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The Linear Arboricity of Planar Graphs with Maximum Degree at Least Five

[1]XIANG TAN, [2]HONGYU CHEN AND [3]JIANLIANG WU

[1]School of Statistics and Mathematics, Shandong University of Finance,
Jinan, Shandong, 250014, P. R. China
[2]School of Science, Shanghai Institute of Technology,
Shanghai, 201418, P. R. China
[1,2,3] School of Mathematics, Shandong University,
Jinan, Shandong, 250100, P. R. China
[1]xtandw@126.com, [2]hongyuchen86@126.com, [3]jlwu@sdu.edu.cn

Abstract. Let G be a planar graph with maximum degree $\Delta \geq 5$. It is proved that $la(G) = \lceil \Delta(G)/2 \rceil$ if (1) any 4-cycle is not adjacent to an i-cycle for any $i \in \{3,4,5\}$ or (2) G has no intersecting 4-cycles and intersecting i-cycles for some $i \in \{3,6\}$.

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1. Introduction

In this paper, all graphs are finite, simple and undirected. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Let G be a graph. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum (vertex) degree and the minimum (vertex) degree, respectively. A k-, k^+- or k^-- vertex is a vertex of degree k, at least k, or at most k, respectively.

A *linear forest* is a graph in which each component is a path. A map φ from $E(G)$ to $\{1, 2, \ldots, t\}$ is called a *t-linear coloring* if the induced subgraph of edges having the same color α is a linear forest for $1 \leq \alpha \leq t$. The *linear arboricity* $la(G)$ of a graph G defined by Harary [6] is the minimum number t for which G has a t-linear coloring.

Akiyama, Exoo and Harary [1] conjectured that $la(G) = \lceil (\Delta(G)+1)/2 \rceil$ for any regular graph G. It is obvious that $la(G) \geq \lceil \Delta(G)/2 \rceil$. So the conjecture is equivalent to the following conjecture.

Conjecture 1.1. *For any graph G, $\lceil \Delta(G)/2 \rceil \leq la(G) \leq \lceil (\Delta(G)+1)/2 \rceil$.*

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The linear arboricity has been determined for complete bipartite graphs [1], complete regular multipartite graphs [11], Halin graphs [7], series-parallel graphs [10] and regular graphs with $\Delta = 3, 4$ [1] and [2], $5, 6, 8$ [4], and 10 [5].

Conjecture 1.1 has already been proved to be true for all planar graphs, see [9] and [13]. Wu also proved in [15] that for a planar graph G with maximum degree Δ, $la(G) = \lceil \Delta(G)/2 \rceil$ if $\Delta(G) \geq 9$. In [8] and [12], it is proved that if G is a planar graph with $\Delta(G) \geq 7$ and without i-cycles for some $i \in \{4, 5, 6\}$, then $la(G) = \lceil \Delta(G)/2 \rceil$. In [14], it's proved that if G is a planar graph with $\Delta(G) \geq 5$ and without 4-cycles, then $la(G) = \lceil \Delta(G)/2 \rceil$. In [3], it is proved that if G is a planar graph with $\Delta(G) \geq 7$ and without adjacent 4-cycles, then $la(G) = \lceil \Delta(G)/2 \rceil$. In this paper, we obtain that if G is a planar graph with $\Delta(G) \geq 5$, $la(G) = \lceil \Delta(G)/2 \rceil$ if

(1) any 4-cycle is not adjacent to an i-cycle, for any $i \in \{3, 4, 5\}$ or
(2) G has no intersecting 4-cycles and intersecting i-cycles for some $i \in \{3, 6\}$.

2. Main results and their proofs

In this section, all graphs are planar graphs which have been embedded in the plane. For a planar graph G, the degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^-- face is a face of degree k, at least k, or at most k, respectively. $F(v) = \{f \in F(G)$: the face f is incident with $v\}$. For $v \in V(G)$, we use $n_i(v)$ to denote the number of i-vertices that are adjacent to v, $f_i(v)$ to denote the number of i-faces incident with v. A k-face with consecutive vertices v_1, v_2, $\ldots$, v_k along its boundary in some direction is often said to be a $(d(v_1), d(v_2), \ldots, d(v_k))$-face.

