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New Characterizations of Some Classes of Finite Groups

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Abstract. Let G be a finite group and $\mathfrak{F}$ a formation of finite groups. We say that a subgroup H of G is $\mathfrak{F}_h$-normal in G if there exists a normal subgroup T of G such that HT is a normal Hall subgroup of G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G. In this paper, we obtain some results about the $\mathfrak{F}_h$-normal subgroups and use them to study the structure of finite groups. Some new characterizations of supersoluble groups, soluble groups and p-nilpotent groups are obtained and some known results are generalized.

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1. Introduction

All groups considered in the paper are finite, the notations and terminology in this paper are standard, as in [4] and [11].

In [15], Wang defined c-normality of a subgroup of a finite group: A subgroup H of a group G is said to be c-normal if there exists a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. In [18], Yang and Guo defined the concept of $\mathfrak{F}_\mathfrak{n}$-supplemented subgroup: A subgroup H of a group G is said to be $\mathfrak{F}_\mathfrak{n}$-supplemented in G if there exists a normal subgroup K of G such that $G = HK$ and $(H \cap K)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G. By using the above subgroups, people has obtained some interesting results (see [5,8,9,15,18]). As a development, we now introduce the following new concept.

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Definition 1.1. *Let $\mathfrak{F}$ be a class of groups and H a subgroup of a group G. H is said to be $\mathfrak{F}_h$-normal in G if there exists a normal subgroup T of G such that HT is a normal Hall subgroup of G and $(H\cap T)H_G/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$.*

Recall that, for a class $\mathfrak{F}$ of groups, a chief factor H/K of a group G is called $\mathfrak{F}$-central (see [12] or [4, Definition 2.4.3]) if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$. The symbol $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathfrak{F}$-central. A subgroup H of G is said to be $\mathfrak{F}$-hypercenter in G if $H \leq Z_\infty^{\mathfrak{F}}(G)$.

A class $\mathfrak{F}$ of groups is called a formation if it is closed under homomorphic image and every group G has a smallest normal subgroup (called $\mathfrak{F}$-residual and denoted by $G^{\mathfrak{F}}$) with quotient is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. We use $\mathfrak{N}$, $\mathfrak{U}$, and $\mathfrak{S}$ to denote the formations of all nilpotent groups, supersoluble groups and soluble groups, respectively. $[A]B$ denotes the semiproduct of two groups A and B.

Obviously, all normal subgroups, c-normal subgroups and $\mathfrak{F}_n$-supplemented subgroups are all $\mathfrak{F}_h$-normal in G, for any non-empty saturated formation $\mathfrak{F}$. For example, if a subgroup H is c-normal in G, then there exists a normal subgroup K such that $G = HK$ and $(H\cap K)H_G/H_G = 1 \leq Z_\infty^{\mathfrak{F}}(G/H_G)$. However, the following example shows that the converse is not true.

Example 1.1. Let $S_3 = [Z_3]Z_2$ be the symmetric group of degree 3 and Z a group of order p, where $p \neq 2, 3$. Let $G = Z \wr S_3 = [K]S_3$ be a regular wreath product, where K is the base group of the regular wreath product G. Then Z_3K is a normal Hall subgroup of G and $Z_3 \cap K = 1$. Hence Z_3 is $\mathfrak{F}_h$-normal in G for any non-empty saturated formation $\mathfrak{F}$. But it is easy to see that Z_3 is not normal, c-normal, and is not $\mathfrak{U}_n$-supplemented in G (In fact, for example, G is the only normal subgroup of G such that $Z_3G = G$ and $(Z_3)_G$=1. But, clearly, $Z_3 \cap G = Z_3 \nleq Z_\infty^{\mathfrak{U}}(G)$. Thus, Z_3 is not $\mathfrak{U}_n$-supplemented).

In this paper, we study the properties of $\mathfrak{F}_h$-normal subgroups and use them to give some new characterizations of some classes of groups. Some previously known results are generalized.

2. Preliminaries

A formation $\mathfrak{F}$ is said to be S-closed (S_n-closed) if it contains every subgroup (every normal subgroup, respectively) of all its group. The following known results are useful in the later.

Lemma 2.1. [6, Lemma 2.1] *Let G be a group and $A \leq G$. Let $\mathfrak{F}$ be a non-empty saturated formation and $Z = Z_\infty^{\mathfrak{F}}(G)$. Then*

(1) *If A is normal in G, then $AZ/A \leq Z_\infty^{\mathfrak{F}}(G/A)$.*
(2) *If $\mathfrak{F}$ is S-closed, then $Z \cap A \leq Z_\infty^{\mathfrak{F}}(A)$.*
(3) *If $\mathfrak{F}$ is S_n-closed and A is normal in G, then $Z \cap A \leq Z_\infty^{\mathfrak{F}}(A)$.*
(4) *If $G \in \mathfrak{F}$, then $Z = G$.*

Lemma 2.2. [16] *If A is a subnormal subgroup of a group G and A is a π-group, then $A \leq O_\pi(G)$.*

Lemma 2.3. [6, Lemma 2.3] *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

Recall that a group G is said to be q-closed if G has a normal Sylow q-subgroup.

Lemma 2.4. [13, Lemma 2.2] *Let G be a group, p and q different prime divisors of $|G|$, and P a non-cyclic Sylow p-subgroup of G. If every maximal subgroup of P (except one) has a q-closed supplement in G, then G is q-closed.*

Lemma 2.5. [17, Theorem II, 3.9] *Let G be a group. If $|G| = 2n$, where n is an odd number, then G is soluble.*

Lemma 2.6. *Let G be a group and $H \leq K \leq G$. Then*

(1) *H is $\mathfrak{F}_h$-normal in G if and only if G has a normal subgroup T such that HT is a normal Hall subgroup of G, $H_G \leq T$ and $H/H_G \cap T/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$.*
(2) *Suppose that H is normal in G. If K is $\mathfrak{F}_h$-normal in G, then K/H is $\mathfrak{F}_h$-normal in G/H.*
(3) *Suppose that H is normal in G. Then for every $\mathfrak{F}_h$-normal subgroup E of G satisfying $(|H|,|E|)=1$, HE/H is $\mathfrak{F}_h$-normal in G/H.*
(4) *If H is $\mathfrak{F}_h$-normal in G and $\mathfrak{F}$ is S-closed, then H is $\mathfrak{F}_h$-normal in K.*
(5) *If H is $\mathfrak{F}_h$-normal in G, K is a normal subgroup of G and $\mathfrak{F}$ is S_n-closed, then H is $\mathfrak{F}_h$-normal in K.*
(6) *If $G \in \mathfrak{F}$, then every subgroup of G is $\mathfrak{F}_h$-normal in G.*

Proof. (1) Assume that H is $\mathfrak{F}_h$-normal in G and let T be a normal subgroup of G such that HT is a normal Hall subgroup of G and $(H \cap T)H_G/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$. Let $T_0 = TH_G$. Then $HT_0 = HTH_G = HT$, $H_G \leq T_0$ and $T_0/H_G \cap H/H_G = (T_0 \cap H)/H_G = (H \cap T)H_G/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$. The converse is clear.

