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Quasirecognition by the Prime Graph of the Group $C_n(2)$, Where $n \neq 3$ is Odd

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Abstract. Let G be a finite group and let $\Gamma(G)$ be the prime graph of G. We assume that n is an odd number. In this paper, we show that if $\Gamma(G) = \Gamma(C_n(2))$, where $n \neq 3$, then G has a unique nonabelian composition factor isomorphic to $C_n(2)$. As consequences of our result, $C_n(2)$ is quasirecognizable by its spectrum and by a new proof the validity of a conjecture of W. J. Shi for $C_n(2)$ is obtained.

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1. Introduction

The *spectrum* $\omega(G)$ of a finite group G is the set of element orders of G, i.e., a natural number n is in $\omega(G)$ if there is an element of order n in G. A finite nonabelian simple group G is called *quasirecognizable by its spectrum*, if each finite group H with $\omega(G) = \omega(H)$ has a unique nonabelian composition factor isomorphic to G (see [4]).

If G is a finite group, we denote by $\pi(G)$ the set of all prime divisors of $|G|$. The *prime graph* (or *Gruenberg-Kegel graph*) $\Gamma(G)$ of G is the graph with vertex set $\pi(G)$ where two distinct vertices p and q are adjacent by an edge (we write $(p, q) \in \Gamma(G)$) if $p.q \in \omega(G)$ and we denote by $s(G)$ the number of connected components of $\Gamma(G)$. Finite groups G satisfying $\Gamma(G) = \Gamma(S)$ have been determined, where S is one of the following groups: A sporadic simple group [7]; a CIT simple group [13]; $PSL(2, q)$ where $q = p^\alpha < 100$ [16]; $PSL(2, p)$ where $p > 3$ is a prime [14]; $G_2(7)$ [25]; ${}^2G_2(q)$ where $q = 3^{2m+1} > 3$ [8, 25]; $L_{16}(2)$ [15]; $PSL(2, q)$ [9, 11]. A finite nonabelian simple group G is *quasirecognizable by its prime graph*, if each finite group P with $\Gamma(P) = \Gamma(G)$ has a unique nonabelian composition factor isomorphic to G [8]. The quasirecognition of the following simple nonabelian groups by their prime graphs have been obtained: Alternating group A_p where p and $p-2$ are primes [17]; $L_{10}(2)$

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[10]; ${}^2F_4(q)$ where $q = 2^{2m+1}$ for some $m \geq 1$ [3]; ${}^2D_p(3)$ where $p = 2^n + 1 \geq 5$ is a prime [5]; $F_4(q)$ where $q = 2^n > 2$ [12]. In this paper, we show that the group $C_n(2)$ is quasirecognizable by its prime graph. In fact, we prove that the following main theorem.

Theorem 1.1. *Let n be an odd number. The simple group $C_n(2)$, where $n \neq 3$, is quasirecognizable by its prime graph.*

It is obvious that the knowledge of $\omega(G)$ determines $\Gamma(G)$, but not vice versa in general and hence, as the first corollary of the main theorem, the group $C_n(2)$, where n is an odd number, is quasirecognizable by its spectrum.

For every n and q, the simple groups $B_n(q)$ and $C_n(q)$ have the same order and these groups are isomorphic if and only if $n = 2$ or q is even. Also, we know that if n is an odd prime, then $s(C_n(2)) = 2$ and if n is an odd non-prime, then $s(C_n(2)) = 1$. In [21], the first example of an infinite series of finite simple groups with connected prime graph which are quasirecognizable by their prime graphs is obtained. Since $G_2(7)$ and some sporadic simple groups are the only finite simple groups with two prime graph components which their quasirecognizability by their prime graphs have been obtained (see [7, 25]), hence as the second corollary of the main theorem, the group $C_n(2)$, where n is an odd prime, is the first example of an infinite series of finite simple groups with two prime graph components which are quasirecognizable by their prime graphs.

Shi in [19], put forward the following conjecture.

Conjecture 1.1. *Let G be a finite group and let M be a finite simple group. Then $G \cong M$ if and only if (i) $|G| = |M|$ and (ii) $\omega(G) = \omega(M)$.*

A series of papers proved that this conjecture is valid for most of finite simple groups (see a survey in [20]) and the last step of the proof of this conjecture is to prove that the conjecture holds for the simple groups $B_n(q)$ and $C_n(q)$. Also, it is just proved that this conjecture is valid for these groups as well [22]. As another corollary of the main theorem, by a new proof the validity of this conjecture is obtained for the group under study. Of course, for the spacial case, i.e., $n = p$, Shi and Chen in [18] proved that this conjecture is valid for the group $C_p(2)$.

Throughout this paper, we use the following notations: By $[x]$ we denote the integer part of x and by $gcd(a_1, a_2, \cdots, a_n)$ we denote the greatest common divisor of numbers $a_1, a_2, \cdots, a_n$. A set of vertices of a graph is called a coclique (or independent), if its elements are pairwise non-adjacent. We denote by $\rho(G)$ and $\rho(r, G)$ a coclique of maximal size in $GK(G)$ and a coclique of maximal size, containing r, in $GK(G)$, respectively. Also, we put $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$. Also, we assume that $q = p^\alpha$, where p is a prime and α is a natural number. All further unexplained notations are standard and can be found, for example, in [6].