Given a t-linear coloring φ and a vertex v of G, we denote $C^i_\varphi(v)$ the set of colors appears i times at v, where $i = 0, 1, 2$. Let $C_\varphi(u, v) = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$, that is, $C_\varphi(u, v)$ is the set of colors that appear at least two times at u and v. A monochromatic path is a path whose edges receive the same color. For two different edges e_1 and e_2 of G, they are said to be in the same color component, denoted by $e_1 \leftrightarrow e_2$ if there is a monochromatic path of G connecting them. Furthermore, if two ends of e_i are known, that is, $e_i = x_i y_i$ $(i = 1, 2)$, then $x_1 y_1 \leftrightarrow x_2 y_2$ denotes more accurately that there is a monochromatic path from x_1 to y_2 passing the edges $x_1 y_1$ and $x_2 y_2$ in G (that is, y_1 and x_2 are internal vertices in the path). Otherwise, we use $x_1 y_1 \not\leftrightarrow x_2 y_2$ (or $e_1 \not\leftrightarrow e_2$) to denote that such monochromatic path connecting them does not exist. Note that $x_1 y_1 \leftrightarrow x_2 y_2$ and $x_1 y_1 \leftrightarrow y_2 x_2$ are different.

Let v be a vertex with $d(v) = d$, denote $f_1, f_2, \ldots, f_d$ be the faces incident with v in a clockwise order, and $v_1, v_2, \ldots, v_d$ be the neighbors of v, where v_i is incident with f_i, f_{i+1}, $i = 1, 2, \ldots, d$. Note that eventually f_1 and f_{d+1} denote the same face.

Theorem 2.1. *Let G be a planar graph with $\Delta(G) \geq 5$. If any 4-cycle is not adjacent to an i-cycle for any $i \in \{3, 4, 5\}$, then $la(G) = \lceil \Delta(G)/2 \rceil$.*

Proof. According to [3], if G is a planar graph with $\Delta(G) \geq 7$ and without adjacent 4-cycles, then $la(G) = \lceil \Delta(G)/2 \rceil$. According to [9] and [13], Conjecture 1.1 is true for all planar graphs. Henceforth, to prove Theorem 2.1, we only need to prove that a planar graph with $\Delta(G) = 6$ and any 4-cycle is not adjacent to an i-cycle for any

$i \in \{3, 4, 5\}$ has a 3-linear coloring. Let $G = (V, E, F)$ be a minimal counterexample to the theorem. First, we prove some lemmas for G.

Lemma 2.1. *For any $uv \in E(G)$, $d_G(u) + d_G(v) \geq 8$.*

Proof. The proof of Lemma 2.1 is similar to that of Lemma 2.2 in [8]. ∎

By Lemma 2.1, we have

(a) $\delta(G) \geq 2$, and the two neighbors of a 2-vertex are 6-vertices, and
(b) any two 3^--vertices are not adjacent, and
(c) any 3-face is incident with three 4^+-vertices, or at least two 5^+-vertices.

In the proofs of the following Lemmas, the notation $xx' \not\leftrightarrow (v, 1)$ denotes there does not exist a path colored with 1 from x to v passing the edge xx'.

Lemma 2.2. *The graph G has the following properties:*

(i) *Each vertex is adjacent to at most two 2-vertices;*
(ii) *there is no $(4, 4, 5^-)$-triangle;*
(iii) *if a vertex u is adjacent to two 2-vertices v, w and incident with a 3-face $uxyu$.*

Then $\min\{d(x), d(y)\} \geq 4$.

Proof. (i) Suppose that v is a vertex adjacent to three 2-vertices x, y, z. let x', y', z' be another neighbors of x, y, z. Since G is minimal, $G^{'} = G - vx$ has a 3-linear coloring φ. Without loss of generality, assume $\varphi(xx') = 1$. If there is a color $c \in C^0_\varphi(v)$, or $c \in C^1_\varphi(v) \setminus \{1\}$, or $c = 1 \in C^1_\varphi(v)$ but $xx' \not\leftrightarrow (v, 1)$, then color directly vx with c. So $C^0_\varphi(v) = \emptyset$, $C^1_\varphi(v) = \{1\}$ and $xx^{'} \leftrightarrow (v, 1)$. This implies that $\varphi(vy) \neq 1$ or $\varphi(vz) \neq 1$. Assume that $\varphi(vy) \neq 1$. Thus we can recolor vy with 1 and color vx with $\varphi(vy)$. So φ is extended to a 3-linear coloring of G, a contradiction. Hence each vertex is adjacent to at most two 2-vertices.

(ii) Suppose G contains a $(4, 4, 5^-)$-face uvw with $d(u) = d(v) = 4$ and $d(w) \leq 5$. Since G is minimal, $G^{'} = G - uv$ has a 3-linear coloring φ. If there is a color α such that $\alpha \notin C_\varphi(u, v)$, or $\alpha \in C^1_\varphi(u) \cap C^1_\varphi(v)$ but $(u, \alpha) \not\leftrightarrow (v, \alpha)$, then we can color uv with α to obtain a 3-linear coloring of G, a contradiction. So $C_\varphi(u, v) = \{1, 2, 3\}$ and for any $\alpha \in C^1_\varphi(u) \cap C^1_\varphi(v)$, we have $(u, \alpha) \leftrightarrow (v, \alpha)$.

