \in Rad^\sigma A$, *there exists an idempotent* $(e^+, e^-) \in A$ *such that* $e^\sigma \in \mathcal{U}_{ii} x^\sigma$ *and* $Id - e^\sigma x^{-\sigma} \in \mathcal{U}_{ii}(Id - x^\sigma x^{-\sigma})$ *for all* $\sigma \in \{+, -\}$.

Proof. (1) $\Longleftrightarrow$ (2) is similar to the proof of Proposition 2.2.
(2) $\Longrightarrow$ (3) Let $x^+ - \langle x^+ x^- x^+ \rangle \in Rad^+ A$. Since $Rad\ A = (Rad^+ A, Rad^- A)$ is an ideal of A, we obtain that:

$$\langle x^+ x^- x^+ \rangle - \langle x^+ x^- x^+ x^- x^+ \rangle \in Rad^+ A,$$
$$\langle x^- x^+ x^- \rangle - \langle x^- x^+ x^- x^+ x^- \rangle \in Rad^- A,$$
$$\langle x^- x^+ x^- x^+ x^- \rangle - \langle x^- x^+ x^- x^+ x^- x^+ x^- \rangle \in Rad^- A.$$

Therefore,

$$x^+ - \langle x^+ x^- x^+ x^- x^+ \rangle \in Rad^+ A$$

and

$$\langle x^- x^+ x^- \rangle - \langle x^- x^+ x^- x^+ x^- x^+ x^- \rangle \in Rad^- A.$$

For the element $(x^+, \langle x^- x^+ x^- \rangle) \in A$, we apply (2). Then we obtain that there exists an idempotent $(e^+, e^-) \in A$ such that

$$\begin{aligned} e^+ - x^+ &\in \mathcal{U}_{11}(x^+ - \langle x^+ x^- x^+ x^- x^+ \rangle) \\ &\subseteq \mathcal{U}_{11}(x^+ - \langle x^+ x^- x^+ \rangle + \mathcal{U}_{11}(\langle x^+ x^- x^+ \rangle - \langle x^+ x^- x^+ x^- x^+ \rangle) \\ &= \mathcal{U}_{11}(x^+ - \langle x^+ x^- x^+ \rangle) + \mathcal{U}_{11} x^+ x^- (x^+ - \langle x^+ x^- x^+ \rangle) \\ &\subseteq \mathcal{U}_{11}(x^+ - \langle x^+ x^- x^+ \rangle). \end{aligned}$$

In view of Proposition 2.2, von Neumann regular elements can be lifted modulo every left ideal contained in $Rad^+ A$.
(3) $\Longrightarrow$ (4) Let $(x^+, x^-) \in A$ be such that $x^\sigma - \langle x^\sigma x^{-\sigma} x^\sigma \rangle \in Rad^\sigma A$ for all $\sigma \in \{+, -\}$. Then

$$\langle x^+ x^- x^+ \rangle - \langle x^+ x^- x^+ x^- x^+ \rangle \in Rad^+ A$$

and

$$\langle x^+ x^- x^+ x^- x^+ \rangle - \langle x^+ x^- x^+ x^- x^+ x^- x^+ \rangle \in Rad^+ A.$$

Hence we obtain that $\langle x^+ x^- x^+ \rangle - \langle x^+ x^- x^+ x^- x^+ x^- x^+ \rangle \in Rad^+ A$.

Now consider the element $(\langle x^+ x^- x^+ \rangle, x^-) \in A$. By (3), there exist a von Neumann regular element $u^+ \in A$ and $\beta \in \mathcal{U}_{11}$ such that

$$u^+ - \langle x^+ x^- x^+ \rangle = \beta(\langle x^+ x^- x^+ \rangle - \langle x^+ x^- x^+ x^- x^+ x^- x^+ \rangle).$$

Then the proof proceeds as that of iii $\Rightarrow$ iv in [2, Theorem 2].
(4) $\Longrightarrow$ (2) is also similar to [2, Theorem 2]. Similarly, we can prove that the equivalent conditions in the last proposition also hold if we replace $Rad^+ A$ in condition (3) with $Rad^- A$. ∎

Theorem 2.1. *Let A be an associative pair with $Rad\ A = (Rad^+ A, Rad^- A)$. The following are equivalent:*

(1) *Idempotents can be lifted modulo $Rad\ A$.*
(2) *Idempotents can be lifted modulo each left ideal contained in $Rad\ A$.*
(3) *von Neumann regular elements can be lifted modulo $Rad^+ A$.*
(4) *von Neumann regular elements can be lifted modulo each left ideal contained in $Rad^+ A$.*

Proof. (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (4) are trivial.
(2) $\Longleftrightarrow$ (4) We apply the last proposition.
(4) $\Longrightarrow$ (3) Let $x^+ \in A^+, x^- \in A^-$ be such that $x^+ - \langle x^+ x^- x^+ \rangle \in Rad^+ A$. As von Neumann regular elements can be lifted modulo $Rad^+ A$ by assumption, there exists a von Neumann regular element $u^+ \in A^+$ such that $u^+ - x^+ \in Rad^+ A$. We may assume that $u^+ = \langle u^+ y^- u^+ \rangle$ for some $y^- \in A^-$. We have that $y^-(u^+ - x^+) \in Rad\ \mathcal{U}_{22}$ by Proposition 2.1, hence $Id - y^-(u^+ - x^+)$ is invertible in $\mathcal{U}_{22}$.