(2) Assume that K is $\mathfrak{F}_h$-normal in G. Then by (1), G has a normal subgroup T such that KT is a normal Hall subgroup of G, $K_G \leq T$ and $K/K_G \cap T/K_G \leq Z_\infty^{\mathfrak{F}}(G/K_G)$. Since $H \trianglelefteq G$ and $H \leq K$, $H \leq K_G$. Hence $H \leq T$ and so T/H is a normal subgroup of G/H. Clearly, KT/H is a normal Hall subgroup of G/H. Since $(T \cap K)/K_G \leq Z_\infty^{\mathfrak{F}}(G/K_G)$, $((T \cap K)/H)/(K_G/H) \leq Z_\infty^{\mathfrak{F}}((G/H)/(K_G/H)) = Z_\infty^{\mathfrak{F}}((G/H)/(K/H)_{G/H})$. Hence $(T/H)/(K/H)_{G/H} \cap (K/H)/(K/H)_{G/H} = (T/H)/(K_G/H) \cap (K/H)/(K_G/H) = ((T \cap K)/H)/(K_G/H) \leq Z_\infty^{\mathfrak{F}}((G/H)/(K/H)_{G/H})$. This shows that K/H is $\mathfrak{F}_h$-normal in G/H.

(3) Assume that H is a normal subgroup of G and E is $\mathfrak{F}_h$-normal in G with $(|H|,|E|)=1$. Then by (1), there exists a normal subgroup T of G such that $E_G \leq T$, ET is a normal Hall subgroup of G and $E/E_G \cap T/E_G \leq Z_\infty^{\mathfrak{F}}(G/E_G)$. If $H \leq T$, then $HET = ET$ is a normal Hall subgroup of G. In order to prove that HE/H is $\mathfrak{F}_h$-normal in G/H, by (2) we only need to show that HE is $\mathfrak{F}_h$-normal in G. Since $H \leq T$, $T \cap HE = H(T \cap E) \leq HZ$, where $Z/E_G = Z_\infty^{\mathfrak{F}}(G/E_G)$. By the G-isomorphism $HZ/HE_G = HE_GZ/HE_G \simeq Z/Z \cap HE_G = Z/E_G(Z \cap H)$, we have $HZ/HE_G \leq Z_\infty^{\mathfrak{F}}(G/HE_G)$. Hence $(HE \cap T)/HE_G = H(T \cap E)/HE_G \leq HZ/HE_G \leq Z_\infty^{\mathfrak{F}}(G/HE_G)$. Let $D = (HE)_G$. By Lemma 2.1(1), $Z_\infty^{\mathfrak{F}}(G/HE_G)(D/HE_G)/(D/HE_G) \leq Z_\infty^{\mathfrak{F}}((G/HE_G)/(D/HE_G))$. Thus $((HE \cap T)/HE_G)\ (D/HE_G)/(D/HE_G) \leq Z_\infty^{\mathfrak{F}}(G/HE_G)(D/HE_G)/(D/HE_G) \leq Z_\infty^{\mathfrak{F}}((G/$

$HE_G)/(D/HE_G))$. It follows that $(HE\cap T)D/D \leq Z_\infty^{\mathfrak{F}}(G/D)$. Therefore HE is $\mathfrak{F}_h$-normal in G. Assume that $H \nleq T$. Obviously, TH/H is a normal subgroup of G/H such that $(HE/H)(TH/H) = ETH/H$ is a normal Hall subgroup of G/H. Now we only need to show that $(EH/H \cap TH/H)(EH/H)_{G/H}/(EH/H)_{G/H} \leq Z_\infty^{\mathfrak{F}}((G/H)/(EH/H)_{G/H})$. Let $D = (HE)_G$. Since $(E\cap T)/E_G \leq Z_\infty^{\mathfrak{F}}(G/E_G) = Z/E_G$, $E\cap T \leq Z$ and $(E\cap T)D/D \leq ZD/D$. By Lemma 2.1(1), $((E\cap T)D/E_G)/(D/E_G) \leq (ZD/E_G)/(D/E_G) = Z_\infty^{\mathfrak{F}}(G/E_G)(D/E_G)/(D/E_G) \leq Z_\infty^{\mathfrak{F}}((G/E_G)/(D/E_G)$. It follows that $(E\cap T)D/D \leq Z_\infty^{\mathfrak{F}}(G/D)$. Since $(|H|,|E|) = 1$,$(|HT:T|,|HT\cap E|) = 1$ and so $HT\cap E \leq T\cap E$. Hence $(HE/H\cap HT/H)(HE/H)_{G/H}/(HE/H)_{G/H} = (H(E\cap T)D/H)/(D/H) \leq ((E\cap T)D/H)/(D/H) \leq Z_\infty^{\mathfrak{F}}((G/H)/(D/H)$. Therefore HE/H is $\mathfrak{F}_h$-normal in G/H.

(4) Assume that H is $\mathfrak{F}_h$-normal in G. Then by (1), G has a normal subgroup T such that HT is a normal Hall subgroup of G, $H_G \leq T$ and $H/H_G \cap T/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$. Let $T_1 = K\cap T$. Then T_1 is a normal subgroup of K and $HT_1 = H(K\cap T) = K\cap HT$ is a normal Hall subgroup of K. Obviously, $T_1/H_G \cap H/H_G = (H\cap T\cap K)/H_G \leq Z/H_G := Z_\infty^{\mathfrak{F}}(G/H_G)\cap K/H_G$. Since $\mathfrak{F}$ is S-closed, by Lemma 2.1(2), $Z/H_G \leq Z_\infty^{\mathfrak{F}}(K/H_G)$. By Lemma 2.1(1), $(Z/H_G)(H_K/H_G)/(H_K/H_G) \leq Z_\infty^{\mathfrak{F}}((K/H_G)/(H_K/H_G))$ and so $(T_1\cap H)H_K/H_K \leq Z_\infty^{\mathfrak{F}}(K/H_K)$. Hence H is $\mathfrak{F}_h$-normal in K.

(5) See the proof of (4).