Suppose that $\varphi(uw) = \varphi(vw) = 1$. If $C^2_\varphi(v) = \emptyset$, then we can recolor uw with $\{2, 3\} \backslash C^2_\varphi(w)$, and color uv with 1. Otherwise, assume $C^2_\varphi(u) = \{2\}$, then $C^2_\varphi(v) = \{3\}$. Since $d(w) \leq 5$, $\mid C^2_\varphi(w) \mid \leq 2$. Without loss of generality, assume that $3 \notin C^2_\varphi(w)$, thus we can recolor uw with 3 and color uv with 1.

Suppose that $\varphi(uw) \neq \varphi(vw)$. Without loss of generality, assume that $\varphi(uw) = 1$ and $\varphi(vw) = 2$. If $3 \in C^2_\varphi(u)$, then $2 \in C^2_\varphi(v)$ and $1 \in C^1_\varphi(u) \cap C^1_\varphi(v)$. As $(u, 1) \leftrightarrow (v, 1)$, we can get $1 \in C^2_\varphi(w)$, thus $2 \notin C^2_\varphi(w)$ or $3 \notin C^2_\varphi(w)$. If $2 \notin C^2_\varphi(w)$, we can recolor uw with 2 and color uv with 1, otherwise we can recolor vw with 3 and color uv with 2. If $3 \in C^2_\varphi(v)$, similarly to the above case, we omit here. In other case, if $1 \in C^2_\varphi(u)$, then $2 \in C^2_\varphi(v)$, we can recolor uw with 2, vw with 1 and color uv with 1. Otherwise, $C^2_\varphi(w) = \{1, 2\}$, then we can recolor uw with 3 and color uv with 1.

By the above steps, φ is extended to a 3-linear coloring of G, a contradiction.

(iii) Suppose that $\min\{d(x), d(y)\} \leq 3$. Without loss of generality, assume that $d(x) \geq d(y)$. By Lemma 2.2, $d(x) \geq d(y) \geq 3$. So $d(y) = 3$. By Lemma 2.1, $d(x) \geq 5$ and $d(u) = 6$. Let $v^{'}, w^{'}$ be another neighbors of v, w, respectively. Since G is minimal, $G^{'} = G - uv$ has a 3-linear coloring φ. Without loss of generality, assume $\varphi(vv^{'}) = 1$. If there is a color $c \in C_{\varphi}^{0}(u)$, or $c \in C_{\varphi}^{1}(u) \backslash \{1\}$, or $c = 1 \in C_{\varphi}^{1}(u)$ but $vv' \not\leftrightarrow (u, 1)$, then color directly uv with c. So $C_{\varphi}^{0}(u) = \emptyset$, $C_{\varphi}^{1}(u) = \{1\}$ and $vv^{'} \leftrightarrow (u, 1)$. If $\varphi(uw) \neq 1$, then $ww^{'} \not\leftrightarrow (u, 1)$ and it follows that we can recolor uw with 1 and color uv with $\varphi(uw)$. So we have $\varphi(uw) = \varphi(ww^{'}) = 1$, $\varphi(ux) \neq 1$ and $\varphi(uy) \neq 1$. Now let's come back to discuss y and x. If $1 \notin C_{\varphi}^{2}(y)$, then we can recolor uy with 1, and color uv with $\varphi(uy)$. Otherwise, we have $\varphi(xy) = 1$ and then recolor ux with 1, xy with $\varphi(ux)$ and color uv with $\varphi(ux)$. Thus φ is extended to a 3-linear coloring of G, a contradiction. ∎

Lemma 2.3. *G has no subgraph isomorphic to one of the configurations in Figure 1(a)–(c). where the vertices marked by • have no other neighbors in G.*

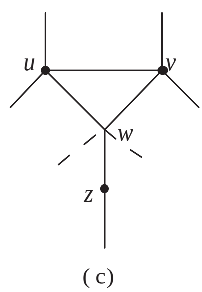

Figure 1. Reducible configurations of Lemma 2.3

Proof. 1(a) Suppose to be contrary, that G has a configuration as depicted in Figure 1(a). Since G is minimal, $G^{'} = G - vx$ has a 3-linear coloring φ. Without loss of generality, assume $\varphi(xx') = 1$. Similarly we have $C_{\varphi}^{0}(v) = \emptyset$, $C_{\varphi}^{1}(v) = \{1\}$ and $xx^{'} \leftrightarrow (v, 1)$. If $\varphi(vy) = 1$, then $\varphi(yy') = 1$(since $xx' \leftrightarrow (v,\ 1)$) and it follows that we can recolor vy' with 1, vy with 1, yy' with $\varphi(vy')$, and color vx with $\varphi(vy')$. Otherwise, we can recolor vy with 1 and color vx with $\varphi(vy)$. Thus we can obtain a 3-linear coloring of G, a contradiction shows that G has no configuration in Figure 1(a).