2. Preliminaries

Lemma 2.1. [21, Proposition 1] *Let G be a finite group, $t(G) \geq 3$, and let K be the maximal normal soluble subgroup of G. Then for every subset ρ of primes in $\pi(G)$ such that $|\rho| \geq 3$ and all primes in ρ are pairwise non-adjacent in $\Gamma(G)$, the intersection $\pi(K) \cap \rho$ contains at most one number. In particular, G is insoluble.*

Lemma 2.2. [21, Theorem] *Let G be a finite group satisfying the two conditions:*

(a) *There exist three primes in $\pi(G)$ pairwise non-adjacent in $\Gamma(G)$; i.e., $t(G) \geq 3$;*

(b) *There exists an odd prime in $\pi(G)$ non-adjacent in $\Gamma(G)$ to the prime 2; i.e., $t(2, G) \geq 2$.*

Then there is a finite nonabelian simple group S such that $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal soluble subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$ and one of the following statements holds:

(1) *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S)=t(2,S)=3$.*

(2) *For every prime $p \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2,S) \geq t(2,G)$.*

Lemma 2.3. [21, Proposition 3] *Let G be a group satisfying the conditions of Lemma 2.2 and let groups K, S, and $\bar{G}$ be as in the claim of Lemma 2.2. Then $t(S) \geq t(G) - 1$. Moreover, for every subset ρ of $\pi(G)$ such that $|\rho| \geq 3$ and all primes in ρ are pairwise non-adjacent in $\Gamma(G)$, at most one prime in ρ divides the product $|K|.|\bar{G}/S|$.*

Lemma 2.4. *Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Then:*

(1) *If $(p, q) \in \Gamma(H)$, then $(p, q) \in \Gamma(G)$;*

(2) *If $(p, q) \in \Gamma(G/N)$, then $(p, q) \in \Gamma(G)$;*

(3) *If $(p, q) \in \Gamma(G)$ and $\{p, q\} \cap \pi(N) = \emptyset$, then $(p, q) \in \Gamma(G/N)$.*

Proof. The proof is straightforward. ∎

Let s be a prime and let m be a natural number. The s-part of m is denoted by m_s, i.e., $m_s = s^t$ if $s^t \mid m$ and $s^{t+1} \nmid m$. If $gcd(s, m) = 1$ and s is odd, then by $e(s, m)$ we mean that $s \mid (m^{e(s,m)} - 1)$ but $s \nmid (m^a - 1)$ for all natural number a with $a < e(s, m)$. If m is odd, we put $e(2, m) = 1$, if $m \equiv 1 \pmod 4$ and $e(2, m) = 2$ if $m \equiv -1 \pmod 4$.

Lemma 2.5. [24, Corollary of Zsigmondy's theorem] *Let q be a natural number greater than 1. For every natural number m there exists a prime r with $e(r, q) = m$, but for the cases $q = 2$ and $m = 1$, $q = 3$ and $m = 1$, and $q = 2$ and $m = 6$.*

The prime s with $e(s, m) = n$ is called a *primitive prime divisor* of $m^n - 1$. It is obvious that $m^n - 1$ can have more than one primitive prime divisor. We denote by $r_n(m)$ some primitive prime divisor of $m^n - 1$.

We write $A_n^\varepsilon(q)$ and $D_n^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$ and $D_n^-(q) = {}^2D_n(q)$. Also, $\nu(n)$ and $\eta(n)$ for an integer n, are defined in [23] as follow:

$$\nu(n) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 4; \\ n/2 & \text{if} \quad n \equiv 2 \pmod 4; \\ 2n & \text{if} \quad n \equiv 1 \pmod 2). \end{cases}, \ \eta(n) = \begin{cases} n & \text{if} \quad n \ \text{ is odd}; \\ n/2 & \text{otherwise}. \end{cases}$$

Lemma 2.6. [23, Proposition 4.1] *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q - 1$, and s be an odd prime distinct from the characteristic. Put $k = e(s, q)$. Then s and r are non-adjacent if and only if one of the following holds:*

(1) *$k = n$, $n_r \leq (q-1)_r$, and if $n_r = (q-1)_r$, then $2 < (q-1)_r$;*
(2) *$k = n-1$ and $(q-1)_r \leq n_r$.*

Lemma 2.7. [23, Proposition 4.2] *Let $G = {}^2A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q+1$, and s be an odd prime distinct from the characteristic. Put $k = e(s,q)$. Then s and r are non-adjacent if and only if one of the following holds:*

(1) *$\nu(k) = n$, $n_r \leq (q+1)_r$, and if $n_r = (q+1)_r$, then $2 < (q+1)_r$;*
(2) *$\nu(k) = n-1$ and $(q+1)_r \leq n_r$.*

Lemma 2.8. [23, Propositions 2.1,2.2] *Let G be a finite simple group of Lie type over a field of order q with characteristic p. Let r and s be odd prime and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$.*

(1) *If $G = A_{n-1}(q)$ and $2 \leq k \leq l$, then r and s are non-adjacent if and only if $k + l > n$ and k does not divide l;*
(2) *If $G = {}^2A_{n-1}(q)$ and $2 \leq \nu(k) \leq \nu(l)$, then r and s are non-adjacent if and only if $\nu(k) + \nu(l) > n$ and $\nu(k)$ does not divide $\nu(l)$.*

Lemma 2.9. [24, Proposition 2.7(5)] *Let $G = E_7(q)$ be a finite simple exceptional group of Lie type over a field of characteristic p, suppose that r, s are odd primes, and assume that $r, s \in \pi(G) \setminus \{p\}$, $k = e(r,q)$ and $l = e(s,q)$, and $1 \leq k \leq l$. Then r and s are non-adjacent if and only if $k \neq l$ and either $l = 5$ and $k = 4$, or $l = 6$ and $k = 5$, or $l \in \{14, 18\}$ and $k \neq 2$, or $l \in \{7, 9\}$ and $k \geq 2$, or $l = 8$ and $k \geq 3$, $k \neq 4$, or $l = 10$ and $k \geq 3$, $k \neq 6$, or $l = 12$ and $k \geq 4, k \neq 6$.*