(6) Assume that $G\in\mathfrak{F}$ and let H be an arbitrary subgroup of G. By Lemma 2.1(4) $Z = Z_\infty^{\mathfrak{F}}(G) = G$ and so by Lemma 2.1(1), $Z_\infty^{\mathfrak{F}}(G/H_G) = G/H_G$. Let $T = G$. Then $(H\cap T)H_G/H_G = H/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$. ∎

Lemma 2.7. *Suppose that G has a unique minimal normal subgroup N and $\Phi(G) = 1$. If N is soluble, then $N = O_p(G) = F(G) = C_G(N)$ for some prime p.*

Proof. Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = NM$. Since N is soluble, N is an abelian p-group for some prime p and $N\cap M \trianglelefteq G$. It follows that $N\cap M = 1$ and so $G = [N]M$. Clearly, $N \leq O_p(G) \leq F(G) \leq C_G(N)$. Let $C = C_G(N)$. If $C \neq N$, then $C = C\cap NM = N(C\cap M)$. It is easy to see that $C\cap M \trianglelefteq G$. Hence $C\cap M = 1$ and consequently $C = N$. This completes the proof. ∎

3. New characterization of supersoluble groups

Theorem 3.1. *A group G is supersoluble if and only if there exists a normal subgroup E of G such that G/E is supersoluble and every maximal subgroup of every non-cyclic Sylow subgroup of E is $\mathfrak{U}_h$-normal in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and consider a counterexample for which $|G||E|$ is minimal. Then:

(1) If N is a non-trivial normal p-subgroup of G contained in E for some prime p, then G/N is supersoluble.

Obviously, $(G/N)/(E/N) \simeq G/E$ is supersoluble. Let T/N be any non-cyclic Sylow q-subgroup of E/N and T_1/N a maximal subgroup of T/N, where q is a prime divisor of $|E/N|$. If $q = p$, then T is a non-cyclic Sylow p-subgroup of E and T_1 is a maximal subgroup of T. By hypothesis, T_1 is $\mathfrak{U}_h$-normal in G. Hence by Lemma 2.6(2), T_1/N is $\mathfrak{U}_h$-normal in G/N. Now suppose that $q \neq p$, then there exists a Sylow q-subgroup Q of E such that $T = QN$. Let $Q_1 = Q \cap T_1$. Then it is easy to see that Q_1 is a maximal subgroup of Q and $T_1 = Q_1 N$. By hypothesis, Q_1 is $\mathfrak{U}_h$-normal in G. Hence by Lemma 2.6(3), T_1/N is $\mathfrak{U}_h$-normal in G/N. This shows that $(G/N, E/N)$ satisfies the hypothesis. The minimal choice of G implies that G/N is supersoluble.

(2) G is soluble.

Since the class $\mathfrak{U}$ of all supersoluble groups is S-closed, by Lemma 2.6(4) we see that the hypothesis is still true for (E, E). If $E < G$, then E is supersoluble by the choice of G. It follows that G is soluble. Now assume that $E = G$ and G is not soluble. Let p be the smallest prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then $p = 2$ by Feit-Thompson's theorem. If P is cyclic, then G is 2-nilpotent by [11, (10.1.9)]. Hence G is soluble, a contradiction. We may therefore assume that P is non-cyclic. Let P_1 be a maximal subgroup of P. Then P_1 is $\mathfrak{F}_h$-normal in G by hypothesis. Therefore there exists a normal subgroup T of G such that P_1T is a normal Hall subgroup of G and $(P_1 \cap T)(P_1)_G/(P_1)_G \leq Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. By (1), we have $(P_1)_G = 1$ and so $P_1 \cap T \leq Z_\infty^{\mathfrak{U}}(G)$. If $Z_\infty^{\mathfrak{U}}(G) \neq 1$, then there exists a minimal normal subgroup H of G contained in $Z_\infty^{\mathfrak{U}}(G)$. Obviously, H is an elementary abelian r-subgroup, for some prime r. By (1), G/H is supersoluble. This implies that G is soluble, a contradiction. Hence $Z_\infty^{\mathfrak{U}}(G) = 1$. It follows that $P_1 \cap T = 1$ and so $T < G$. Obviously, (T, T) satisfies the hypothesis and hence T is supersoluble by the minimal choice of G and Lemma 2.6(4). Suppose that q is the largest prime divisor of $|T|$ and T_q is a Sylow q-subgroup of T. Then T_q char $T \trianglelefteq G$. It follows that $T_q \trianglelefteq G$. By (1), G/T_q is supersoluble. Consequently G is soluble.

(3) G has a unique minimal normal subgroup N contained in E, $G = [N]M$ for some maximal subgroup M of G, and $N = O_p(E) = F(E) = C_E(N)$, for some prime $p \in \pi(G)$.

Let N be a minimal normal subgroup of G contained in E. By (2), N is an elementary abelian p-subgroup for some prime p. By (1), G/N is supersoluble. Since the class $\mathfrak{U}$ of all supersoluble groups is a saturated formation, N is a unique minimal normal subgroup of G contained in E and $N \nsubseteq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $N \nleq M$. Clearly $\Phi(E) = 1$, $G = [N]M$ and $N \subseteq O_p(E) \leq F(E)$. Let $F = F(E)$. Then $F = F \cap NM = N(F \cap M)$. Since $\Phi(E) = 1$, $F(E)$ is abelian by (2). Hence $F \cap M \trianglelefteq G$ and so $F \cap M = 1$. Consequently, $F = N$. Since E is soluble, $N \leq C_E(N) = C_E(F(E)) \leq F(E) = F$. It follow that $N = O_p(E) = F(E) = C_E(N)$. Thus (3) holds.

(4) N is a Sylow p-subgroup of E and N is not cyclic.

If N is cyclic, then by (1) and Lemma 2.3, we have that G is supersoluble, a contradiction. Hence N is not cyclic. Let q be the largest prime divisor of $|E|$ and

Q is a Sylow q-subgroup of E. Then QN/N is a Sylow q-subgroup of E/N. Since G/N is supersoluble by (1), E/N is supersoluble and so $QN/N \trianglelefteq E/N$. It follows that $QN \trianglelefteq E$. Let P be a Sylow p-subgroup of E. If $p = q$, then $P = Q = QN \trianglelefteq E$. Therefore by (3), $N = O_p(E) = P$ is the Sylow p-subgroup of E. Assume that $q > p$. Then clearly $QP = QNP$ is a subgroup of E. If $QP < G$, then by Lemma 2.6(4), (QP, QP) satisfies the hypothesis. The minimal choice of (G, E) implies that QP is supersoluble. Consequently $Q \trianglelefteq QP$ and so $QN = Q \times N$. It follows that $Q \leq C_E(N) = N$, a contradiction.