1(b) Suppose G has a configuration as depicted in Figure 1(b). By the minimality of G, $G^{'} = G - uw$ has a 3-linear coloring φ. If there is a color c such that $c \notin C_{\varphi}(u, w)$, then color directly uw with c, so $C_{\varphi}(u, w) = \{1, 2, 3\}$.

Suppose $\varphi(wx^{'}) = \varphi(wy^{'})$. Without loss of generality, let $\varphi(wx^{'}) = \varphi(wy^{'}) = 1$. Since $d_{G'}(u) = 4$, we have $C_{\varphi}^{0}(u) = \{1\}$. If $1 \notin C_{\varphi}^{2}(v)$, then recolor uv with 1 and color uw with $\varphi(uv)$. Otherwise, we have $\varphi(vx) = \varphi(vy) = 1$. Thus we can recolor

ux with 1, vx with $\varphi(ux)$ and color uw with $\varphi(ux)$. It follows that G is 3-linear colorable, a contradiction shows that G has no configuration in Figure 1(b).

Suppose $\varphi(wx^{'}) \neq \varphi(wy^{'})$. Without loss of generality, let $\varphi(wx^{'}) = 1$, $\varphi(wy^{'}) = 2$, then $C^1_\varphi(u) = \{1, 2\}$. If $c = 1 \in C^1_\varphi(u)$ but $wx^{'} \not\leftrightarrow (u, 1)$ or $c = 2 \in C^1_\varphi(u)$ but $wy^{'} \not\leftrightarrow (u, 2)$, then color directly uw with c. Otherwise, if $\varphi(uv) = 3$, then $|\{1, 2\} \cap C^2_\varphi(v)| \leq 1$, assume $1 \in C^0_\varphi(v) \cup C^1_\varphi(v)$, and then we can recolor uv with 1 and color uw with $\varphi(uv)$. Otherwise, assume $\varphi(uv) = 1$. Since $wx^{'} \leftrightarrow (u, 1)$, we have $\varphi(vy) = 1$ or $\varphi(vx) = 1$. We recolor uv with 2, and color uw with 1. So φ is extended to a 3-linear coloring of G, a contradiction shows that G has no configuration in Figure 1(b).

1(c) Suppose to be contrary, G has a configuration as depicted in Figure 1(c). By the minimality of G, $G^{'} = G - uv$ has a 3-linear coloring φ. If there is a color α such that $\alpha \notin C_\varphi(u, v)$, or $\alpha \in C^1_\varphi(u) \cap C^1_\varphi(v)$, but $(u, \alpha) \not\leftrightarrow (v, \alpha)$, then we can color uv with α to obtain a 3-linear coloring of G, a contradiction. So $C_\varphi(u, v) = \{1, 2, 3\}$, and for any $\alpha \in C^1_\varphi(u) \bigcap C^1_\varphi(v)$, we have $(u, \alpha) \leftrightarrow (v, \alpha)$.

Suppose $\varphi(uw) = \varphi(vw) = 1$, then $\varphi(wz) \neq 1$. Without loss of generality, we can assume $\varphi(wz) = 2$. If $C^2_\varphi(v) = \emptyset$, and $(v, 2) \leftrightarrow wz$, then we can recolor uw with 2, wz with 1, and color uv with 1. If $C^2_\varphi(v) = \emptyset$, but $(v, 2) \not\leftrightarrow wz$, then we can recolor vw with 2, wz with 1, and color uv with 1. If $C^2_\varphi(v) \neq \emptyset$, we can assume $C^2_\varphi(v) = 3$, then $C^2_\varphi(u) = 2$, then we can recolor wv with 2,wz with 1, and color uv with 1.

Suppose $\varphi(uw) = 2, \varphi(vw) = 1$.

Case 1. $3 \in C^2_\varphi(v)$.

If $\varphi(wz) = 1$, then we can recolor uw with 1, wz with 2, and color uv with 2. If $\varphi(wz) = 2$, then we can recolor vw with 2,wz with 1, and color uv with 1. Otherwise, we recolor uw with 3, vw with 2, wz with 1, and color uv with 1.

Case 2. $3 \notin C^2_\varphi(v)$.

If $\varphi(wz) = 1$, then we can recolor uw with 1, wz with 2, and color uv with 2. If $\varphi(wz) = 2$, then we can recolor vw with 2, wz with 1, and color uv with 1. Otherwise $\varphi(wz) = 3$. If $(u, 3) \leftrightarrow zw$, then we can recolor wz with 2, uw with 3, and color uv with 2. Otherwise we can recolor vw with 3, wz with 1, and color uv with 1.

