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Generalized Fuzzy h-Ideals of Hemirings

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Abstract. The concept of $(\in, \in\vee\mathrm{q})$-fuzzy h-ideals of hemirings is introduced and some characterizations are described. We show that a hemiring S is h-hemiregular if and only if for any $(\in, \in\vee\mathrm{q})$-fuzzy right h-ideal F and $(\in, \in\vee\mathrm{q})$-fuzzy left h-ideal G, $F \circ_{0.5} G = F \cap_{0.5} G$. Finally, the concept of implication-based fuzzy h-ideals of hemirings is considered.

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1. Introduction

Hemirings, as semirings with zero and commutative addition, appear in a natural manner in some applications to the theory of automata and formal languages (see [9, 12, 21]). It is a well known result that regular languages form so-called star semirings. According to the well known theorem of Kleene, the languages, or sets of words, recognized by finite-state automata are precisely those that are obtained from letters of input alphabets by the application of the operations sum (union), product, and star (Kleene closure). If a language is represented as a formal series with the coefficients in a Boolean semiring, then the Kleene theorem can be well described by the Schützenberger Representation theorem. Moreover, if the coefficient semiring is a field, then the corresponding syntactic algebra of the series has a finite rank if and only if the series are rational (see [22, 23] for details).

Many-valued logic has been proposed to model phenomena in which uncertainty and vagueness are involved. One of the most general classes of the many-valued logic is the BL-logic defined as the logic of continuous t-norms. But in fact, BL-logic is a commutative lattice-ordered semiring. So, Łukasiewicz logic, Gödel logic and Product logic, as special cases of BL-logic, are special cases of semirings. The class of K-fuzzy semirings $(K \cup \{+\infty\}, \min, \max)$, where K denotes a subset of the

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power set of R which is closed under the operations min, $+$, or max, has many interesting applications. Min-max-plus computations (and suitable semirings) are used in several areas, e.g., in differential equations. Continuous timed Petri nets can be modelled by using generalized polynomial recurrent equations in the "$(\min, +)$ semiring" (see [2]). It is interesting to observe that the fuzzy calculus, which is used for artificial intelligence purposes, indeed involves essentially "(min, max) semirings" (see [5] for more details and references). Moreover, the same hemirings can be used to study fundamental concepts of the automata theory such as nondeterminism (cf. [21]). Many other applications with references can be found in a guide to the literature on semirings and their applications [8].

Ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals if S is a ring and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. Henriksen defined in [11] a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the semiring S is a ring then a complex in S is a k-ideal if and only if it is a ring ideal. Another more restricted class of ideals has been given in hemirings by Iizuka [14]. However, in an additively commutative semiring S, ideals of a semiring coincide with "ideals" of a ring, provided that the semiring is a hemiring. We now call this ideal an h-ideal of the hemiring S. The properties of h-ideals and also k-ideals of hemirings were thoroughly investigated by Torre [17] and by using the h-ideals and k-ideals, Torre established some analogous ring theorems for hemirings. General properties of fuzzy k-ideals of semirings are described in [30]. Recently, Jun [15] considered the fuzzy setting of h-ideals of hemirings. As a continuation of this paper, Zhan *et al.* [29] discussed the h-hemiregular hemirings by fuzzy h-ideals. Other important results connected with fuzzy h-ideals of hemirings were obtained in [6, 7, 10, 13, 18, 19, 24, 27, 28].

After the introduction of fuzzy sets by Zadeh [26], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in \vee \mathrm{q})$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [1] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [20]. In fact, the $(\in, \in \vee \mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. With this objective in view, Davvaz [3] applied this theory to near-rings and obtained some useful results. Further, Davvaz and Corsini [4] redefined fuzzy H_v-submodule and many valued implications. Recently, Ma *et al.* [18] also discussed the properties of generalized interval-valued fuzzy h-ideals of hemirings. For more details, the reader is referred to [3, 4, 18].

The work of this paper is organized as follows. In Section 2, we first recall some basic definitions and results of hemirings. Since the concept of $(\in, \in \vee q)$-fuzzy h-ideals is an important and useful generalization of ordinary fuzzy h-ideals of hemiring, some fundamental aspects of $(\in, \in \vee q)$-fuzzy h-ideals of hemirings will be discussed in Section 3. Finally, in Section 4, we consider the concept of implication-based fuzzy h-ideals of hemirings.

2. Preliminaries

Recall that a *semiring* is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

$$a(b+c) = ab + ac \quad \text{and} \quad (a+b)c = ac + bc$$

are satisfied for all $a, b, c \in S$.

By *zero* of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called a *hemiring*.

A *left ideal* of a semiring is a subset A of S closed with respect to the addition and such that $SA \subseteq A$.

A left ideal A of a hemiring S is called a *left h-ideal* if for any $x, z \in S$ and $a, b \in A$ from $x + a + z = b + z$, it follows $x \in A$. A *right h-ideal* is defined analogously.

We now recall some fuzzy logic concepts. A fuzzy set is a function $\mu : S \to [0, 1]$.

Definition 2.1. [15] *A fuzzy set F of a hemiring S is called a fuzzy left* (*resp., right*) *h-ideal if it satisfies:*

(F1a) $\forall x, y \in S,\ F(x+y) \geq \min\{F(x), F(y)\}$,
(F1b) $\forall x, y \in S,\ F(xy) \geq F(y)\ (resp., F(xy) \geq F(x))$,
(F1c) $\forall a, b, x, z \in S,\ x + a + z = b + z \longrightarrow F(x) \geq \min\{F(a), F(b)\}$.