Lemma 2.10. [24, Proposition 2.4] *Let G be one of simple groups of Lie type, $B_n(q)$ or $C_n(q)$, over a field of characteristic p. Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $1 \leq \eta(k) \leq \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k) + \eta(l) > n$ and k, l satisfy to (2.1):*

$$l/k \text{ is not an odd natural number.} \tag{2.1}$$

Lemma 2.11. [23, Propositions 3.1] *Let G be a finite simple classical group of Lie type defined over a field of characteristic p. Let $r \in \pi(G)$ and $r \neq p$. Then r and p are non-adjacent if and only if one of the following holds:*

(1) *$G = A_{n-1}(q)$, r is odd, and $e(r,q) > n-2$;*
(2) *$G = {}^2A_{n-1}(q)$, r is odd, and $\nu(e(r,q)) > n-2$;*
(3) *$G = C_n(q)$, $\eta(e(r,q)) > n-1$;*
(4) *$G = B_n(q)$, $\eta(e(r,q)) > n-1$;*
(5) *$G = D_n^{\varepsilon}(q)$, where $\varepsilon \in \{\pm\}$, $\eta(e(r,q)) > n-2$;*
(6) *$G = A_1(q)$, $r = 2$;*
(7) *$G = A_2^{\varepsilon}(q)$, where $\varepsilon \in \{\pm\}$, $r = 3$ and $(q - \varepsilon 1)_3 = 3$.*

3. Proof of the main theorem

Proof of Theorem 1.1. Let G be a finite group with $\Gamma(G) = \Gamma(C_n(2))$, where n is an odd number and $n \neq 3$. By the results of the previous section, we prove the main theorem in four parts. Since we use Tables 2–8 in [23] many times, we bring only these tables without the reference number [23].

Part A. Let $n \geq 17$. By Tables 4 and 8 we have:

(1) $t(C_n(2)) = [(3n+5)/4] \geq 14$, $\rho(C_n(2)) = \{r_{2i}(2) | (n+1)/2 \leq i \leq n\} \cup \{r_i(2) | (n+1)/2 \leq i \leq n, i \equiv 1 \pmod 2\}$,

(2) $t(2, C_n(2)) = 3$, $\rho(2, C_n(2)) = \{2, r_n(2), r_{2n}(2)\}$.

Since $t(G) = t(C_n(2)) \geq 14$ and $t(2, G) = t(2, C_n(2)) = 3$, the conclusion of Lemma 2.2 holds for G. Let S be the nonabelian simple group which is obtained in that lemma. If S is a simple sporadic or an exceptional group of Lie type, then $t(S) \leq t(E_8(q)) = 12$ (see Table 4 in [24] and Table 2 in [23]). But this is impossible, because $t(S) \geq t(G) - 1 \geq 13$. Hence, S is either an alternating group or a classical group of Lie type. We will prove that $S \cong C_n(2)$ in two steps:

Step I. The simple group S can not be an alternating group $A_m, m \geq 5$.

If $S \cong A_m$, where $m \geq 5$, then since $t(S) \geq 13$, by Table 3, we can see that $m \geq 7$ and $\rho(2, S) = \tau(2, m) \cup \{2\}$, where $\tau(2, m) = \{s \mid s \text{ is a prime}, m - 3 \leq s \leq m\}$. Also, by Lemma 2.2(2), we can see that, $\rho(2, C_n(2)) = \rho(2, G) \subseteq \rho(2, S)$. Hence, $\{r_n(2), r_{2n}(2)\} \subseteq \tau(2, m)$. On the other hand, by the definition of $\tau(2, m)$, $m \geq 7$, we can assume that if $x_1 \neq x_2 \in \tau(2, m)$, then $x_1 - x_2 = \varepsilon$, where $\varepsilon \in \{\pm 2\}$. Thus, $r_n(2) - r_{2n}(2) = \varepsilon$, where $\varepsilon \in \{\pm 2\}$. Also, by Fermat's little theorem, $2n = e(r_{2n}(2), 2) \mid (r_{2n}(2) - 1)$ and $n = e(r_n(2), 2) \mid (r_n(2) - 1)$. Therefore, $n \mid \varepsilon$ which implies that $n = 1$ and this is impossible.

Step II. If S is a classical Lie type group, then we will prove that $S \cong C_n(2)$. We prove this, with a case by case analysis.

Case 1. S can not be a simple group of type $A_{n'-1}(q)$, where $q = p^{\alpha}$.

If $S \cong A_{n'-1}(q)$, since $t(S) \geq 13$, by Table 8 we can see that $n' \geq 25$. Then by Tables 4 and 6, we distinguish the following two subcases:

(i) If $p = 2$, then by Table 4, $\rho(2, S) = \{2, r_{n'-1}(2^{\alpha}), r_{n'}(2^{\alpha})\}$ and since $\rho(2, C_n(2)) \subseteq \rho(2, S)$, we conclude that each number in the set $\{r_{n'-1}(2^{\alpha}), r_{n'}(2^{\alpha})\}$ is a primitive prime divisor of $2^n - 1$ or $2^{2n} - 1$.

If $r_{n'-1}(2^{\alpha}) = r_n(2), r_{n'}(2^{\alpha}) = r_{2n}(2)$, then we can see that $(n'-1)\alpha = n, n'\alpha = 2n$. Therefore, $n' = 2$, which is a contradiction.

If $r_{n'-1}(2^{\alpha}) = r_{2n}(2), r_{n'}(2^{\alpha}) = r_n(2)$, then we can see that $(n'-1)\alpha = 2n, n'\alpha = n$. Therefore, $n' = -1$, which is a contradiction.