Thus φ is extended to a 3-linear coloring of G, a contradiction shows that G has no configuration in Figure 1(c). ∎

Lemma 2.4. *By the choice of G, we have the following observations.*

(a) *Any vertex is incident with at most $\lfloor d(v)/2 \rfloor$ 4^--faces.*
(b) *If a 2-vertex v is incident with a 5-face, then its another incident face must be a 5^+-face.*
(c) *If a 2-vertex v is incident with a 3-face, then its another incident face must be a 6^+-face.*
(d) *If a 6-face f is adjacent to a $(2, 6, 6)$-face, then f is incident with at most two 2-vertices.*

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(d(v)-4) + \sum_{f\in F}(d(f)-4) = -8 < 0. \tag{2.1}$$

We define ch to be the initial charge. Let $ch(x) = d(x) - 4$ for each $x \in V(G) \cup F(G)$. In the following, we will reassign a new charge denoted by $ch^{'}(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch^{'}(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -8. \tag{2.2}$$

In the following, we will show that $ch^{'}(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (2.2), completing the proof.

The discharging rules are defined as follows.

R1-1. Let v be a 2-vertex. If v is incident with a 3-face, then v receives 1 from its another incident face, and receives $\frac{1}{2}$ from each of its neighbors. If v is incident with a 4-face, then v receives $\frac{2}{3}$ from its another incident face, and receives $\frac{2}{3}$ from each of its neighbors. Otherwise v receives $\frac{1}{2}$ from each of its incident 5^+-face, and receives $\frac{1}{2}$ from each of its neighbors.

R1-2. Let v be a 3-vertex, then v receives $\frac{1}{2}$ from each of its incident 5^+-face.

R1-3. Let f be a 3-face $uvwu$ with $d(u) \leq d(v) \leq d(w)$. If $d(u) = d(v) = 4$, then f receives $\frac{1}{4}$ from each of u and v, and receives $\frac{1}{2}$ from w. If $d(u) \leq 3$, then f receives $\frac{1}{2}$ from each of v and w. Otherwise f receives $\frac{1}{3}$ from each of u, v and w.

R1-4. Let v be a 4-vertex, let f be a 5^+-face and v_1, v_2 be two neighbors of v incident with f. If $d(v_1) = d(v_2) = 4$, then v receives $\frac{1}{5}$ from f. If $\min\{d(v_1), d(v_2)\} \geq 5$, then v receives $\frac{1}{3}$ from f. Otherwise v receives $\frac{1}{4}$ from f.

Let f be a face of G. If $d(f) = 3$, then $ch(f) = 3 - 4 = -1$. Thus

$$ch'(f) \geq -1 + \min\left\{2 \times \frac{1}{2}, \frac{1}{4} \times 2 + \frac{1}{2}, \frac{1}{3} \times 3\right\} = 0.$$

If $d(f) = 4$, then $ch'(f) = ch(f) = 4 - 4 = 0$.

If $d(f) = 5$, then $n_{3^-}(f) \leq 2$ by Lemma 2.1, and $ch(f) = 5 - 4 = 1$. If $n_{3^-}(f) = 2$, then $n_4(f) = 0$, and it follows that $ch'(f) = 1 - \frac{1}{2} \times 2 = 0$ by R1-1 and R1-2. If $n_{3^-}(f) = 1$, then $n_4(f) \leq 2$, and it follows that $ch'(f) \geq 1 - \frac{1}{2} - \max\{\frac{1}{4} \times 2, \frac{1}{3}\} = 0$ by Lemma 2.1 and R1-4. If $n_{3^-}(f) = 0$, then

$$ch'(f) \geq 1 - \max\left\{\frac{1}{5} \times 5, \frac{1}{4} \times 2 + \frac{1}{5} \times 2, \frac{1}{3} \times 3\right\} = 0.$$

If $d(f) = 6$, then $ch'(f) = 6 - 4 = 2$ and $n_{3^-}(f) \leq 3$. If $n_{3^-}(f) = 3$, then $n_4(f) = 0$, and it follows that $ch'(f) \geq 2 - \max\{1 + \frac{1}{2} \times 2, \frac{2}{3} \times 3\} = 0$ by Lemma

2.3. If $n_{3^-}(f) = 2$, then $n_4(f) \leq 1$ by Lemma 2.1, thus

$$ch'(f) \geq 2 - \max\left\{1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{2}{3}, \frac{2}{3} \times 2 + \frac{1}{3}\right\} = \frac{1}{6} > 0.$$

If $n_{3^-}(f) = 1$, then

$$ch'(f) \geq 2 - \max\left\{1 + \frac{1}{4} \times 2 + \frac{1}{5}, 1 + \frac{1}{3} \times 2, 1 + \frac{1}{3}\right\} = \frac{3}{10} > 0.$$

If $n_{3^-}(f) = 0$, then $ch'(f) \geq 2 - \frac{1}{3} \times 6 = 0$.