Note that a fuzzy left (resp., right) h-ideal F of a hemiring S satisfies the inequality $F(0) \geq F(x)$ for all $x \in S$.

From the Transfer Principle in fuzzy set theory, cf. [16], it follows that a fuzzy set F defined on X can be characterized by level subsets, i.e. by sets of the form

$$U(F; t) = \{x \in X \mid F(x) \geq t\},$$

where $t \in [0, 1]$. Namely, as it is proved in [16], for any algebraic system $\mathfrak{A} = (X, \mathbb{F})$, where $\mathbb{F}$ is a family of operations (also partial) defined on X, the Transfer Principle can be formulated in the following way.

Lemma 2.1. *A fuzzy set F defined on $\mathfrak{A}$ has the property $\mathcal{P}$ if and only if all non-empty level subsets $U(F; t)$ have the property $\mathcal{P}$.*

As a simple consequence of the above property, we obtain the following theorem, which was first proved in [15].

Theorem 2.1. *A fuzzy set F of a hemiring S is a fuzzy left* (*resp., right*) *h-ideal of S if and only if each non-empty level subset $U(F; t)$ is a left* (*resp., right*) *h-ideal of S.*

For any subset A of a hemiring S, χ_A will denote the characteristic function of A.

Theorem 2.2. [15] *A non-empty subset A of a hemiring S is a left (resp., right) h-ideal of S if and only if χ_A is a fuzzy left (resp., right) h-ideal of S.*

Definition 2.2. [29] *A hemiring S is said to be h-hemiregular if for each $a \in S$, there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$.*

The *h-closure* $\overline{A}$ of A in a hemiring S is defined as

$$\overline{A} = \{x \in S \mid x + a_1 + z = a_2 + z \text{ for some } a_1, a_2 \in A,\ z \in S\}.$$

It is clear that if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}} = \overline{A}$ for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$.

Lemma 2.2. [29] *If A and B are, respectively, right and left h-ideals of a hemiring S, then $\overline{AB} \subseteq A \cap B$.*

Lemma 2.3. [29] *A hemiring S is h-hemiregular if and only if for any right h-ideal A and any left h-ideal B, we have $\overline{AB} = A \cap B$.*

Definition 2.3. [15] *Let F and G be fuzzy sets in a hemiring S. Then the h-product of F and G is defined by*

$$(F \circ_h G)(x) = \sup_{x + a_1b_1 + z = a_2b_2 + z} (\min\{F(a_1), \mu(a_2), G(b_1), G(b_2)\})$$

and $(F \circ_h G)(x) = 0$ if x cannot be expressed as $x + a_1b_1 + z = a_2b_2 + z$.

Definition 2.4. [15] *Let F and G be fuzzy sets of a hemiring S, we define $F \cap G$ by $(F \cap G)(x) = \min\{F(x), G(x)\}$, for all $x \in S$.*

In [29], we obtained the following important result.

Theorem 2.3. *A hemiring S is h-hemiregular if and only if for any fuzzy right h-ideal F and fuzzy left h-ideal G, we have $F \circ_h G = F \cap G$.*

3. $(\in, \in\vee\mathrm{q})$-fuzzy h-ideals

A fuzzy set F of a hemiring S of the form

$$F(y) = \begin{cases} t(\neq 0) & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t and is denoted by x_t*. A fuzzy point x_t is said to *belong to* (resp. be *quasi-coincident with*) a fuzzy set F, written as $x_t \in F$ (resp. x_tqF) if $F(x) \geq t$ (resp. $F(x) + t > 1$). If $x_t \in F$ or (resp. and) x_tqF, then we write $x_t \in\vee\mathrm{q}$(resp. $\in\wedge\mathrm{q}$) F . The symbol $\overline{\in\vee\mathrm{q}}$ means $\in\vee\mathrm{q}$ does not hold. Using the notion of " belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy subsets, the concept of (α, β)-fuzzy subsemigroup, where α and β are any two of $\{\in, q, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$ with $\alpha \neq \in\wedge\mathrm{q}$, was introduced in [1]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in, \in\vee\mathrm{q})$-fuzzy subgroup.

In what follows, S is always a hemiring.

Definition 3.1. *A fuzzy set F of S is said to be an $(\in, \in\vee\mathrm{q})$-fuzzy left (resp., right) h-ideal of S if for all $t, r \in (0, 1]$ and $x, y \in S$,*

(F2a) $x_t \in F$ *and* $y_r \in F$ *imply* $(x+y)_{\min\{t,r\}} \in \vee qF$,
(F2b) $y_r \in F$ *implies* $(xy)_r \in\vee\, \mathrm{q}F$ *(resp.* $(yx)_r \in\vee\, \mathrm{q}F$*)*,
(F2c) $a_t \in F$ *and* $b_r \in F$ *imply* $x_{\min\{t,r\}} \in\vee\, \mathrm{q}F$*, for all* $a, b, x, z \in S$ *with* $x+a+z = b+z$.

Example 3.1. Consider the hemiring $(N_0, +, \cdot)$, where N_0 is the set of all non-negative integers. Define a fuzzy set F of N_0 by

$$F(x) = \begin{cases} 0.6 & \text{if } x \in \langle 4\rangle, \\ 0.8 & \text{if } x \in \langle 2\rangle - \langle 4\rangle, \\ 0.2 & \text{otherwise.} \end{cases}$$

One can easily check that F is an $(\in, \in\vee\,\mathrm{q})$- fuzzy left h-ideal of N_0.