(ii) If $p \neq 2$, then $gcd(2, p) = 1$. Let $t = e(p, 2)$, then $t \geq 2$ and by Lemma 2.5, $t \neq 6$. In follows, we consider t in different cases:

a. If $t = 2$, then $p = 3$. Since $n' \geq 25$ and $|S| = |A_{n'-1}(q)| = 1/gcd(n', q-1)$ $q^{(n'-1)n'/2} \prod_{i=2}^{n'} (q^i - 1)$, thus $13 = r_3(3^{\alpha})$ or $r_1(3^{\alpha}) \in \pi(S)$. Now we calculate $t(13, S)$. If $13 = r_3(3^{\alpha})$, then by Lemmas 2.6, 2.11(1), we can see that $(2, 13) \in \Gamma(S)$, $(r_1(q), 13) \in \Gamma(S)$ and $(3, 13) \in \Gamma(S)$. Therefore, if $13 \neq x \in \rho(13, S)$, then x is an odd number distinct from $3, 13$ and $r_1(q)$ and if $e(x, q) = l$, then by Lemma 2.8(1), we conclude that $l + 3 > n'$ and $3 \nmid l$. Therefore, $l \in \{n', n'-1, n'-2\}$

and $3 \nmid l$. Since $n', n'-1, n'-2$ are three consecutive numbers, then 3 divides exactly one of them and we have exactly two choices for l and hence, two elements of the set $\{r_{n'}(q), r_{n'-1}(q), r_{n'-2}(q)\}$ can be chosen for x. Also, by Lemma 2.8(1), we can see that this set is independent. Thus, $t(13, S) = 3$. If $13 = r_1(3^\alpha)$, then by Lemma 2.6, we can see that $t(13, S) \leq 3$. On the other side, $13 \in \pi(S) \subseteq \pi(C_n(2))$ and we can consider $\rho(13, C_n(2))$. Since $\eta(e(13, 2)) = 6$, by Lemma 2.10, we can see that $\{r_n(2), r_{n-2}(2), r_{n-4}(2), r_{2n}(2), r_{2(n-2)}(2), r_{2(n-4)}(2)\} \subseteq \rho(13, C_n(2))$ and also, since 6 divides at most one of $n-1, n-3$ and $n-5$, thus at least two elements of the set $\{r_{2(n-1)}(2), r_{2(n-3)}(2), r_{2(n-5)}(2)\}$ are in $\rho(13, C_n(2))$. Hence, $t(13, C_n(2)) \geq 8$. Now by assuming $\rho = \rho(13, C_n(2))$ in Lemma 2.3, we can see that $t(13, S) \geq |\rho(13, C_n(2)) \cap \pi(S)| \geq t(13, C_n(2)) - 1$, and hence, $3 \geq t(13, S) \geq t(13, C_n(2)) - 1 \geq 8 - 1 = 7$, which is impossible.

b. If $t = 3$, then $p = 7$ and similar to the subcase (ii)(a), we can see that $t(19, S) \leq 3$. On the other hand, $19 \in \pi(C_n(2))$ and $\eta(e(19, 2)) = 9$. By Lemma 2.10, we can see that $\{r_{2(n-1)}(2), r_{2(n-3)}(2), r_{2(n-5)}(2), r_{2(n-7)}(2)\} \subseteq \rho(19, C_n(2))$ and also, since 9 divides at most one of $n, n-2, n-4, n-6, n-8$, thus at least four elements of the set $\{r_{2n}(2), r_{2(n-2)}(2), r_{2(n-4)}(2), r_{2(n-6)}(2), r_{2(n-8)}(2)\}$ are in $\rho(19, C_n(2))$. Hence $t(19, C_n(2)) \geq 8$. Thus similarly to the subcase (ii)(a), we get a contradiction.

c. If $t = 4$, then $p = 5$ and similar to the subcase (ii)(a), we can see that $t(31, S) \leq 3$. On the other hand, $31 \in \pi(C_n(2))$ and $\eta(e(31, 2)) = 5$. By Lemma 2.10, we see that $\{r_{2(n-1)}(2), r_{2(n-3)}(2)\} \subseteq \rho(31, C_n(2))$ and also, since 5 divides at most one of $n, n-2$ and $n-4$, thus at least four elements of $\{r_n(2), r_{2n}(2), r_{n-2}(2), r_{2(n-2)}(2), r_{n-4}(2), r_{2(n-4)}(2)\}$ are in $\rho(31, C_n(2))$. Hence $t(31, C_n(2)) \geq 6$ and we can get a contradiction similarly to the subcase(ii)(a).

d. If $t = 5$, then $p = 31$ and it is enough to choose 331 instead of 13 in the subcase (ii)(a) and use $\rho(331, C_n(2))$. Since $\eta(e(331, 2)) = 15$, by Lemma 2.10, we can see that $\{r_{2(n-1)}(2), r_{2(n-3)}(2), r_{2(n-5)}(2), r_{2(n-7)}(2)\} \subseteq \rho(331, C_n(2))$ and also, since 15 divides at most one of $n, n-2$ and $n-4$, thus at least four elements of $\{r_n(2), r_{2n}(2), r_{n-2}(2), r_{2(n-2)}(2), r_{n-4}(2), r_{2(n-4)}(2)\}$ are in $\rho(331, C_n(2))$. Hence, $t(331, C_n(2)) \geq 8$ and we can get a contradiction similarly to the subcase (ii)(a).

e. If $t = 7$, then $p = 127$ and it is enough to choose 5419 instead of 13 in the subcase (ii)(a). On the other hand, since $5419 \in \pi(C_n(2))$ and $\eta(e(5419, 2)) = 21$ and $|C_n(2)| = 2^{n^2} \prod_{i=1}^{n} (2^{2i} - 1)$, therefore we can assume that $n \geq 21$ and by Lemma 2.10, we see that $\{r_{2(n-1)}(2), r_{2(n-3)}(2), r_{2(n-5)}(2), r_{2(n-7)}(2), r_{2(n-9)}(2)\} \subseteq \rho(5419, C_n(2))$ and hence, $t(5419, C_n(2)) \geq 5$. Thus similar to the subcase (ii)(a), we get a contradiction .