Now assume that $G = QP = E$. Then obviously $Q \ntrianglelefteq G$. Clearly, $N < P$. Since N is not cyclic, P is not cyclic. We claim that every maximal subgroup of P has a q-closed supplement in G. Let P_1 be an arbitrary maximal subgroup of P. If $(P_1)_G \neq 1$, then by (3), $N \leq (P_1)_G \leq P_1$ and $G = NM = P_1M$, where $M \simeq G/N$ is supersoluble and so M is q-closed. If $(P_1)_G = 1$, then since N is the unique minimal normal subgroup of G and N is not cyclic, $Z_\infty^{\mathfrak{U}}(G) = 1$. Now by hypothesis, there exists a normal subgroup T of G such that P_1T is a normal Hall subgroup of G and $P_1 \cap T \leq Z_\infty^{\mathfrak{U}}(G) = 1$. Assume $P_1T < G$. Since P_1T is a normal Hall subgroup of G, we have $P_1T = P \trianglelefteq G$ and so $P = O_p(G) = N$, a contradiction. Hence $G = P_1T$ and $P_1 \cap T = 1$. In this case, every Sylow p-subgroup of T is a cyclic group of order p. Hence, obviously, (T, T) satisfies the hypothesis of the theorem. The minimal choice of (G, E) implies that T is supersoluble. Consequently T is q-closed. Thus our claim holds. Therefore, by Lemma 2.4, $Q \trianglelefteq G$. This contradiction shows that $N = P$. Thus, (4) holds.

(5) The final contradiction.

Let P be a Sylow p-subgroup of G. Then by (3), $N \subseteq P$ and clearly $N \nsubseteq \Phi(P)$. Therefore there exists a maximal subgroup P_1 of P with $N \nleq P_1$. Consequently $P = NP_1$. Let $N_1 = N \cap P_1$. Since $|N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$, $N_1 = N \cap P_1$ is a maximal subgroup of N. By (3) and (4), $N_1 \neq 1$ and $(N_1)_G = 1$. By the hypothesis, N_1 is $\mathfrak{U}_h$-normal in G. Hence there exists a normal subgroup T of G such that N_1T is a normal Hall subgroup of G and $N_1 \cap T \leq Z_\infty^{\mathfrak{U}}(G)$. If $N_1T = G$, then $N = N \cap N_1T = N_1(N \cap T)$. This implies that $N \cap T \neq 1$. Obviously $N \cap T \trianglelefteq G$. Hence $N \cap T = N$ and so $N \leq T$. Hence $1 \neq N_1 \leq Z_\infty^{\mathfrak{U}}(G) \cap N \leq N$. Since $Z_\infty^{\mathfrak{U}}(G) \cap N \trianglelefteq G$, $Z_\infty^{\mathfrak{U}}(G) \cap N = N$ and so $N \leq Z_\infty^{\mathfrak{U}}(G)$. It follows from (1) that G is supersoluble, a contradiction. Hence we may assume that $N_1T < G$. Since $N \cap T \trianglelefteq G$, $N \cap T = 1$ or N. If $N \cap T = 1$, then $N_1 = N_1(N \cap T) = N \cap N_1T \trianglelefteq G$, which is impossible. If $N \cap T = N$, then $N \leq T$ and so $N_1 \leq T$. This implies that $N_1 \leq Z_\infty^{\mathfrak{U}}(G) \cap N$. By the same argument as above, we see that $N \leq Z_\infty^{\mathfrak{U}}(G)$ and consequently G is supersoluble, a contradiction again. The final contradiction completes the proof. ∎

Corollary 3.1. *Let $\mathfrak{F}$ be an S-closed saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup E of G such that $G/E \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of E is $\mathfrak{F}_h$-normal in G.*

Proof. The necessity is obvious, we only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample with $|G||E|$ is minimal.

By Lemma 2.6(4) and our Theorem 3.1, we see that $E \in \mathfrak{U}$. Let p be the largest prime divisor of $|E|$ and E_p a Sylow p-subgroup of E. Then E_p char $E \trianglelefteq G$ and so $E_p \trianglelefteq G$. Let N be a minimal normal subgroup of G contained in E_p. Obviously, $(G/N)/(E/N) \simeq G/E \in \mathfrak{F}$. By Lemma 2.6(2), we see that the hypothesis is still true for G/N (with respect to E/N). The choice of G implies that $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, N is the only minimal normal subgroup of G contained in E_p and $N \nleq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = [N]M$. Then it is easy to see that $N = O_p(E) = E_p$ (see the proof (3) of Theorem 3.1). If N is cyclic, then $G \in \mathfrak{F}$ by Lemma 2.3, which contradicts the choice of G. Thus we may assume that N is not cyclic. Let M_p be a Sylow p-subgroup of M and put $P = NM_p$. Then P is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of P such that $M_p \leq P_1$. Then $P = NP_1$. Analogy to the proof (5) of Theorem 3.1, we can obtain that $N \leq Z_\infty^{\mathfrak{F}}(G)$. This is impossible. ∎

The following results follows directly from our Theorem 3.1 and Corollary 3.1.

Corollary 3.2. [9] *Let $\mathfrak{F}$ be an S-closed saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of E is c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 3.3. [7, VI. Theorem 10.3] *A group G is supersoluble if every Sylow subgroup of G is cyclic.*

Corollary 3.4. [14] *Let G be a group with a normal subgroup E such that G/E is supersoluble. If every maximal subgroup of every Sylow subgroup of E is normal in G, then G is supersoluble.*

Corollary 3.5. [15] *Let G be a group with a normal subgroup E such that G/E is supersoluble. If every maximal subgroup of every Sylow subgroup of E is c-normal in G, then G is supersoluble.*

Theorem 3.2. *Let $\mathfrak{F}$ be an S-closed saturated formation containing all supersoluble groups and G a group. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathfrak{F}$ and every cyclic subgroup of E of prime order or 4 are $\mathfrak{U}_h$-normal in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample with $|G||E|$ is minimal. Then, obviously, $E = G^{\mathfrak{F}}$. By Lemma 2.6(4), it is easy to see the hypothesis still holds for (H, H), where H is any subgroup of E. This shows that every subgroup of E is supersoluble by the choice of G. It follows from [7, VI. Theorem 9.6] that E is soluble. Let M be any maximal subgroup of G not containing E. Then $M/M \cap E \simeq ME/E \in \mathfrak{F}$. Hence the hypothesis still true for $(M, M \cap E)$ by Lemma 2.6(4). The minimal choice of G implies that $M \in \mathfrak{F}$. Then, by [4, Theorem 3.4.2], $E = G^{\mathfrak{F}}$ is a p-subgroup for some prime p and the following conditions hold:

(1) $E/\Phi(E)$ is a G-chief factor and so it is an elementary abelian p-group.
(2) E is a group with exponent p or 4(if $p = 2$ and E is non-abelian).
(3) $\Phi(E) = E \cap \Phi(G) \leq Z(E)$, where $Z(E)$ is the center of E.