Theorem 3.1. *The conditions* $(F2a)$, $(F2b)$ *and* $(F2c)$ *in Definition 3.1, are equivalent to the following conditions respectively:*
(F3a) $\forall x, y \in S, F(x+y) \geq \min\{F(x), F(y), 0.5\}$
(F3b) $\forall x, y \in S, F(xy) \geq \min\{F(y), 0.5\}$ *(resp.* $F(xy) \geq \min\{F(x), 0.5\}$*)*
(F3c) *for all* $a, b, x, z \in S, x+a+z = b+z$ *implies* $F(x) \geq \min\{F(a), F(b), 0.5\}$.

Proof. (F2a) $\Longrightarrow$ (F3a) Suppose that $x, y \in S$, we consider the following cases:
(a) $\min\{F(x), F(y)\} < 0.5$,
(b) $\min\{F(x), F(y)\} \geq 0.5$.

Case (a): Assume that there exist $x, y \in S$ such that $F(x+y) < \min\{F(x), F(y), 0.5\}$, then $0 < t \leq 0.5, x_t \in F$ and $y_t \in F$, but $(x+y)_t \overline{\in} F$. Since $F(x+y)+t \leq 1$, we have $(x+y)_t \overline{q} F$. It follows that $(x+y)_t \overline{\in\vee\,\mathrm{q}} F$, contradiction.

Case (b): Assume that there exist $x, y \in S$ such that $F(x+y) < 0.5$, then $x_{0.5} \in F$ and $y_{0.5} \in F$, but $(x+y)_{0.5} \overline{\in\vee\,\mathrm{q}} F$, a contradiction. Hence (F3a) holds.

(F3a) $\Longrightarrow$ (F2a) Let $x_t \in F$ and $y_r \in F$, then $F(x) \geq t$ and $F(y) \geq r$. Now, we have

$$F(x+y) \geq \min\{F(x), F(y), 0.5\} \geq \min\{t, r, 0.5\}.$$

If $\min\{t, r\} > 0.5$, then $F(x+y) \geq 0.5$, which implies $F(x+y) + \min\{t, r\} > 1$. If $\min\{t, r\} \leq 0.5$, then $F(x+y) \geq \min\{t, r\}$. Therefore $(x+y)_{\min\{t,r\}} \in\vee\,\mathrm{q}F$.

Similarly, we can prove (F2b) $\Longleftrightarrow$ (F3b) and (F2c) $\Longleftrightarrow$ (F3c). ∎

By Definition 3.1 and Theorem 3.1, we immediately deduce the following:

Corollary 3.1. *A fuzzy set* F *of* S *is an* $(\in, \in\vee\,\mathrm{q})$*-fuzzy left* (*resp. right*) h*-ideal of* S *if and only if the conditions* $(F3a)$, $(F3b)$ *and* $(F3c)$ *in Theorem 3.1 hold.*

Corollary 3.2. *Every fuzzy left* (*resp., right*) h*-ideal* F *of* S *is an* $(\in, \in\vee\,\mathrm{q})$*-fuzzy left* (*resp. right*) h*-ideal.*

Remark 3.1. The converse of Corollary 3.2 may not be true. We know that F is an $(\in, \in\vee\,\mathrm{q})$-fuzzy left h-ideal of S in Example 3.1, but it is not a fuzzy left h-ideal of S because $F(0) = 0.6 \not\geq 0.8 = F(2)$.

The following proposition is obvious and we omit the proof.

Proposition 3.1. *A non-empty subset A of S is a left (resp. right) h-ideal of S if and only if χ_A is an $(\in, \in\vee\mathrm{q})$-fuzzy left (resp. right) h-ideal of S.*

Definition 3.2. *Let F and G be fuzzy sets of S. Then 0.5-product of F and G is defined by*

$$(F \circ_{0.5} G)(x) = \sup_{x+a_1b_1+z=a_2b_2+z} (\min\{F(a_1), F(a_2), G(b_1), G(b_2), 0.5\})$$

and $(F \circ_{0.5} G)(x) = 0$ if x cannot be expressed as $x + a_1b_1 + z = a_2b_2 + z$.

Lemma 3.1. *If F and G are $(\in, \in\vee\mathrm{q})$-fuzzy left (resp. right) h-ideals of S, then so is $F \cap_{0.5} G$, where $F \cap_{0.5} G$ is defined by $(F \cap_{0.5} G)(x) = \min\{F(x), G(x), 0.5\}$, for all $x \in S$.*

Proof. We only consider the case of fuzzy left h-ideals, and the proof of fuzzy right h-ideals is similar.

For $x, y \in S$,

$$\begin{aligned}(F \cap_{0.5} G)(x+y) &= \min\{F(x+y), G(x+y), 0.5\}\\ &\geq \min\{\min\{F(x), F(y), 0.5\}, \min\{G(x), G(y), 0.5\}, 0.5\}\\ &= \min\{\min\{F(x), G(x), 0.5\}, \min\{F(y), G(y), 0.5\}, 0.5\}\\ &= \min\{(F \cap_{0.5} G)(x),\end{aligned}$$

$$\begin{aligned}(F \cap_{0.5} G)(y), 0.5\}, (F \cap_{0.5} G)(xy) &= \min\{F(xy), G(xy), 0.5\}\\ &\geq \min\{\min\{F(y), 0.5\}, \min\{G(y), 0.5\}, 0.5\}\\ &= \min\{\min\{F(y), G(y), 0.5\}, 0.5\}\\ &= \min\{(F \cap_{0.5} G)(y), 0.5\},\end{aligned}$$

let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then

$$\begin{aligned}(F \cap_{0.5} G)(x) &= \min\{F(x), G(x), 0.5\}\\ &\geq \min\{\min\{F(a), F(b), 0.5\}, \min\{G(a), G(b), 0.5\}, 0.5\}\\ &= \min\{\min\{F(a), G(a), 0.5\}, \min\{F(b), G(b), 0.5\}, 0.5\}\\ &= \min\{(F \cap_{0.5} G)(a), (F \cap_{0.5} G)(b), 0.5\}.\end{aligned}$$