If $d(f) \geq 7$. Let $d(f) - r$ denote the number of 4-vertices incident with f, then the number of 3^--vertices incident with f is not larger than $\lfloor r/2 \rfloor$ by Lemma 2.1. If $d(f) = 7$, then

$$ch'(f) = 3 - \max\left\{3 \times 1, 2 \times 1 + \frac{1}{3} \times 2, 1 + \frac{1}{3} \times 4, \frac{1}{3} \times 7\right\} \geq 0.$$

If $d(f) \geq 8$, then

$$ch'(f) \geq d(f) - 4 - \frac{1}{3} \times (d(f) - r) - \left\lfloor \frac{r}{2} \right\rfloor \times 1 \geq \frac{2d(f)}{3} - \frac{r}{6} - 4 \geq \frac{d(f)}{2} - 4 \geq 0.$$

Let v be a vertex of G. If $d(v) = 2$, then $ch(v) = 2 - 4 = -2$. And it follows that $ch'(v) \geq -2 + \min\{1 + \frac{1}{2} \times 2, \frac{2}{3} + \frac{2}{3} \times 2, \frac{1}{2} \times 2 + \frac{1}{2} \times 2\} = 0$ by R1-1.

If $d(v) = 3$, then $ch(v) = 3 - 4 = -1$ and $f_{5^+}(v) \geq 2$ by Lemma 2.3. And it follows that $ch'(v) \geq -1 + \frac{1}{2} \times 2 = 0$ by R1-2.

If $d(v) = 4$, then $ch(v) = 4 - 4 = 0$ and $f_3(v) \leq 2$. If $f_3(v) = 2$, then

$$ch'(v) \geq \min\left\{\frac{1}{4} \times 2 - \frac{1}{4} \times 2, \frac{1}{5} + \frac{1}{3} - \frac{1}{4} \times 2, \frac{1}{3} + \frac{1}{4} - \frac{1}{3} - \frac{1}{4}, \frac{1}{3} \times 2 - \frac{1}{3} \times 2\right\} = 0$$

by R1-3 and R1-4. If $f_3(v) = 1$, then $ch'(v) \geq \frac{1}{5} \times 2 - \frac{1}{3} = \frac{1}{15} > 0$. If $f_3(v) = 0$, then $ch'(v) \geq 0$.

If $d(v) = 5$, then $ch(v) = 5 - 4 = 1$ and $f_3(v) \leq 2$. Thus

$$ch'(v) \geq 1 - \max\left\{\frac{1}{2} \times 2, \frac{1}{3} \times 2\right\} = 0$$

by R1-3.

If $d(v) = 6$, then $ch(v) = 6 - 4 = 2$ and $f_3(v) \leq 3$. If $f_3(v) = 3$, then $f_4(v) = 0$ and $n_2(v) \leq 1$ by Lemma 2.3. And it follows that $ch'(v) \geq 2 - \frac{1}{2} - \frac{1}{2} \times 3 = 0$. If $f_3(v) < 3$, then $n_2(v) \leq 2$ by Lemma 2.2. And if $n_2(v) = 2$, then the 3-face incident with v must be a $(6, 4^+, 5^+)$-face by Lemma 2.2 and Lemma 2.3. Thus

$$ch'(v) \geq 2 - \max\left\{\frac{2}{3} \times 2 + \frac{1}{3} \times 2, \frac{2}{3} + \frac{1}{2} \times 2, \frac{1}{2} \times 2\right\} = 0$$

by R1-1 and R1-3.

Hence we complete the proof of the Theorem 2.1. ∎

Theorem 2.2. *Let G be a planar graph with $\Delta(G) \geq 5$. If G has no intersecting 4-cycles and intersecting i-cycles for some $i \in \{3, 6\}$, then $la(G) = \lceil \Delta(G)/2 \rceil$.*

Proof. According to [3], if $\Delta(G) \geq 7$ and without adjacent 4-cycles, then $la(G) = \lceil \Delta(G)/2 \rceil$. According to [9] and [13], Conjecture 1.1 is true for all planar graphs. Henceforth, to prove Theorem 2.2, we only need to prove that a planar graph with $\Delta(G) = 6$ and without intersecting 4-cycles and intersecting i-cycles for some $i \in \{3, 6\}$ has a 3-linear coloring. let $G = (V, E, F)$ be a minimal counterexample to the theorem.

The Lemmas from 2.1 to 2.3 are also true for Theorem 2.2.

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -12 < 0. \tag{2.3}$$

We define ch to be the initial charge. Let $ch(x) = 2d(x) - 6$ for each $x \in V(G)$ and $ch(x) = d(x) - 6$ for each $x \in F(G)$. In the following, we will reassign a new charge denoted by $ch^{'}(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x \in V(G) \cup F(G)} ch^{'}(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12. \tag{2.4}$$

In the following, we will show that $ch^{'}(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (2.4), completing the proof.