We claim that $|E/\Phi(E)| = p$. Assume that this is not true. Let $\Phi = \Phi(E)$, X/Φ be a subgroup of E/Φ of prime order, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then by (2), $|L| = p$ or

$|L| = 4$. By hypothesis, L is $\mathfrak{U}_h$-normal in G. Hence there exists a normal subgroup T of G such that LT is a normal Hall subgroup of G and $(L \cap T)L_G/L_G \leq Z_\infty^{\mathfrak{U}}(G/L_G)$. Then, since $L \leq E$ is p-group, $E \leq LT$.

We first assume that $|L| = 4$. Since $X/\Phi = L\Phi/\Phi \simeq L/L \cap \Phi$ is of prime order, $L \cap \Phi \neq 1$. Let H be a maximal subgroup of L. Since L is a cyclic group, $H = L \cap \Phi \leq \Phi$. Suppose that L is not normal in G, then $L_G = H$ or $L_G = 1$. Assume that $L_G = H$. If $L \leq \Phi(G)$, then $L \leq E \cap \Phi(G) = \Phi$ by (3), a contradiction. Therefore $L \nleq \Phi(G)$ and so there exists a maximal subgroup M of G such that $G = LM$. Since $L_G = H \leq \Phi \leq \Phi(G) \leq M$, $|G : M| = 2$. Hence $M \trianglelefteq G$ and so G/M is a cyclic group. It follows that $L \leq E = G^{\mathfrak{F}} \leq M$. This contradiction shows that $L_G = 1$. Then $L \cap T \leq Z_\infty^{\mathfrak{U}}(G)$. Since $|L| = 4$, $L \cap T = L$ or $L \cap T = H$ or $L \cap T = 1$. If $L \cap T = L$, then $L \leq T$ and so $L \leq Z_\infty^{\mathfrak{U}}(G)$. By Lemma 2.1(1), $1 \neq L\Phi/\Phi \leq Z_\infty^{\mathfrak{U}}(G)\Phi/\Phi \leq Z_\infty^{\mathfrak{U}}(G/\Phi)$. It follows that $1 \neq L\Phi/\Phi \leq Z_\infty^{\mathfrak{U}}(G/\Phi) \cap E/\Phi$. But since E/Φ is a chief factor, $E/\Phi \leq Z_\infty^{\mathfrak{U}}(G/\Phi)$ and consequently $|E/\Phi| = 2$, a contradiction. Hence $L \nsubseteq T$, and $L \cap T = H$ or $L \cap T = 1$. If $LT = G$, then $G/T = LT/T \simeq L/L \cap T$ is cyclic and so $G/T \in \mathfrak{F}$. It follows that $L \leq E = G^{\mathfrak{F}} \leq T$ and consequently $T = G$, a contradiction. Hence $LT < G$. Since $LT \trianglelefteq G$, $LT/T \trianglelefteq G/T$. Therefore LT/T is $\mathfrak{U}_h$-normal in G/T and $(G/T)/(LT/T) \simeq G/LT \simeq (G/E)/(LT/E) \in \mathfrak{F}$. If $L \cap T = H$, then $LT/T \simeq L/L \cap T = L/H$ is a group of order 2. In this case, obviously, $(G/T, LT/T)$ satisfies the hypothesis. By the choice of G, $G/T \in \mathfrak{F}$. It follows that $L \leq E = G^{\mathfrak{F}} \leq T$, a contradiction. If $L \cap T = 1$, then $LT/T \simeq L/L \cap T = L$ is a cyclic group of order 4. Hence HT/T char $LT/T \trianglelefteq G/T$ and so $HT/T \trianglelefteq G/T$. It follows that HT/T is $\mathfrak{U}_h$-normal in G/T. Hence $(G/T, LT/T)$ satisfies the hypothesis. The minimal choice of G implies that $G/T \in \mathfrak{F}$ and thereby $L \leq E = G^{\mathfrak{F}} \leq T$, a contradiction again. Those contradictions show that L is normal in G when $|L| = 4$. Since E/Φ is a chief factor, $E/\Phi = L\Phi/\Phi = X/\Phi$ is a cyclic group of order 2. This contradiction shows that $|E/\Phi| = 2$ when $|L| = 4$.

Now assume that $|L|$ is a prime. If L is not normal in G, then $L_G = 1$ and so $L \cap T \leq Z_\infty^{\mathfrak{U}}(G)$. Obviously $L \cap T = L$ or $L \cap T = 1$. If $L \cap T = L$, then $L \leq T$. It follows that $L \leq Z_\infty^{\mathfrak{U}}(G)$. By Lemma 2.1(1), $1 \neq L\Phi/\Phi \leq Z_\infty^{\mathfrak{U}}(G/\Phi) \cap E/\Phi$. Since E/Φ is a chief factor, $E/\Phi \leq Z_\infty^{\mathfrak{U}}(G/\Phi)$ and consequently $|E/\Phi| = p$, a contradiction. Assume that $L \cap T = 1$. If $LT = G$, then $G/T \simeq L$ is cyclic and so $G/T \in \mathfrak{F}$. This implies that $L \leq E = G^{\mathfrak{F}} \leq T$, a contradiction again. Assume $LT < G$. Clearly, $(G/T)/(LT/T) \simeq G/LT \simeq (G/E)/(LT/E) \in \mathfrak{F}$. Since $LT/T \trianglelefteq G/T$, LT/T is $\mathfrak{U}_h$-normal in G/T. Hence $(G/T, LT/T)$ satisfies hypothesis. The choice of G implies that $G/T \in \mathfrak{F}$. This implies also that $L \leq E = G^{\mathfrak{F}} \leq T$, a contradiction. Those contradictions show that $|E/\Phi| = p$ when $|L| = p$.