Therefore $F \cap_{0.5} G$ is an $(\in, \in\vee\mathrm{q})$-fuzzy left h-ideal of S. ∎

Lemma 3.2. *If F and G are an $(\in, \in\vee\mathrm{q})$-fuzzy right h-ideal and an $(\in, \in\vee\mathrm{q})$-fuzzy left h-ideal of S, respectively, then $F \circ_{0.5} G \subseteq F \cap_{0.5} G$.*

Proof. Let F and G be an $(\in, \in\vee\mathrm{q})$-fuzzy right h-ideal and an $(\in, \in\vee\mathrm{q})$-fuzzy left h-ideal of S, respectively. The proof is obvious if $(F \cap_{0.5} G)(x) = 0$. Otherwise, for every $a_i, b_i \in S, i = 1, 2$, satisfying $x + a_1b_1 + z = a_2b_2 + z$, we have

$$\begin{aligned}F(x) &\geq \min\{F(a_1b_1), F(a_2b_2), 0.5\}\\ &\geq \min\{\min\{F(a_1), 0.5\}, \min\{F(a_2), 0.5\}, 0.5\}\\ &= \min\{F(a_1), F(a_2), 0.5\},\end{aligned}$$

as F is an $(\in, \in\vee\mathrm{q})$-fuzzy right h-ideal of S, and

$$G(x) \geq \min\{G(a_1b_1), G(a_2b_2), 0.5\}$$

$$\begin{aligned}&\geq \min\{\min\{F(b_1), 0.5\}, \min\{G(b_2), 0.5\}, 0.5\}\\ &= \min\{G(b_1), G(b_2), 0.5\},\end{aligned}$$

as G is an $(\in, \in \vee \mathrm{q})$-fuzzy left h-ideal of S.

Thus,

$$\begin{aligned}(F \circ_{0.5} G)(x) &= \sup_{x+a_1b_1+z=a_2b_2+z} (\min\{F(a_1), F(a_2), G(b_1), G(b_2), 0.5\})\\ &= \sup_{x+a_1b_1+z=a_2b_2+z} (\min\{F(a_1), F(a_2), 0.5\}, \min\{G(b_1), G(b_2), 0.5\}, 0.5)\\ &\leq \min\{F(x), G(x), 0.5\}\\ &= (F \cap_{0.5} G)(x).\end{aligned}$$

Consequently, $F \circ_{0.5} G \subseteq F \cap_{0.5} G$. ▮

Now, we describe the characterization of h-hemiregular hemirings by $(\in, \in \vee \mathrm{q})$-fuzzy h-ideals, which is a generalization of Theorem 2.3.

Theorem 3.2. *A hemiring S is h-hemiregular if and only if for any $(\in, \in \vee \mathrm{q})$-fuzzy right h-ideal F and $(\in, \in \vee \mathrm{q})$-fuzzy left h-ideal G, $F \circ_{0.5} G = F \cap_{0.5} G$.*

Proof. Let S be an h-hemiregular hemiring, then $F \circ_{0.5} G \subseteq F \cap_{0.5} G$ by Lemma 3.2. For any $a \in S$ there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$. Thus

$$\begin{aligned}(F \circ_{0.5} G)(a) &= \sup_{a+ax_1a+z=ax_2a+z} (\min\{F(ax_1), F(ax_2), G(a), 0.5\})\\ &\geq \min\{F(ax_1), F(ax_2), G(a), 0.5\}\\ &\geq \min\{F(a), G(a), 0.5\}\\ &= (F \cap_{0.5} G)(a),\end{aligned}$$

i.e., $F \cap_{0.5} G \subseteq F \circ_{0.5} G$, whence $F \circ_{0.5} G = F \cap_{0.5} G$.

Conversely, let C and D be, respectively right and left h-ideal of S. Then, by Proposition 3.1, their characteristic functions χ_C and χ_D are, respectively, fuzzy right h-ideal and fuzzy left h-ideal. Moreover, by Lemma 2.2, $\overline{CD} \subseteq C \cap D$. Let $a \in C \cap D$. Then $\chi_C(a) = 1 = \chi_D(a)$. Thus

$$(\chi_C \circ_{0.5} \chi_D)(a) = (\chi_C \cap_{0.5} \chi_D)(a) = \min\{\chi_C(a), \chi_D(a), 0.5\} = 0.5.$$

So, $\min\{\chi_C(a_1), \chi_D(b_1), \chi_C(a_2), \chi_D(b_2), 0.5\} = 0.5$ for some a_1, a_2, b_1, b_2 satisfying the equality $a + a_1b_1 + z = a_2b_2 + z$. hence $\chi_C(a_i) = 1 = \chi_D(b_i)$ for $i = 1, 2$, which implies $a_i \in C$ and $b_i \in D$. This proves $a \in \overline{CD}$, which implies, $C \cap D \subseteq \overline{CD}$. Thus, $\overline{CD} = C \cap D$. It follows from Lemma 2.3 that S is h-hemiregular. ▮

Next, we characterize $(\in, \in \vee \mathrm{q})$-fuzzy h-ideals by their level h-ideals.