f. For $t \geq 8$, if t is an odd number, set $\rho = \{r_{2(n-1)}(2), r_{2(n-3)}(2), r_{2(n-5)}(2), r_{2(n-7)}(2)\}$. Since n and t are odd numbers and $n \geq 17$, by Lemma 2.10, we can see that $\rho \subseteq \rho(p, C_n(2)) \setminus \{p\}$ and since $S \leq G/K$, by Lemma 2.4(1,2), $\rho \cap \pi(S) \subseteq \rho(p, S) \setminus \{p\}$ and hence, by Table 4, $|\rho \cap \pi(S)| \leq 2$. But, by Lemma 2.1 we conclude that $|\rho \cap \pi(S)| \geq |\rho| - 1 = 3$ and this is a contradiction. Also, if t is an even number except $10, 14$, where $t/2$ is an odd number, similar to the previous argument, we get a contradiction. If $t = 10$, then $p = 11$ and it is enough to follow the subcase (ii)(b) for getting a contradiction. If $t = 14$, then $p = 43$ and similar to the subcase (ii)(a), we can see that $t(631, S) \leq 3$. On the other hand, $631 \in \pi(C_n(2))$ and $\eta(e(631, 2)) = 45$ and $|C_n(2)| = 2^{n^2} \prod_{i=1}^{n}(2^{2i} - 1)$, therefore we can assume that $n \geq 45$ and by Lemma 2.10, we see that $\{r_{2(n-1)}(2), r_{2(n-3)}(2), r_{2(n-5)}(2), r_{2(n-7)}(2), r_{2(n-9)}(2)\} \subseteq \rho(631, C_n(2))$ and hence, $t(631, C_n(2)) \geq 5$. Thus similar to the subcase (ii)(a), we get a contradiction.

If t and $t/2$ are even, it is enough to replace ρ with the set $\{r_n(2), r_{2n}(2), r_{n-2}(2), r_{2(n-2)}(2)\}$ in the previous argument and get a contradiction.

Hence, by (i),(ii), we have shown that $S \ncong A_{n'-1}(q)$. Also, similar to the case 1, we can see that S can not be a simple group of type $D_{n'}(q)$ or ${}^2D_{n'}(q)$. We omit the details for convenience.

Case 2. S can not be a simple group of type ${}^2A_{n'-1}(q)$.

If $S \cong {}^2A_{n'-1}(q)$, since $t(S) \geq 13$, by Table 8, we can see that $n' \geq 25$. Then by Tables 4 and 6, we consider cases (i) and (ii) in follows:

(i) If $p = 2$, then by Table 4, we can assume four different cases for n' as follows:

If $n' \equiv 0 \pmod 4$, then $\rho(2, S) = \{2, r_{2n'-2}(2^\alpha), r_{n'}(2^\alpha)\}$ and hence, $r_n(2) \in \{r_{2n'-2}(2^\alpha), r_{n'}(2^\alpha)\}$. Therefore, $n = 2(n' - 1)\alpha$ or $n = n'\alpha$ and since n is odd and $4 \mid n'$, we get a contradiction.

If $n' \equiv 1 \pmod 4$, then $\rho(2, S) = \{2, r_{n'-1}(2^\alpha), r_{2n'}(2^\alpha)\}$ and similar to the above procedure, we get a contradiction.

If $n' \equiv 2 \pmod 4$, then $\rho(2, S) = \{2, r_{2n'-2}(2^\alpha), r_{n'/2}(2^\alpha)\}$ and since $\rho(2, C_n(2)) \subseteq \rho(2, S)$, $\{r_{2n'-2}(2^\alpha), r_{n'/2}(2^\alpha)\} = \{r_n(2), r_{2n}(2)\}$ and hence, we can assume that either $r_{2n'-2}(2^\alpha) = r_n(2)$ and $r_{n'/2}(2^\alpha) = r_{2n}(2)$, or $r_{2n'-2}(2^\alpha) = r_{2n}(2)$ and $r_{n'/2}(2^\alpha) = r_n(2)$. Therefore, $2 \mid n$ or $n' = 2$, which is impossible.

If $n' \equiv 3 \pmod 4$, then $\rho(2, S) = \{2, r_{(n'-1)/2}(2^\alpha), r_{2n'}(2^\alpha)\}$ and similar to the above procedure, we get a contradiction.

(ii) If $p \neq 2$, then by Table 6 and $n' \geq 25$, we see that $2 < n'_2 = (q + 1)_2$ and $\rho(2, S) = \{2, r_{2n'-2}(q), r_{n'}(q)\}$. Similar to case 1, we want to get a contradiction by considering $t = e(p, 2)$ in different cases. We know that $t \geq 2$ and by Lemma 2.5, $t \neq 6$.

a. If $t = 2$, then $p = 3$ and by $2 < n'_2 = (q + 1)_2$, we have 4 divides $3^\alpha + 1$ and n' and hence, α is odd and since $e(5, 3) = 4$, $e(5, 3^\alpha) = 4$. Thus by Lemma 2.8(2), $(5, r_{n'}(3^\alpha)) \in \Gamma(S)$. But $e(5, 2) = 4$ and n is odd, hence by Lemma 2.10, we can see that $(5, r_n(2)), (5, r_{2n}(2)) \notin \Gamma(C_n(2))$, which is a contradiction. Also, by the same procedure, we conclude that $t \notin \{3, 5, 7, 10, 14\}$. Moreover, since 4 does not divide $q + 1$, we can easily see that $t \neq 4$.

b. If $t \geq 8$ and $t \neq 10, 14$, by Table 4, $t(p, S) = 3$ and similar to the subcase (ii)(f) of case 1, we get a contradiction.