Hence, in any case, our claim holds, that is, $E/\Phi = L\Phi/\Phi$ is a cyclic group of prime order. Since $G/E \simeq (G/\Phi)/(E/\Phi) \in \mathfrak{F}$ and E/Φ is cyclic, by Lemma 2.3, we obtain $G/\Phi \in \mathfrak{F}$. This implies that $G \in \mathfrak{F}$ since $\mathfrak{F}$ is a saturated formation. The final contradiction completes the proof. ∎

Corollary 3.6. *A group G is supersoluble if and only if every cyclic subgroup of G of prime order or order 4 are $\mathfrak{U}_h$-normal in G.*

Corollary 3.7. [15] *If all cyclic subgroups of a group G with prime order or order 4 are c-normal in G, then G is supersoluble.*

Corollary 3.8. [1] *Let $\mathfrak{F}$ be an S-closed saturated formation containing $\mathfrak{U}$ and G a group. If all minimal subgroups and all cyclic subgroups of order 4 of $G^{\mathfrak{F}}$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 3.9. [2] *Let G be a group of odd order. If all cyclic subgroups of a group G with prime order or order 4 are normal in G, then G is supersoluble.*

Corollary 3.10. [10] *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and all subgroups of prime order or order 4 of H are c-normal in G.*

4. New characterization of soluble groups

Theorem 4.1. *A group G is soluble if and only if every minimal subgroup of G is $\mathfrak{S}_h$-normal in G.*

Proof. In view of Lemma 2.6(6), we only need to prove that G is soluble if every minimal subgroup of G is $\mathfrak{S}_h$-normal in G. Assume that this is false and let G be a counterexample of minimal order.

Let $p = p_1, p_2, \ldots, p_t = q$ be all primes dividing $|G|$ such that $p_1 > p_2 > \ldots > p_t$. Then in view of Burnside $p^a q^b$-Theorem, we have that $t > 2$. By Lemma 2.6(4), the hypothesis holds for every subgroup of G and so every maximal subgroup of G is soluble by the choice of G. Let R be the largest soluble normal subgroup of G. Then $Z_\infty^{\mathfrak{S}}(G) \leq R$. We claim that $R \neq 1$. If $R = 1$, then G is a non-abelian simple group. Let L be a minimal subgroup of G with $|L|$ is the smallest prime dividing $|G|$. Then, clearly, $Z_\infty^{\mathfrak{S}}(G) = 1$. By hypothesis, L is $\mathfrak{S}_h$-normal in G. Hence there exists a normal subgroup K of G such that LK is a normal Hall subgroup of G and $(L \cap K)L_G/L_G \leq Z_\infty^{\mathfrak{S}}(G/L_G)$. Since G is a simple group, $L_G = 1$ and $K = G$. Hence $L = L \cap K \leq Z_\infty^{\mathfrak{S}}(G) = 1$, a contradiction. Thus $R \neq 1$. Obviously, R is the unique proper normal subgroup of G such that G/R is a non-abelian simple group. Let H/K be a chief factor of G such that $H \leq Z_\infty^{\mathfrak{S}}(G)$. Then $[H/K](G/C_G(H/K))$ is soluble (see [4, Lemma 2.4.2]). Clearly $C_G(H/K) \trianglelefteq G$. If $C_G(H/K) < G$, then $C_G(H/K)$ is soluble and consequently G is soluble. This contradiction shows that $C_G(H/K) = G$. This implies that $Z_\infty^{\mathfrak{S}}(G) = Z_\infty(G)$ is the hypercenter of G. If $R \not\leq \Phi(G)$, then $G = RE$ for some maximal subgroup E of G and so $G/R \simeq E/E \cap R$ is soluble. It follows that G is soluble, which contradicts the choice of G. Thus $R \leq \Phi(G)$ and hence every prime dividing $|G|$ is also a divisor of G/R. Suppose that some minimal subgroup L of G has a complement E in G. Then by Lemma 2.6(4), we see that E is a soluble maximal subgroup of G. Hence $R \leq E$ and $(E/R)_{G/R} = 1$. By considering the permutation representation of G/R on the right coset of E/R, we see that G/R is isomprphic to some subgroup of the symmetric group $S_{|L|}$ of degree $|L|$. Hence $|L| = p$ is the largest prime dividing $|G|$. This induces that if H is a minimal subgroup of G with $|H| \neq p$, then H has no a complement in G. But, by hypothesis, H is $\mathfrak{S}_h$-normal in G. So there exists a normal subgroup K of G such that HK is a normal Hall subgroup of G and $(H \cap K)H_G/H_G \leq Z_\infty^{\mathfrak{S}}(G/H_G)$. If $HK = G$, then it is easy to see that $H \leq Z_\infty^{\mathfrak{S}}(G) \leq R$. Since $t > 2$, for some odd prime $r \neq p$ dividing $|G|$, all subgroups

H of order r are contained in $Z_{\infty}^{\mathfrak{S}}(G) = Z_{\infty}(G)$. Clearly, G is not r-nilpotent and so by [7, IV. Theorem 5.4] and [4, Theorem 3.4.11], G has a r-closed Schmidt subgroup $A = [A_r]D$, where A_r is a Sylow r-subgroup of A of exponent r and $A_r/\Phi(A_r)$ is a eccentric chief factor of A. Let $X/\Phi(A_r)$ be a subgroup of $A_r/\Phi(A_r)$ of prime order, $x \in X\backslash\Phi(A_r)$ and $L = \langle x\rangle$. Then $|L| = r$ and so from above we know that $L \leq Z_{\infty}(G)$. But then $L \leq Z_{\infty}(A)$ and hence $X/\Phi(A_r) \leq Z_{\infty}(A/\Phi(A_r))$. It follows that the factor $A_r/\Phi(A_r)$ is central, a contradiction. If $HK \neq G$, then by Lemma 2.6(2), G/HK satisfies the hypothesis. The minimal choice of G implies that G/HK is soluble, and consequently G is soluble. The finial contradiction completes the proof. ∎

The following results now follows directly from our Theorem 3.2.

Corollary 4.1. [7, Theorem IV.5.7] *If all minimal subgroups of a group G are normal in G, then G is soluble.*

Corollary 4.2. *If all minimal subgroups of a group G are c-normal in G, then G is soluble.*

Corollary 4.3. [18, Theorem 3.1] *A group G is soluble if and only if every minimal subgroup of G is $\mathfrak{S}_n$-supplemented in G.*

Theorem 4.2. *Let G be a group and N a nonidentity normal subgroup of G. Then N is soluble if and only if every maximal subgroup of G not containing N is $\mathfrak{S}_h$-normal in G.*

Proof. Suppose that every maximal subgroup M of G with $N \nleq M$ is $\mathfrak{S}_h$-normal in G. Let R be a minimal normal subgroup of G. Assume that M/R is a maximal subgroup of G/R such that $NR/R \nleq M/R$. Then $N \nsubseteq M$. By hypothesis, M is $\mathfrak{S}_h$-normal in G. Then M/R is $\mathfrak{S}_h$-normal in G/R by Lemma 2.6(2). Thus, by induction, NR/R is soluble. If $R \cap N = 1$, then $N \simeq NR/R$ is soluble. Hence we may assume that every minimal normal subgroup of G is contained in N. It is easy to see that $(N/R, G/R)$ satisfies the hypothesis. Hence by induction again, N/R is soluble. Since the class of all soluble groups is closed under subdirect product, R is a unique minimal normal subgroup of G.