Theorem 3.3. *A fuzzy set F of S is an $(\in, \in \vee \mathrm{q})$-fuzzy left (resp. right) h-ideal of S if and only if $U(F; t)(\neq \emptyset)$ is a left (resp. right) h-ideal of S for all $0 < t \leq 0.5$.*

Proof. We only consider the case of $(\in, \in \vee \mathrm{q})$-fuzzy left h-ideals, and the proof of $(\in, \in \vee \mathrm{q})$-fuzzy right h-ideals is similar. Let F be an $(\in, \in \vee \mathrm{q})$-fuzzy left h-ideal of S and $0 < t \leq 0.5$. Let $x, y \in U(F; t)$, then $F(x) \geq t$ and $F(y) \geq t$. Now we have $F(x+y) \geq \min\{F(x), F(y), 0.5\} \geq \min\{t, 0.5\} = t$. which implies, so $x+y \in U(F; t)$. Now, for every $x \in U(F; t)$ and $r \in S$, we have $F(rx) \geq \min\{F(x), 0.5\} \geq \min\{t, 0.5\} = t$, which implies $rx \in U(F;t)$, and so $RU(F;t) \subseteq U(F;t)$. Hence $U(F;t)$ is a left ideal of S.

Now, let $x, z \in S$ and $a, b \in U(F;t)$ be such that $x + a + z = b + z$. Then $F(x) \geq \min\{F(a), F(b), 0.5\} \geq \min\{t, t, 0.5\} = t$, and so, $x \in U(F;t)$. Therefore $U(F;t)$ is a left h-ideal of S.

Conversely, let F be a fuzzy set of S such that $U(F;t)(\neq \emptyset)$ is a left h-ideal of S for all $0 < t \leq 0.5$. For every $x, y \in S$, we can write $F(x) \geq \min\{F(x), F(y), 0.5\} = t_0$, $F(y) \geq \min\{F(x), F(y), 0.5\} = t_0$, then $x, y \in U(F;t_0)$, and so $x + y \in U(F;t_0)$. Hence $F(x+y) \geq \min\{F(x), F(y), 0.5\}$. Also, we have $F(y) \geq \min\{F(y), 0.5\} = s_0$. Hence $y \in U(F;s_0)$, and so $xy \in U(F;s_0)$, for every $x \in S$. Thus $F(xy) \geq s_0 = \min\{F(y), 0.5\}$.

Finally, let $a, b, x, z \in S$ be such that $x+a+z = b+z$. Then we can write $F(a) \geq \min\{F(a), F(b), 0.5\} = r_0$, $F(b) \geq \min\{F(a), F(b), 0.5\} = r_0$. Then $a, b \in U(F;r_0)$, and so $x \in U(F;r_0)$, that is, $F(x) \geq r_0$. Thus, $F(x) \geq \min\{F(a), F(b), 0.5\}$.

Therefore, F is an $(\in, \in\vee\mathrm{q})$-fuzzy left h-ideal of S. ∎

Naturally, a corresponding result should be considered when $U(F;t)$ is an h-ideal of S for all $0.5 < t \leq 1$.

Definition 3.3. *A fuzzy set F of S is said to be an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy left (resp., right) h-ideal of S if for all $t, r \in (0,1]$ and for all $x, y \in S$,*

- (F4a) $(x+y)_{\min\{t,r\}}\overline{\in}F$ *implies* $x_t\overline{\in}\vee\overline{q}F$ *or* $y_r\overline{\in}\vee\overline{q}F$,
- (F4b) $(xy)_r\overline{\in}F$ *(resp.,* $(yx)_r\overline{\in}F$*) implies* $y_r\overline{\in}\vee\overline{q}F$,
- (F4c) $x_{\min\{t,r\}}\overline{\in}F$ *imply* $a_t\overline{\in}\vee\overline{q}F$ *or* $b_r\overline{\in}\vee\overline{q}F$, *for all* $a, b, x, z \in S$ *with* $x+a+z = b+z$.

Theorem 3.4. *A fuzzy set F of S is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy left (resp., right) h-ideal of S if and only if it satisfies:*

- (F5a) $\forall x, y \in S, \max\{F(x+y), 0.5\} \geq \min\{F(x), F(y)\}$
- (F5b) $\forall x, y \in S, \max\{F(xy), 0.5\} \geq F(y)$ *(resp.,* $\max\{F(xy), 0.5\} \geq F(x)$)
- (F5c) *for all* $a, b, x, z \in S, x+a+z = b+z$ *implies* $\max\{F(x), 0.5\} \geq \min\{F(a), F(b), 0.5\}$

Proof. (F4a) $\Longrightarrow$ (F5a) If there exist $x, y \in S$ such that $\max\{F(x+y), 0.5\} < t = \min\{F(x), F(y)\}$, then $0.5 < t \leq 1$, $F(x+y) < t$ and $x, y \in U(F;t)$. Thus, $(x+y)_t \in F$. By (F4a), we have $x_t\overline{q}\ F$ or $y_t\overline{q}\ F$. Then ($t \leq F(x)$ and $t + F(x) \leq 1$) or ($t \leq F(y)$ and $t + F(y) \leq 1$). Thus, $t \leq 0.5$, contradiction.

(F5a) $\Longrightarrow$ (F4a) Let $(x+y)_{\min\{t,r\}} \in F$, then $F(x+y) < \min\{t, r\}$.