Hence, by (i),(ii), we have shown that $S \ncong {}^2A_{n'-1}(q)$.

Case 3. If $S \cong C_{n'}(q)$. Since $t(S) \geq 13$ and $t(2,S) \geq 3$, by Tables 4,6 and 8, we have the following information:

(1) n' is odd and $n' \geq 17$,
(2) $p = 2$ and $\rho(2,S) = \{2, r_{n'}(2^\alpha), r_{2n'}(2^\alpha)\}$.

In this case, for proving $S \cong C_n(2)$, it is enough to show that $n = n'$ and $\alpha = 1$. Since $\rho(2, C_n(2)) \subseteq \rho(2,S)$, $\{r_n(2), r_{2n}(2)\}$=$\{r_{n'}(2^\alpha), r_{2n'}(2^\alpha)\}$. If $r_n(2) = r_{2n'}(2^\alpha)$, then $n = 2n'\alpha$, which is impossible, because n is odd. Therefore, $r_n(2) = r_{n'}(2^\alpha)$, and it implies that $n = n'\alpha$.

Now by contradiction, we prove that $\alpha = 1$ and hence, $n = n'$. Let $\rho = \{r_{2(n-1)}(2), r_{2(n-2)}(2), r_{2(n-4)}(2)\}$. By Lemma 2.10, $\rho \subseteq \rho(C_n(2))$. We claim that $\rho \cap \pi(S) = \emptyset$.

We know that $|S| = 2^{\alpha n'^2} \prod_{i=1}^{n'} (2^{2\alpha i} - 1)$. If $r_{2(n-1)}(2) \in \pi(S)$, then there exists an integer m, such that $m \geq 0$ and $r_{2(n-1)}(2) \mid (2^{2(n'-m)\alpha} - 1)$, which implies that $e(r_{2(n-1)}(2), 2) \mid 2(n'-m)\alpha$ and hence, $n-1 \mid (n'\alpha - m\alpha) = n - m\alpha$. Thus $m\alpha \leq 1$ and since $\alpha > 1$, we conclude that $m = 0$ and hence, $n - 1 \mid n$, which is impossible for $n \geq 3$. Therefore, $r_{2(n-1)}(2) \notin \pi(S)$. Also, by the same argument, we can see that $r_{2(n-2)}(2), r_{2(n-4)}(2) \notin \pi(S)$. Thus our assertion is proved.

Since $|\rho| = 3$ and $\rho \cap \pi(S) = \emptyset$, by Lemma 2.3, we get a contradiction and this contradiction proves that $\alpha = 1$ and hence, $S \cong C_n(2)$.

Part B. $9 \leq n \leq 15$. By Tables 4 and 8, we have:

(1) $\rho(2, C_n(2)) = \{2, r_n(2), r_{2n}(2)\}$;
(2) $\rho(C_9(2)) = \{r_5(2), r_7(2), r_9(2), r_{10}(2), r_{12}(2), r_{14}(2), r_{16}(2), r_{18}(2)\}$;
(3) $\rho(C_{11}(2)) = \{r_7(2), r_9(2), r_{11}(2), r_{12}(2), r_{14}(2), r_{16}(2), r_{18}(2), r_{20}(2), r_{22}(2)\}$;
(4) $\rho(C_{13}(2)) = \{r_7(2), r_9(2), r_{11}(2), r_{13}(2), r_{14}(2), r_{16}(2), r_{18}(2), r_{20}(2), r_{22}(2), r_{24}(2), r_{26}(2)\}$;
(5) $\rho(C_{15}(2)) = \{r_9(2), r_{11}(2), r_{13}(2), r_{15}(2), r_{16}(2), r_{18}(2), r_{20}(2), r_{22}(2), r_{24}(2), r_{26}(2), r_{28}(2), r_{30}(2)\}$.

Since $t(G) = t(C_n(2)) \geq t(C_9(2)) = 8$ and $t(2,G) = t(2, C_n(2)) = 3$, the conclusion of the Lemma 2.2 holds for G. Let S be the nonabelian simple group which is obtained in that lemma. Since $\rho(2, C_n(2)) \subseteq \rho(2,S)$, similar to step I, part A, we can conclude that S can not be an alternating group. Also, since $n \in \{9, 11, 13, 15\}$, by Table 2 and the numbers $r_9(2) = 73$, $r_{22}(2) = 683$, $r_{13}(2) = 8191$ and $r_{30}(2) = 331$, we can see that S can not be a sporadic group as well. Thus, S can be a classical or an exceptional Lie type group and we will consider them separately.

Case 1. If S is a classical Lie type group, then S can be one of the groups $A_{n'-1}(q), {}^2A_{n'-1}(q), C_{n'}(2^\alpha), D_{n'}(q), {}^2D_{n'}(q)$. Since $n \geq 9$, by Table 8, we can see that for all of these groups $n' \geq 8$. If $p = 2$, then we can prove that $S \cong C_n(2)$ similarly to part A, and if $p \neq 2$, then we consider different cases for $t = e(p,2)$ and get a contradiction as follow:

If $t = 2$, then $p = 3$. Since $n' \geq 8$, $\{r_1(q), r_2(q), r_5(q), r_{10}(q)\} \subseteq \pi(S)$ and we know that $e(61,3) = 10$, thus $61 \in \pi(S)$. On the other hand, $e(61,2) = 60$ and since

$n \leq 15$, $61 \notin \pi(C_n(2)) = \pi(G)$, which is a contradiction. Also, if $t = 3, 4, 5, 8$, by replacing 61 with $2801, 71, 17351, 88741$, respectively we get a contradiction. Thus, $t \geq 7$ and $t \neq 8$. In this case, by Table 4, we have $t(p, S) = 3$ and according to $\rho(C_n(2))$, we can easily choose a six-element independent set in $\Gamma(C_n(2))$, namely ρ such that $p \notin \rho$ and get a contradiction by Lemma 2.3. Consequently, if S is a classical Lie type group, then $S \cong C_n(2)$.