Suppose that R is not soluble. Let $E = N_G(P)$, where P is a Sylow p-subgroup of R and $p \in \pi(R)$. Then by Frattini argument, we have $G = RE$. Obviously $E \neq G$. Let M be a maximal subgroup of G such that $E \leq M$. Then $R \nleq M$ and hence $N \nleq M$. Let G_p be a Sylow p-subgroup of G such that $P = R \cap G_p$. Then $P \trianglelefteq G_p$. Therefore $G_p \leq E$ and consequently p dose not divide $|G : M|$.

Since M is $\mathfrak{S}_h$-normal in G, there exists a normal subgroup T of G such that MT is a normal Hall subgroup of G and $(M \cap T)M_G/M_G \leq Z_{\infty}^{\mathfrak{S}}(G/M_G)$. Since R is the unique minimal normal subgroup of G, $M_G = 1$. If $MT < G$, then $M = MT \trianglelefteq G$ and so $R \leq M$, a contradiction. Hence $MT = G$. Assume that $M \cap T = 1$. Then $|T| = |G : M|$. But since $R \leq T$ and $G = RM$, $R = T$ and p divides $|R| = |G : M|$, a contradiction again. Thus $M \cap T \neq 1$ and so $Z_{\infty}^{\mathfrak{S}}(G) \neq 1$. Therefore $R \leq Z_{\infty}^{\mathfrak{S}}(G)$ and consequently R is soluble. This induce that N is soluble.

Conversely, assume that N is soluble. Let M be a maximal subgroup of G such that $N \nleq M$ and let $1 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{t-1} \leq N_t = N$, where $N_i/N_{i-1} (i = 1, 2, \cdots t)$ is a chief factor of G. Since N is soluble, N_i/N_{i-1} is abelian. We may choose an index i such that $N_i \nleq M$ and $N_{i-1} \leq M$. Then $N_i/N_{i-1} \cap M/N_{i-1} \unlhd G/N_{i-1}$ and $N_i \cap M = N_{i-1} \leq M_G$. Now $MN_i = G$ and $(M \cap N_i)M_G/M_G = 1 \leq Z_\infty^{\mathfrak{S}}(G/M_G)$. This means that M is $\mathfrak{S}_h$-normal in G. The proof is completed. ∎

Corollary 4.4. *Let G be a group. Then G is soluble if and only if every maximal subgroup of G is $\mathfrak{S}_h$-normal in G.*

Corollary 4.5. [15] *Let G be a group. Then G is soluble if and only if every maximal subgroup of G is c-normal in G.*

Corollary 4.6. [18] *Let G be a group. Then G is soluble if and only if every maximal subgroup of G is $\mathfrak{S}_n$-supplemented in G.*

Theorem 4.3. *A group G is soluble if and only if one of following conditions holds:*

(a) *There exists a maximal subgroup P_1 of some Sylow 2-subgroup P of G such that P_1 is $\mathfrak{S}_h$-normal in G.*

(b) *P is $\mathfrak{S}_h$-normal in G, for some Sylow 2-subgroup P of G.*

Proof. In view of Lemma 2.6(6), we only need to prove the "if" part.

(a) Suppose that there exists a maximal subgroup P_1 of some Sylow 2-subgroup P of G such that P_1 is $\mathfrak{S}_h$-normal in G. We prove that G is soluble. Assume that the assertion is not true and let G be a counterexample of minimal order. Then obviously $P \neq 1$ and $P_1 \neq 1$. In fact, if $P = 1$, then G is a group of odd order. By Feit-Thompson theorem, G is soluble. If $P_1 = 1$, then $|G| = 2n$, where n is an odd number, and G is also soluble by Lemma 2.5.

Since P_1 is $\mathfrak{S}_h$-normal in G, there exists a normal subgroup K of G such that P_1K is a normal Hall subgroup of G and $(P_1 \cap K)(P_1)_G/(P_1)_G \leq Z_\infty^{\mathfrak{S}}(G/P_G)$. If $(P_1)_G \neq 1$, then it is clear that the hypotheses still holds for the quotient group $G/(P_1)_G$ by Lemma 2.6(2) and so $G/(P_1)_G$ is soluble by the choice of G. It follows that G is soluble, a contradiction. Thus we may assume that $(P_1)_G = 1$. In this case, $P_1 \cap K \leq Z_\infty^{\mathfrak{S}}(G)$. Assume that $P_1K = G$. If $P_1 \cap K = 1$, then $|K| = 2n$ where n is an odd number and $G/K \simeq P_1$. By Lemma 2.5, K is soluble and consequently G is also soluble, a contradiction. Thus $P_1 \cap K \neq 1$ and so $Z_\infty^{\mathfrak{S}}(G) \neq 1$. Therefore, there exists a minimal normal subgroup R of G contained in $Z_\infty^{\mathfrak{S}}(G)$. It follows that R is an elementary abelian p-subgroup, for some prime p. By Lemma 2.6(2), we can easily see that G/R satisfies the hypotheses. Hence G/R is soluble and so G is soluble, a contradiction again. Now assume that $P_1K < G$. Then G/P_1K is a group of order $2m$, where m is an odd number. Hence by Lemma 2.5, G/P_1K is soluble. It is easy to see that P_1K satisfies the hypotheses by Lemma 2.6(4). The minimal choice of G implies that P_1K is soluble. It follows that G is soluble. The contradiction completes the proof.

(b) The proof is the same as (a) and we hence omit the proof. ∎

Corollary 4.7. *Let G be a group. If some maximal subgroup of some Sylow 2-subgroup of G is c-normal in G, then G is soluble.*

Corollary 4.8. *Let G be a group. If some Sylow 2-subgroup of G is c-normal in G, then G is soluble.*

Corollary 4.9. [18] *A group G is soluble if and only if one of following conditions holds:*

(a) *There exists a maximal subgroup P_1 of some Sylow 2-subgroup P of G such that P_1 is $\mathfrak{S}_n$-supplemented in G.*
(b) *P is $\mathfrak{S}_n$-supplemented in G, for some Sylow 2-subgroup of G.*

Corollary 4.10. [18] *Let G be a group and P a Sylow p-subgroup of G, where p is a minimal prime divisor of $|G|$. If there exists a Sylow p-subgroup P of G (or P has a maximal subgroup P_1 of P) such that P (or P_1, respectively) is $\mathfrak{S}_n$-supplemented in G, then G is soluble.*