- (a) If $F(x+y) \geq \min\{F(x), F(y)\}$, then $\min\{F(x), F(y)\} < \min\{t, r\}$, and consequently, $F(x) < t$ or $F(y) < r$. It follows that $x_t\overline{\in}\ F$ or $y_r\overline{\in}\ F$. Thus, $x_t\overline{\in}\vee\overline{q}\ F$ or $y_r\overline{\in}\vee\overline{q}\ F$.
- (b) If $F(x+y) < \min\{F(x), F(y)\}$, then by (F5a), $0.5 \geq \min\{F(x), F(y)\}$. Putting $x_t \in F, y_r \in F$, then $t \leq F(x) \leq 0.5$ or $r \leq F(y) \leq 0.5$. It follows that $x_t\overline{q}\ F$ or $y_r\overline{q}\ F$, and thus $x_t\overline{\in}\vee\overline{q}\ F$ or $y_r\overline{\in}\vee\overline{q}\ F$.

Similarly, we can prove (F4b) $\Longleftrightarrow$ (F5b) and (F4c) $\Longleftrightarrow$ (F5c). This complete the proof. ∎

Lemma 3.3. *Let F be a fuzzy set of S. Then $U(F;t)(\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $0.5 < t \leq 1$ if and only if F satisfies (F5a), (F5b) and (F5c).*

Proof. We only consider the case of left h-ideals, and the proof of right h-ideals is similar. Assume that $U(F;t)(\neq \emptyset)$ is a left h-ideal of S. Suppose that for some $x, y \in S$, $\max\{F(x+y), 0.5\} < \min\{F(x), F(y)\} = t$, then $0.5 < t \leq 1$, $F(x+y) < t$, and $x, y \in U(F;t)$. Since $x, y \in U(F;t)$ and $U(F;t)$ is a left h-ideal, so $x+y \in U(F;t)$ or $F(x+y) \geq t$, which is a contradiction with $F(x+y) < t$. Hence (F5a) holds. Similarly, we can prove (F5b) and (F5c) hold.

Conversely, suppose that condition (F5a), (F5b) and (F5c) hold. We show that $U(F;t)$ is a left h-ideal of S. Assume that $0.5 < t < 1, x, y \in U(F;t)$ and $a \in S$. Then

$$0.5 < t \leq \min\{F(x), F(y)\} \leq \max\{F(x+y), 0.5\} < F(x+y),$$
$$0.5 < t < F(x) \leq \max\{F(ax), 0.5\} < F(ax),$$

and so $x + y \in U(F;t)$ and $ax \in U(F;t)$. Hence $U(F;t)$ is a left ideal of S. Now, assume that $0.5 < t \leq 1$, let $x, z \in S$ and $a, b \in U(F;t)$ be such that $x + a + z = b + z$. Then $0.5 < t \leq \min\{F(a), F(b)\} \leq \max\{F(x), 0.5\} < F(x)$, which implies $x \in U(F;t)$. Therefore $U(F;t)$ is a left h-ideal of S. ∎

Theorem 3.5. *A fuzzy set F of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp., right) h-ideal of X if and only if $U(F;t)(\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $0.5 < t \leq 1$.*

Proof. It is an immediate consequence of Theorem 3.4 and Lemma 3.3. ∎

Let F be a fuzzy set of a hemiring S and $J = \{t | t \in (0,1]$ and $U(F;t)$ is an empty set or a left (resp., right) h-ideal of $S\}$. In particular, if $J = (0,1]$, then F is an ordinary fuzzy left (resp., right) h-ideal of S (Theorem 2.1); if $J = (0, 0.5]$, F is an $(\in, \in \vee \mathrm{q})$-fuzzy left (resp., right) h-ideal of S(Theorem 3.3); if $J = (0.5, 1]$, F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp., right) h-ideal of S (Theorem 3.5).

In [25], Yuan *et al.* gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and Bhkat and Das'fuzzy subgroup. Based on [25], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy left (resp., right) h-ideals with thresholds in the following way.

Definition 3.4. *Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$, then a fuzzy set F of S is called a fuzzy left (resp., right) h-ideal with thresholds $(\alpha, \beta]$ of S if for all $x, y \in S$,*

(F6a) $\max\{F(x+y), \alpha\} \geq \min\{F(x), F(y), \beta\}$,

(F6b) $\max\{F(xy), \alpha\} \geq \min\{F(y), \beta\}$ *(resp.,* $\max\{F(xy), \alpha\} \geq \min\{F(x), \beta\}$*)*,

(F6c) *for all $a, b, x, z \in S, x+a+z = b+z$ implies* $\max\{F(x), \alpha\} \geq \min\{F(a), F(b), \beta\}$.

Now, we characterize fuzzy left (resp., right) h-ideals with thresholds by their level left (resp., right) h-ideals.

Theorem 3.6. *A fuzzy set F of S is an fuzzy left (resp., right) h-ideal with thresholds $(\alpha, \beta]$ of S if and only if $U(F;t)(\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $\alpha < t \leq \beta$.*

Proof. The proof is similar to the proof of Theorem 3.3 and Lemma 3.3. ∎

Remark 3.2.

(1) By Definition 3.4, we have the following result: If F is a fuzzy left (resp., right) h-ideal with thresholds $(\alpha, \beta]$ of S, then we can conclude that

(i) F is an ordinary fuzzy left (resp., right) h-ideal when $\alpha = 0, \beta = 1$;
(ii) F is an $(\in, \in \vee \mathrm{q})$- fuzzy left (resp., right) h-ideal when $\alpha = 0, \beta = 0.5$;
(iii) F is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp., right) h-ideal when $\alpha = 0.5, \beta = 1$.

(2) By Definition 3.4, we can define other fuzzy left (resp., right) h-ideals of S, such as fuzzy ideal with thresholds $(0.2, 0.8]$, with thresholds $(0.4, 0.6]$ of S, etc.