Case 2. If S is an exceptional Lie type group. Since $n \geq 9$, $t(S) \geq t(C_9(2)) - 1 = 7$ and by Table 4 in [24], $S \cong E_7(q)$ *or* $E_8(q)$. If $p \neq 2$, by Table 5, we have $t(p, S) = 5$. Thus, all of the statements in part B(case 1) for $p \neq 2$, can be used here and get a contradiction. Therefore, $p = 2$. By Table 5, $\rho(2, E_7(2^\alpha)) = \{2, r_7(2^\alpha), r_9(2^\alpha), r_{14}(2^\alpha), r_{18}(2^\alpha)\}$ and $\rho(2, E_8(2^\alpha)) = \{2, r_{15}(2^\alpha), r_{20}(2^\alpha), r_{24}(2^\alpha), r_{30}(2^\alpha)\}$. Thus, according to $\rho(2, C_{11}(2))$, $\rho(2, C_{13}(2))$ and $\rho(2, C_n(2)) \subseteq \rho(2, S)$, for $n = 11, 13$, we conclude that S can not be an exceptional Lie type group. Also, if $n = 15$, then $t(S) \geq t(C_{15}(2)) - 1 = 11$ and since $\rho(2, C_{15}(2)) \subseteq \rho(2, S)$, thus S can only be isomorphic to $E_8(2)$. Let $\rho = \{r_{11}(2), r_{13}(2), r_{22}(2), r_{26}(2)\}$. By Lemma 2.10, ρ is an independent set in $\Gamma(C_{15}(2))$ and by $|E_8(2)|$, we can see that $\rho \cap \pi(E_8(2)) = \emptyset$ and this is a contradiction by Lemma 2.3. Hence, for n=15, S can not be an exceptional Lie type group. If $n = 9$, then by $\rho(2, C_9(2)) \subseteq \rho(2, S)$, we have $S \cong E_7(2)$ and since $|Out(E_7(2))| = 1$ and $S \leq G/K \leq \mathrm{Aut}(S)$, thus $G/K \cong E_7(2)$. Moreover, by $|C_9(2)|$ and $|E_7(2)|$, we have $r_{16}(2) \in \pi(C_9(2)) = \pi(G)$ and $r_{16}(2) \notin \pi(E_7(2))$. Hence, by using Lemma 2.1 for $\rho = \rho(C_9(2))$ and $\rho = \rho(r_4(2), C_9(2))$, we can see that $\{r_7(2), r_4(2)\} \cap \pi(K) = \emptyset$. On the other hand, by [24, Proposition 2.7(5)] and Lemma 2.10, $(r_7(2), r_4(2)) \in \Gamma(C_9(2))$ and $(r_7(2), r_4(2)) \notin \Gamma(E_7(2))$. Now we can get a contradiction by Lemma 2.4(3).

Part C. If $n = 7$, then by $|C_7(2)|$ and Tables 4 and 8, we have the following information:

(1) $\pi(C_7(2)) = \{2, 3, 5, 7, 11, 13, 17, 31, 43, 127\}$;
(2) $\rho(2, C_7(2)) = \{2, 43, 127\}$;
(3) $\rho(C_7(2)) = \{11, 13, 17, 31, 43, 127\}$.

Since $t(G) = t(C_7(2)) = 6$ and $t(2, G) = t(2, C_7(2)) = 3$, the conclusion of the Lemma 2.2 holds for G. Let S be the nonabelian simple group which is obtained in that lemma. Thus, $t(S) \geq 5$, $t(2, S) \geq 3$ and $\rho(2, C_7(2)) \subseteq \rho(2, S)$. Similarly to step I, part A, we can conclude that S can not be an alternating group. Also, since $127 \in \pi(C_7(2))$, according to the orders of sporadic groups, S is not a sporadic group as well. Thus, S can be a classical or an exceptional Lie type group and we will consider them separately:

Case 1. If S is a classical Lie type group, then S can be one of the groups $A_{n'-1}(q), {}^2A_{n'-1}(q), C_{n'}(2^\alpha), D_{n'}(q), {}^2D_{n'}(q)$. If $p = 2$, then since $\rho(2, C_7(2)) \subseteq \rho(2, S)$ and 7 is a prime number, similarly to part A, we can conclude that $S \cong C_7(2)$. If $p \neq 2$, then $S \leq G/K$ implies that $|\pi(S)| \leq |\pi(C_7(2))| = 10$. Thus, we can find an upper bound for n'. For instance, if $S \cong A_{n'-1}(q)$, then according to $|S|$, we conclude that $n' \leq 10$. Also, since $t(S) \geq 5$, by Table 8, we can see that $n' \geq 11$ and this is impossible. For the other cases, by finding an upper bound for n' and using

$\rho(2, C_7(2)) \subseteq \rho(2, S)$ and Fermat's little theorem, we can get a contradiction. For the sake of convenience, we omit the details. Therefore, if S is a classical Lie type group, then $S \cong C_7(2)$.