First we assume that G has no intersecting 3-cycles and intersecting 4-cycles. The discharging rules are defined as follows.

R2-1. Each 2-vertex receives 1 from each of its neighbors.

R2-2. From each 4-vertex to each of its incident k-faces f, where $3 \leq k \leq 5$, transfer 1, if $k = 3$, 1/2, if $k = 4$, 1/5, if $k = 5$.

R2-3. From each 5^+-vertex to each of its incident k-faces f, where $3 \leq k \leq 5$, transfer 3/2, if $k = 3$, 1, if $k = 4$, 1/3, if $k = 5$.

Let f be a face of G. If $d(f) = 3$, then

$$ch'(f) \geq ch(f) + \min\left\{\frac{3}{2} \times 2, 2 + \frac{3}{2}\right\} = 0.$$

If $d(f) = 4$, then

$$ch'(f) \geq ch(f) + \min\left\{\frac{1}{2} \times 4, 2 \times 1\right\} = 0.$$

If $d(f) = 5$, then

$$ch'(f) \geq ch(f) + \min\left\{\frac{1}{5} \times 5, \frac{1}{3} \times 3\right\} = 0.$$

If $d(f) \geq 6$, then $ch'(f) = d(f) - 6 \geq 0$.

Let v be a vertex of G. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$ by R2-1. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. If $d(v) = 4$, then

$$ch'(v) \geq ch(v) - 1 - \frac{1}{2} - \frac{1}{5} \times 2 = \frac{1}{10} > 0$$

by R2-2. If $d(v) = 5$, then

$$ch'(v) \geq ch(v) - \frac{3}{2} - 1 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$$

by R2-3. If $d(v) = 6$, then

$$ch'(v) \geq ch(v) - 2 - \frac{3}{2} - 1 - \frac{1}{3} \times 4 = \frac{1}{6} > 0$$

by R2-1 and R2-3. Hence we complete the case that G has no intersecting 3-cycles and intersecting 4-cycles.

Now we assume that G has no intersecting 4-cycles and intersecting 6-cycles. The discharging rules are defined as follows.

R3-1. Each 2-vertex receives 1 from each of its neighbors.

R3-2. Let f be a 3-face $uvwu$ with $d(u) \leq d(v) \leq d(w)$. If $d(u) \leq 3$, then f receives $\frac{3}{2}$ from each of v and w. If $d(u) = d(v) = 4$, then f receives $\frac{3}{4}$ from each of u and v, receives $\frac{3}{2}$ from w. If $d(u) = 4$, $d(v) \geq 5$, then f receives $\frac{3}{4}$ from u, receives $\frac{9}{8}$ from each of v and w. If $\delta(f) \geq 5$, then f receives 1 from each of u, v and w.

R3-3. Let f be a 4-face $uvwz$ with $d(u) \leq d(v) \leq d(w) \leq d(z)$. If $d(v) \leq 3$, then f receives 1 from each of w and z. If $d(u) \leq 3$, $d(v) \geq 4$, then f receives $\frac{1}{2}$ from v and $\frac{3}{4}$ from each of w and z. If $d(u) \geq 4$, then f receives $\frac{1}{2}$ from each of u, v, w and z.

R3-4. For a 5-face f and its incident vertex v, f receives $1/5$ if $d(v) = 4$, $1/3$ if $d(v) \geq 5$.

R3-5. For a 5^+-vertex v, v receives $(d(f) - 6)/n$ from each of its incident 7^+-face f. (n denotes the number of 5^+-vertices incident with f).

Let f be a face of G. If $d(f) = 3$, then

$$ch'(f) \geq ch(f) + \min\left\{\frac{3}{2} \times 2, \frac{3}{4} \times 2 + \frac{3}{2}, \frac{3}{4} + \frac{9}{8} \times 2, 1 \times 3\right\} = 0$$

by R3-2 and Lemma 2.1. If $d(f) = 4$, then

$$ch'(f) \geq ch(f) + \min\left\{1 \times 2, \frac{3}{4} \times 2 + \frac{1}{2}, \frac{2}{3} \times 3, \frac{1}{2} \times 4\right\} = 0$$

by R3-3. If $d(f) = 5$, then

$$ch'(f) \geq ch(f) + \min\left\{\frac{1}{3} \times 3, \frac{1}{5} \times 5\right\} = 0$$

by R3-4. If $d(f) = 6$, then $ch'(f) = ch(f) = 0$. If $d(f) \geq 7$, then

$$ch'(f) \geq ch(f) - \frac{ch(f)}{n} \times n = 0$$

by R3-5.