(3) However, the fuzzy left (resp., right) h-ideal with thresholds of S may not be an ordinary fuzzy left (resp., right) h-ideal, may not be an $(\in, \in \vee \mathrm{q})$-fuzzy left (resp., right) h-ideal, and may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp., right) h-ideal, respectively, as shown by the following example.

Example 3.2. Let $S = \{0, 1, 2, 3\}$ be a hemiring with Cayley tables as follows:

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

.	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

One can easily check that $\{0\}, \{0, 1, 2\}$ and $\{0, 1, 2, 3\}$ are all left h-ideals of S. Define a fuzzy set F of S by $F(0) = 0.6, F(1) = 0.8, F(2) = 0.4$ and $F(3) = 0.2$. Then, we have

$$U(F; t) = \begin{cases} \{0, 1, 2, 3\} & \text{if } \; 0 < t \leq 0.2, \\ \{0, 1, 2\} & \text{if } \; 0.2 < t \leq 0.4, \\ \{0, 1\} & \text{if } \; 0.4 < t \leq 0.6, \\ \{1\} & \text{if } \; 0.6 < t \leq 0.8, \\ \emptyset & \text{if } \; 0.8 < t \leq 1. \end{cases}$$

Thus, F is both a fuzzy left h-ideal with thresholds (0.2,0.4] and a fuzzy left h-ideal with thresholds (0,0.2] of S. But F could neither are fuzzy left h-ideal and $(\in, \in \vee \mathrm{q})$-fuzzy left h-ideal of S, nor is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left h-ideal of S.

4. Implication-based fuzzy h-ideals

Fuzzy logic is an extension of set theoretic variables in terms of the linguistic variable truth. Some operators, like $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic can also be defined by using the truth tables. Also, the extension principle can be used to derive definitions of the operators.

In the fuzzy logic, we denote the truth value of fuzzy proposition P by $[P]$. In the following, we display the fuzzy logical and corresponding set-theoretical notions:

$[x \in F] = F(x)$;
$[x \overline{\in} F] = 1 - F(x)$;
$[P \wedge Q] = \min\{[P], [Q]\}$;
$[P \vee Q] = \max\{[P], [Q]\}$;
$[P \rightarrow Q] = \min\{1, 1 - [P] + [Q]\}$;
$[\forall x P(x)] = \inf[P(x)]$;
$\models P$ if and only if $[P] = 1$ for all valuations.

Of course, various implication operators can be similarly defined. We only show a selection of them in the following table, where x denotes the degree of truth (or degree of membership) of the premise and y denotes the respective values for the consequence, and I the resulting degree of truth for the implication:

Name	Definition of Implication Operators
Early Zadeh	$I_m(x,y) = \max\{1-x, \min\{x,y\}\}$
Lukasiewicz	$I_a(x,y) = \min\{1, 1-x+y\}$
Standard Star(Godel)	$I_g(x,y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ y & \text{if} \quad x > y \end{cases}$
Contraposition of Godel	$I_{cg}(x,y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 1-x & \text{if} \quad x > y \end{cases}$
Gaines-Rescher	$I_{gr}(x,y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{if} \quad x > y \end{cases}$
Kleene-Dienes	$I_b(x,y) = \max\{1-x, y\}$

The "quality" of these implication operators could be evaluated either by empirical or by axiomatic methods.

In the following definition, we consider the implication operators in the Lukasiewicz system of continuous-valued logic.

Definition 4.1. *A fuzzy set F of S is called a fuzzifying left (resp., right) h-ideal of S if it satisfies:*

(F7a) *for any $x, y \in S$, $\models [x \in F] \wedge [y \in F] \rightarrow [x+y \in F]$,*
(F7b) *for any $x, y \in S$, $\models [y \in F] \rightarrow [xy \in F]$ (resp., $\models [x \in F] \rightarrow [xy \in F]$),*
(F7c) *for any $a, b, x, z \in S$ with $x+a+z = b+z$, $\models [a \in F] \wedge [b \in F] \rightarrow [x \in F]$.*

Clearly, Definition 4.1 is equivalent to Definition 2.1. Therefore a fuzzifying left (resp., right) h-ideal is an ordinary fuzzy left (resp., right) h-ideal.

Now, we introduce the concept of t-tautology, i.e.,

$\models_t P$ if and only if $[P] \geq t$, for all valuations.

Next, we can extend the concept of implication-based fuzzy left (resp., right) h-ideals in the following way.

Definition 4.2. *Let F be a fuzzy set of S and $t \in (0,1]$ is a fixed number. Then F is called a t-implication-based fuzzy left (resp., right) h-ideal of S if the following conditions hold:*

(F8a) *for any $x, y \in S$, $\models_t [x \in F] \wedge [y \in F] \rightarrow [x+y \in F]$,*
(F8b) *for any $x, y \in S$, $\models_t [y \in F] \rightarrow [xy \in F]$ (resp., $\models [x \in F] \rightarrow [xy \in F]$),*
(F8c) *for any $a, b, x, z \in S$ with $x+a+z = b+z$, $\models_t [a \in F] \wedge [b \in F] \rightarrow [x \in F]$.*

Now, if I is an implication operator, then we have the following corollary.

Corollary 4.1. *A fuzzy set F of S is a t-implication-based fuzzy left (resp., right) h-ideal of S if and only if it satisfies:*

(F9a) *for any $x, y \in S, I(F(x) \wedge F(y), F(x+y)) \geq t$,*
(F9b) *for any $x, y \in S$, $I(F(y), F(xy)) \geq t$ (resp., $I(F(x), F(xy)) \geq t$),*
(F9c) *for any $a, b, x, z \in S, x+a+z = b+z$ implies, $I(F(a) \wedge F(b), F(x)) \geq t$.*

Let F be a fuzzy set of S. Then we have the following results.