Case 2. If S is an exceptional Lie type group of characteristic $p \neq 2$, then according to Tables 7 and 9, S can be one of the groups $F_4(q), E_6(q), {}^2E_6(q), E_7(q), E_8(q), {}^2G_2(3^{2n+1})$. By the same procedure which has been mentioned in case 1 for $p \neq 2$, we can see that the only possibility for S is the group ${}^2G_2(3^{2n+1})$. Let $q = 3^{2n+1}$. By Table 7, $\rho(2, {}^2G_2(q)) = \{2, s_1, s_2\}$, where s_1 divides $q - \sqrt{3q} + 1$ and s_2 divides $q + \sqrt{3q} + 1$. Since $\rho(2, C_7(2)) \subseteq \{2, s_1, s_2\}$, thus $43 \in \{s_1, s_2\}$ and it implies that $43 \mid q^3 + 1 \mid q^6 - 1$. Therefore, $42 = e(43, 3) \mid 6(2n + 1)$. So there exists an odd number k, such that $2n + 1 = 7k$. Since $q^3 + 1 \mid |{}^2G_2(q)|$, we conclude that $3^{21} + 1 \mid |{}^2G_2(q)|$ and hence, $547 \in \pi(S) \setminus \pi(C_7(2))$ which is a contradiction. Also, if S is an exceptional Lie type group of characteristic 2, then by the same procedure for $p \neq 2$, we can get a contradiction.

Part D. If $n = 5$, then by $|C_5(2)|$ and Tables 4 and 8, we have the following information:

(1) $\pi(C_5(2)) = \{2, 3, 5, 7, 11, 17, 31\}$;
(2) $\rho(2, C_5(2)) = \{2, 11, 31\}$;
(3) $\rho(C_5(2)) = \{7, 11, 17, 31\}$.

Since $t(G) = t(C_5(2)) = 4$ and $t(2, G) = t(2, C_5(2)) = 3$, the conclusion of the Lemma 2.2 holds for G. Let S be the nonabelian simple group which is obtained in that lemma. If the second statement holds for S, then $\rho(2, C_5(2)) \subseteq \rho(2, S)$ and orders of sporadic groups conclude that S can not be a sporadic group. Moreover, by the same procedure in part C and using Lemma 2.3 for $\rho(C_5(2))$, we can see that $S \cong C_5(2)$, so we omit the details for convenience. If the first statement holds for S, then $S \cong A_7$ or $A_1(q)$ for some odd q. By using Lemma 2.3 for $\rho = \rho(C_5(2))$, we conclude that $S \ncong A_7$. Also, if $S \cong A_1(q)$, where q is an odd number, then Lemma 2.3 implies that at least one of the sets $A = \{7, 11, 31\}, B = \{7, 11, 17\}, C = \{11, 31, 17\}, D = \{31, 17, 7\}$ is a subset of $\pi(S)$.

If $A \subseteq \pi(S)$, then since the set $\rho = \{p, r_1(q), r_2(q)\}$ is the only coclique in $\Gamma(S)$, we have $A = \rho$. Thus, $p \in \{7, 11, 31\}$. If $p = 7$ and $q = p^\alpha$, then $11 \in \{r_1(q), r_2(q)\}$ and hence, $11 \mid 7^{2\alpha} - 1$. Thus, Fermat's little theorem implies that $5 \mid \alpha$. On the other hand, since $7^\alpha - 1 = q - 1 \mid |S|$ and $7^5 - 1 \mid q - 1$ we conclude that $2801 \in \pi(S) \setminus \pi(C_5(2))$ which is a contradiction. Since the same procedure can be used for all cases of p and the sets B, C and D to get a contradiction, we omit the rest for convenience.

Consequently, S can not be an exceptional Lie type group. ∎

As a consequence of the main theorem we have the following corollaries:

Corollary 3.1. *Let G be a finite group and let n be an odd number with $n \neq 3$. If $|G| = |C_n(2)|$ and $\Gamma(G) = \Gamma(C_n(2))$, then $G \cong C_n(2)$.*

Proof. Since $\Gamma(G) = \Gamma(C_n(2))$, by Lemma 2.2, there exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal soluble subgroup

K of G and according to the main theorem, we have $S \cong C_n(2)$. Moreover, since $|Out(C_n(2))| = 1$ (see [6]) and $|G| = |C_n(2)|$ the conclusion holds. ∎

Corollary 3.2. *Let G be a finite group and let $n \neq 3$ be an odd number. If $|G| = |C_n(2)|$ and $\omega(G) = \omega(C_n(2))$, then $G \cong C_n(2)$.*

Remark 3.1. The *non-commuting graph $\nabla(G)$ of an arbitrary group G is the graph with vertex set $G \setminus Z(G)$ where two distinct vertices x and y are adjacent if $[x, y] \neq 1$. In* [1]*, the authors put forward the following conjecture.*

Conjecture 3.1 (AAM's Conjecture)**.** *If G is a finite nonabelian simple group and H is a group such that $\nabla(G) \cong \nabla(H)$, then $G \cong H$.*

A series of papers proved that this conjecture is true for several finite simple groups. As a consequence of the main theorem, by using the results in [2]*, we can conclude that AAM's Conjecture is valid for the group under study.*

Corollary 3.3. *Let G be a finite group and let n be an odd number with $n \neq 3$. If $\nabla(G) \cong \nabla(C_n(2))$, then $G \cong C_n(2)$.*

Proof. According to Lemma 1.1 and Theorem 2 in [2], we conclude that $|G| = |C_n(2)|$ and $\Gamma(G) = \Gamma(C_n(2))$. Thus the conclusion holds by Corollary 3.1. ∎

Remark 3.2. If G is a finite group with $\Gamma(G) = \Gamma(C_3(2))$, then one of the following holds:

(a) G is solvable and G is a Frobenius group or 2-Frobenius group;
(b) $G/O_{\pi(G)} \cong C_3(2), J_2, A_2(4), A_2(4).2_1, A_2(4).2_3, D_4(2), {}^2A_2(5), {}^2A_2(5).2, {}^2A_3(3), {}^2A_3(3).2_2, {}^2A_3(3).2_3, A_7, A_7.2, A_8, A_8.2, A_9, A_1(7), A_1(7).2, A_1(8), A_1(8).3, {}^2A_2(3)$, or ${}^2A_2(3).2$, where $\pi \subseteq \{2, 3, 5\}$.

Proof. See [7, Theorem 3(6)]. ∎

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