Let $\alpha = Id - y^-(u^+ - x^+) \in \mathcal{U}_{22}$ and $f^+ = u^+ \alpha = u^+(Id - y^-(u^+ - x^+)) = u^+ y^- x^+ \in \mathcal{U}_{11} x^+$.

Note that $f^+ = u^+ y^- u^+ \alpha = \langle f^+ \alpha^{-1} y^- f^+ \rangle$, hence f^+ is a von Neumann regular element in A^+. Since $u^+ - x^+ \in Rad^+ A$, we can write $x^+ = u^+ + j^+$ for some $j^+ \in Rad^+ A$.

Let

$$\begin{aligned}\gamma &= x^+x^- - u^+y^-x^+x^- = (Id - u^+y^-)x^+x^- \\ &= (Id - u^+y^-)(u^+ + j^+)x^- = (Id - u^+y^-)j^+x^-.\end{aligned}$$

In view of Proposition 2.1, $j^+x^- \in Rad\,\mathcal{U}_{11}$. Hence $\gamma \in Rad\,\mathcal{U}_{11}$ since $Rad\,\mathcal{U}_{11}$ is an ideal of $\mathcal{U}_{11}$. Therefore, we obtain that there exist a von Neumann regular element $f^+ \in \mathcal{U}_{11}x^+$ and $\beta = Id \in \mathcal{U}_{11}$ such that

$$(Id - f^+x^-) - \beta(Id - x^+x^-) = \gamma \in Rad\,\mathcal{U}_{11}.$$

Thus we conclude that von Neumann regular elements can be lifted modulo each left ideal contained in $Rad^+ A$ by Proposition 2.2.

(1) $\Longrightarrow$ (3) If $x^+ \in A^+, x^- \in A^-$ are such that $x^+ - \langle x^+x^-x^+\rangle \in Rad^+ A$, then

$$\langle x^+x^-x^+\rangle - \langle x^+x^-x^+x^-x^+\rangle \in Rad^+ A,$$
$$\langle x^-x^+x^-\rangle - \langle x^-x^+x^-x^+x^-x^+x^-\rangle \in Rad^- A.$$

Consider the element $(x^+, \langle x^-x^+x^-\rangle) \in A$. By (1), there exists an idempotent $(e^+, e^-) \in A$ such that

$$e^+ - x^+ \in Rad^+ A, \quad e^- - \langle x^-x^+x^-\rangle \in Rad^- A.$$

Then e^+ is a von Neumann regular element lifting x^+ in A^+ modulo $Rad^+ A$, as desired.

Let R be a ring with unit. The set $\mathcal{R} = (R, R)$ can form a natural associative pair over the center of R, denoted by $C(R)$. The $\langle\ \rangle$ mappings are defined as $\langle xyz\rangle = x{\cdot}y{\cdot}z$ for any $x, y, z \in R$, where $\cdot$ denotes the product in R. It is straightforward to show that $Rad\ \mathcal{R} = (Rad\ R, Rad\ R)$. ∎

Proposition 2.4. *Let R be a ring with unit and $\mathcal{R} = (R, R)$. be the corresponding associative pair defined as above. Then idempotents can be lifted modulo $Rad\ \mathcal{R} = (Rad\ R, Rad\ R)$ if and only if idempotents can be lifted modulo $Rad\ R$.*

Proof. Let $x \in R$ be such that $x - x^2 \in Rad\ R$. Then $x - x^3 \in Rad\ R$. Consider the element $(x, x) \in \mathcal{R}$. Since idempotents can be lifted modulo $Rad\ \mathcal{R} = (Rad\ R, Rad\ R)$, then idempotents can be lifted modulo each left ideal contained in $Rad\ \mathcal{R}$. By Proposition 2.3, we obtain that there exists an idempotent element $(e^+, e^-) \in \mathcal{R}$ such that $e^+ - x = \lambda(x - x^3)$ and $e^- - x = \mu(x - x^3)$ for some $\lambda, \mu \in R$. Note that $x - x^3 \in Rad\ R$, thus we have

$$x - e^+e^- = x - (x + \lambda(x - x^3))(x + \mu(x - x^3)) \in Rad\ R.$$

Therefore, the idempotent e^+e^- lifts x modulo $Rad\ R$.