Theorem 4.1.

(i) *Let $I = I_{gr}$. Then F is an 0.5-implication-based fuzzy left (resp., right) h-ideal of S if and only if F is a fuzzy left (resp., right) h-ideal with thresholds $(\alpha = 0, \beta = 1]$ of S;*

(ii) *Let $I = I_g$. Then F is an 0.5-implication-based fuzzy left (resp., right) h-ideal of S if and only if F is a fuzzy left (resp., right) h-ideal with thresholds $(\alpha = 0, \beta = 0.5]$ of S;*

(iii) *Let $I = I_{cg}$. Then F is an 0.5-implication-based fuzzy left (resp., right) h-ideal of S if and only if F is a fuzzy left (resp., right) h-ideal with thresholds $(\alpha = 0.5, \beta = 1]$ of S.*

Proof. We only prove (ii) and the proofs of (i) and (iii) are similar. In the proof of (ii) we only consider the case that F is an 0.5-implication-based fuzzy left h-ideal of S and the proof that F is an 0.5-implication-based fuzzy right h-ideal of S is similar.

Let F be a 0.5-implication-based fuzzy left h-ideal of S, then by Corollary 4.1, we have

(a) $I_g(F(x) \wedge F(y), F(x + y)) \geq 0.5$,
(b) $I_g(F(y), F(xy)) \geq 0.5$,
(c) $I_g(F(a) \wedge F(b), F(x)) \geq 0.5$ with $x + a + z = b + z$.

From (a), we have $F(x+y) \geq \min\{F(x), F(y)\}$ or $\min\{F(x), F(y)\} > F(x+y) \geq 0.5$. Thus, $F(x + y) \geq \min\{F(x), F(y), 0.5\}$, which implies, $\max\{F(x + y), 0\} = F(x + y) \geq \min\{F(x), F(y), 0.5\}$.

From (b), we have $F(xy) \geq F(y)$ or $F(y) > F(xy) \geq 0.5$. Thus, $F(xy) \geq \min\{F(y), 0.5\}$, which implies, $\max\{F(xy), 0\} = F(xy) \geq \min\{F(y), 0.5\}$.

From (c), for any $a, z, x, z \in S$ with $x + a + z = b + z$, we have $F(x) \geq \min\{F(a), F(b)\}$ or $\min\{F(a), F(b)\} > F(x) \geq 0.5$. Thus, $F(x) \geq \min\{F(a), F(b), 0.5\}$, which implies, $\max\{F(x), 0\} = F(x) \geq \min\{F(a), F(b), 0.5\}$.

This proves that F is a fuzzy left h-ideal with thresholds $(\alpha = 0, \beta = 0.5]$ of S.

Conversely, if F is a fuzzy left h-ideal with thresholds $(\alpha = 0, \beta = 0.5]$ of S, then we have

(a) $F(x + y) = \max\{F(x + y), 0\} \geq \min\{F(x), F(y), 0.5\}$,
(b) $F(xy) = \max\{F(xy), 0\} \geq \min\{F(y), 0.5\}$,
(c) for any $a, z, x, z \in S$ with $x + a + z = b + z$, $F(x) = \max\{F(x), 0\} \geq \min\{F(a), F(b), 0.5\}$.

From (a), if $\min\{F(x), F(y), 0.5\} = \min\{F(x), F(y)\}$, then $I_g(F(x) \wedge F(y), F(x+y)) = 1 \geq 0.5$. Otherwise, $I_g(F(x) \wedge F(y), F(x + y)) \geq 0.5$.

From (b), if $\min\{F(y), 0.5\} = F(y)$, then $I_g(F(y), F(xy)) = 1 \geq 0.5$. Otherwise, $I_g(F(y), F(xy)) \geq 0.5$.

From (c), if $\min\{F(a), F(b), 0.5\} = \min\{F(a), F(b)\}$, for any $a, z, x, z \in S$ with $x + a + z = b + z$, then $I_g(F(a) \wedge F(b), F(x)) = 1 \geq 0.5$. Otherwise, $I_g(F(a) \wedge F(b), F(x)) \geq 0.5$.

Therefore, F is a 0.5-implication-based fuzzy left h-ideal of S. ∎

5. Conclusions

To investigate the structure of an algebraic system, it is clear that (fuzzy) ideals with special properties play an important role. In this paper, we considered the notions of $(\in, \in\vee\mathrm{q})$-fuzzy left (resp., right) h-ideals of hemirings and investigated the relationship among these generalized fuzzy left (resp., right) h-ideals of hemirings. Finally, we investigated the concept of implication-based fuzzy left (resp., right) h-ideals of hemirings. It is our hope that this work would serve as a foundation for further study of the theory of hemirings.

In the notions of an (α, β)-fuzzy left (resp., right) h-ideal of hemirings, where α, β is any one of $\in, q, \in\vee\mathrm{q}$ or $\in\wedge\mathrm{q}$, we can consider twelve different types of such structures resulting from three choices of α and four choices of β. But, in this report, we mainly discuss the $(\in, \in\vee\mathrm{q})$-type. In our opinion the future study of fuzzy sets in hemirings can be connected with

(1) focusing on other types and their relationships among them;
(2) investigating prime (semiprime) (α, β)-fuzzy left (resp., right) h-ideals;
(3) establishing a fuzzy spectrum of a hemiring;
(4) considering quotient hemirings via (α, β)-fuzzy left (resp., right) h-ideals.

The obtained results can be used to solve some social networks problems and to decide whether the corresponding graph is balanced or clusterable.

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