Let v be a vertex of G. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 \times 1 = 0$ by R3-1. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. If $d(v) = 4$, then $f_3(v) \le 2$, and it follows that

$$ch'(v) \ge ch(v) - \max\left\{2 \times \frac{3}{4} + \frac{1}{2}, 2 \times \frac{3}{4} + \frac{1}{5}\right\} = 0$$

by R3-2, R3-3 and R3-4.

If $d(v) = 5$, then $f_3(v) \le 3$. If $f_3(v) = 3$, then $f_4(v) = f_5(v) = 0$ and $n_3(v) \le 1$ by Lemma 2.3. Thus

$$ch'(v) \ge ch(v) - \frac{3}{2} \times 2 - \frac{9}{8} + \frac{1}{7} = \frac{1}{56} > 0$$

by Lemma 2.2. If $f_3(v) = 2$, then

$$ch'(v) \ge ch(v) - \max\left\{\frac{3}{2} + \frac{9}{8} + 1 + 3, 2 \times \frac{3}{2} + \frac{1}{3} \times 2\right\} = \frac{1}{24} > 0.$$

If $f_3(v) \le 1$, then

$$ch'(v) \ge ch(v) - \frac{3}{2} - 1 - \frac{1}{3} \times 3 = \frac{1}{2} > 0.$$

If $d(v) = 6$, then $f_3(v) \le 3$ and $n_2(v) \le 2$ by Lemma 2.2.

Suppose $f_3(v) = 3$, then it must be one of the following cases.

Figure 2. Case 1.1 and Case 1.2

Case 1.1 In this case, $n_2(v) \le 1$, $f_4(v) = 0$ and $f_5(v) \le 1$. Then

$$ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{1}{3} = \frac{1}{6} > 0.$$

Case 1.2 In this case, $n_2(v) \le 1$. If $n_2(v) = 0$, then $f_5(v) = 0$, thus $ch'(v) \ge ch(v) - \frac{3}{2} \times 3 - 1 = \frac{1}{2} > 0$. If $n_2(v) = 1$, assume f_1 is the (6, 6, 2)-face and v_1 is the 2-vertex. Then f_2 must not be a 4-face or a 6-face. If f_2 is a 5-face, then $f_4(v) = 0$, thus

$$ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{1}{3} = \frac{1}{6} > 0.$$

If f_2 is a 7-face, then $f_4(v) = f_5(v) = 0$, thus

$$ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 = \frac{1}{2} > 0.$$

If f_2 is a 8^+-face, then $f_5(v) = 0$, thus

$$ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - 1 + \frac{2}{7} + \frac{1}{7} = \frac{5}{28} > 0.$$

Suppose $f_3(v) = 2$, then it must be one of the following cases.

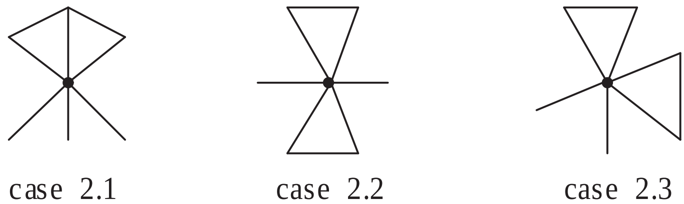

Figure 3. Case 2.1, Case 2.2 and Case 2.3

Case 2.1 In this case, $f_4(v) = 0$, then $ch'(v) \geq ch(v) - 2 - \frac{3}{2} \times 2 - \frac{1}{3} \times 3 = 0$.

Case 2.2 In this case, if $n_2(v) = 2$, then the 3-face incident with v must be a $(6, 4^+, 5^+)$-face by Lemma 2.2 and Lemma 2.3. Thus

$$ch'(v) \geq ch(v) - \max\left\{2 + \frac{9}{8} \times 2 + \frac{3}{4} + \frac{1}{3} \times 3, 1 + \frac{3}{2} \times 2 + 1 + \frac{1}{3} \times 3, \frac{3}{2} \times 2 + 1 + \frac{1}{3} \times 3\right\} = 0.$$

Case 2.3 In this case,

$$ch'(v) \geq ch(v) - \max\left\{2 + \frac{9}{8} \times 2 + \frac{3}{4} + \frac{1}{3} \times 3, 2 + \frac{9}{8} \times 2 + 1 + \frac{1}{3} \times 2, 1 + \frac{3}{2} \times 2 + 1 + \frac{1}{3} \times 3, \frac{3}{2} \times 2 + 1 + \frac{1}{3} \times 3\right\} = 0.$$

Suppose $f_3(v) \leq 1$, then $ch'(v) \geq ch(v) - 2 - \frac{3}{2} - 1 - \frac{1}{3} \times 4 = \frac{1}{6} > 0$.

Hence we complete the proof of Theorem 2.2. ∎

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