Conversely, let $(x^+, x^-) \in \mathcal{R}$ be such that $x^+ - \langle x^+x^-x^+\rangle \in Rad^+\mathcal{R} = Rad\ R$. Then $x^+x^- - x^+x^-x^+x^- \in Rad\ R$. In view of [5, Theorem 2.4, Lemma 2.3(b)(iv)], idempotents can be lifted modulo every left ideal contained in $Rad\ R$, and hence there exists an idempotent $e = e^2 \in R$ such that $e \in Rx^+x^-$ and $1 - e \in R(1 - x^+x^-)$.

Assume that $e = \nu x^+x^-$ and $1 - e = \omega(1 - x^+x^-)$ for some $\nu, \omega \in R$. Then $u^+ = e\nu x^+$ is a von Neumann regular element of $\mathcal{R}^+ = R$ with $u^+ = \langle u^+x^-u^+\rangle$, verifying $u^+ \in \mathcal{U}_{11}x^+$ and $Id - u^+x^- = \omega(1 - x^+x^-)$. In view of Proposition 2.2, von Neumann regular elements can be lifted modulo every left ideal contained in

$Rad^+\mathcal{R} = Rad\ R$. Hence idempotents can be lifted modulo $Rad\ \mathcal{R}$ by Theorem 2.1, as desired. ∎

Combined with [5, Theorem 2.4], the last two propositions have the following pleasant consequence.

Corollary 2.1. *Let R be a ring with unit and $\mathcal{R} = (R, R)$ be the corresponding associative pair. The following are equivalent:*

(1) *Idempotents can be lifted modulo $Rad\ R$.*
(2) *Idempotents can be lifted modulo each left ideal contained in $Rad\ R$.*
(3) *Idempotents can be lifted modulo $Rad\ \mathcal{R}$.*
(4) *Idempotents can be lifted modulo each left ideal contained in $Rad\ \mathcal{R}$.*
(5) *von Neumann regular elements can be lifted modulo $Rad\ R$.*
(6) *von Neumann regular elements can be lifted modulo each left ideal contained in $Rad\ R$.*

Theorem 2.2. *Let A be an associative pair with its standard embedding $\mathcal{U}_A$. If $Rad\,\mathcal{U}_A$ is a lifting ideal of $\mathcal{U}_A$, then A has the same property.*

Proof. Let

$$e = \begin{pmatrix} 1_{\mathcal{U}_{11}} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{U}_A,$$

then $e\mathcal{U}_A e \simeq \mathcal{U}_{11}$.

By [8, Proposition 1.10], if $\mathcal{U}_A$ satisfies that idempotents can be lifted modulo $Rad\,\mathcal{U}_A$, then idempotents also can be lifted modulo $Rad\,\mathcal{U}_{11}$.

Let $x^+ \in A^+, x^- \in A^-$ be such that $x^+ - \langle x^+x^-x^+\rangle \in Rad^+A$. By Proposition 2.1, $x^+x^- - x^+x^-x^+x^- \in Rad\,\mathcal{U}_{11}$. Since idempotents can be lifted modulo $Rad\,\mathcal{U}_{11}$, by Corollary 2.1, idempotents also can be lifted modulo every left ideal contained in $Rad\,\mathcal{U}_{11}$. In view of [5, Lemma 2.3], there exists an idempotent $\alpha = \alpha^2 = \beta x^+x^- \in \mathcal{U}_{11}x^+x^-$ for some $\beta \in \mathcal{U}_{11}$ such that $Id - \alpha \in \mathcal{U}_{11}(Id - x^+x^-)$.

Thus $\alpha\beta x^+ \in \mathcal{U}_{11}x^+$ is a von Neumann regular element such that

$$Id - \alpha\beta x^+x^- = Id - \alpha \in \mathcal{U}_{11}(Id - x^+x^-).$$

Therefore, von Neumann regular elements can be lifted modulo every left ideal contained in Rad^+A by Proposition 2.2. In view of Proposition 2.3, we conclude that idempotents lift in A modulo $Rad\ A$, as asserted. ∎

Question. If A is a unital associative pair, does the converse of the above theorem hold?

Let R be a ring with unit, $\mathcal{R} = (R, R)$ be the corresponding associative pair. One can check that the standard embedding of $\mathcal{R}$ is $M_2(R)$, the 2×2 matrix ring of R. If the answer is affirmative, then we obtain that if a ring R is such that idempotent can be lifted modulo $J(R)$, then $M_2(R)$ has the same property. Incidentally, Nicholson posed the same question in [7, p. 363].

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