5. New characterization of p-nilpotent groups

Theorem 5.1. *Let p be a prime number dividing the order of a group G with $(|G|, p-1) = 1$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every maximal subgroup of P is $\mathfrak{U}_h$-normal in G.*

Proof. The necessity is obvious by Lemma 2.6(6). We only need to prove the sufficiency. Assume that the assertion is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then we may choose a minimal normal subgroup N of G such that $N \leq O_{p'}(G)$. Clearly, $(|G/N|, p-1) = 1$ and PN/N is a Sylow p-subgroup of G/N. Assume that L/N is a maximal subgroup of PN/N. Then, obviously, $L/N = P_1N/N$, where P_1 is some maximal subgroup of P. By hypothesis and Lemma 2.6(3), P_1N/N is $\mathfrak{U}_h$-normal in G/N. This shows that G/N(with respect to PN/N) satisfies the hypothesis. By the choice of G, G/N is p-nilpotent and consequently G is p-nilpotent, a contradiction. Hence $O_{p'}(G) = 1$.

(2) G is soluble.

Suppose that G is not soluble. Then $p = 2$ by the well-known Feit-Thompson Theorem. Assume that $O_2(G) \neq 1$. Let $P_1/O_2(G)$ be a maximal subgroup of $P/O_2(G)$. By hypothesis and Lemma 2.6(2), $P_1/O_2(G)$ is $\mathfrak{U}_h$-normal in $G/O_2(G)$. The minimal choice of G implies that $G/O_2(G)$ is 2-nilpotent and so G is soluble, a contradiction. Now let $O_2(G) = 1$ and P_1 a maximal subgroup of P. Then $(P_1)_G = 1$. By hypothesis, P_1 is $\mathfrak{U}_h$-normal in G. Hence there exists $K \trianglelefteq G$ such that P_1K is a normal Hall subgroup of G and $P_1 \cap K \leq Z_\infty^{\mathfrak{U}}(G)$. Obviously, $K \neq 1$. If $Z_\infty^{\mathfrak{U}}(G) \neq 1$, then there exists a minimal normal subgroup H of G contained in $Z_\infty^{\mathfrak{U}}(G)$ with prime order. But by (1) and $O_2(G) = 1$, we have that H=1, a contradiction. If $Z_\infty^{\mathfrak{U}}(G) = 1$, then $P_1 \cap K = 1$ and $2^2 \nmid |K|$. Hence, by [11, (10.1.9)], K has a normal Hall $2'$-subgroup T. Since T char $K \trianglelefteq G$, $T \trianglelefteq G$. Hence by (1), $T = 1$. This means that $K \leq O_2(G) = 1$, a contradiction again. Hence (2) holds.

(3) If K is a subgroup of G with a Sylow p-subgroup K_p of order p, then K is p-nilpotent.

Since $N_K(K_p)/C_K(K_p)$ is isomorphic with some subgroup of $Aut(K_p)$ and $|Aut(K_p)| = p-1$, by $(|G|, p-1) = 1$, we see that $N_K(K_p) = C_K(K_p)$. Hence K is

p-nilpotent by Burnside theorem.

(4) $O_p(G)$ is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. By (1) and (2), N is an elementary abelian p-group and $N \leq O_p(G)$. By Lemma 2.6(2), G/N satisfies the hypotheses. The minimal choice of G implies G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $\Phi(G) = 1$. By Lemma 2.7, we see that $O_p(G) = N$. Hence (4) holds.

(5) The final contradiction.

By (4), there exists a maximal subgroup M of G such that $G = [O_p(G)]M$. Let $P = O_p(G)M_p$ is a Sylow p-subgroup of G, where M_p is some Sylow p-subgroup of M and P_1 be a maximal subgroup of P such that $M_p \leq P_1$. By hypotheses, there exists a normal subgroup K of G such that P_1K is a normal Hall subgroup of G and $(P_1 \cap K)(P_1)_G/(P_1)_G \leq Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. Since $O_p(G) \nsubseteq P_1$ and $O_p(G)$ is the unique minimal normal subgroup of G, $(P_1)_G = 1$. Therefore $P_1 \cap K \leq Z_\infty^{\mathfrak{U}}(G)$.

If $P_1K < G$, then by Lemma 2.6(4), P_1K satisfies the hypotheses. The minimal choice of G implies that P_1K is p-nilpotent. Obviously, the normal p-complement H of P_1K is a normal subgroup of G. It follows from (1) that $H = 1$ and so $P_1K = P \trianglelefteq G$. Therefore $P = O_p(G)$ is the unique minimal normal subgroup of G and $K = P$. This means that $P_1 = P_1 \cap K \leq Z_\infty^{\mathfrak{U}}(G)$. If $P_1 \neq 1$, then $Z_\infty^{\mathfrak{U}}(G) \neq 1$. Hence $P \leq Z_\infty^{\mathfrak{U}}(G)$ and thereby $|P| = p$. If $P_1 = 1$, then we also have $|P| = p$. Thus $Aut(P)$ is a cyclic group of order $p-1$. Then since $(|G|, p-1) = 1$, we have $N_G(P) = C_G(P)$. By using the well known Burnside Theorem, we obtain that G is p-nilpotent, a contradiction.

Now assume that $P_1K = G$. If $P_1 \cap K = 1$, then every Sylow p-subgroup of K is a group of order p. Therefore K is p-nilpotent by (3). Let $K_{p'}$ be a normal p-complement of K. Then $K_{p'} \trianglelefteq G$. But by (1), $K_{p'} = 1$. Hence $|K| = p$. It follows that G is a p-group, a contradiction. Hence $P_1 \cap K \neq 1$, which implies that $Z_\infty^{\mathfrak{U}}(G) \neq 1$. Since $O_p(G)$ is the unique minimal normal subgroup of G, $O_p(G) \leq Z_\infty^{\mathfrak{U}}(G)$ and so $|O_p(G)| = p$. By Lemma 2.7, $C_G(O_p(G)) = O_p(G)$. Hence $M \simeq G/O_p(G) = N_G(O_p(G))/C_G(O_p(G))$ is a cyclic group of order $p-1$. However, since $(|G|, p-1) = 1$, $M = 1$. It follows that $G = O_p(G)$. The final contradiction completes the proof. ∎

The following results now follows immediately from Theorem 5.1.

Corollary 5.1. *Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is $\mathfrak{U}_h$-normal in G, then G is p-nilpotent.*

Corollary 5.2. [5] *Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Theorem 5.2. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is $\mathfrak{U}_h$-normal in G.*

Proof. The necessity is clear. We only need to prove the sufficiency. If $p = 2$, then G is p-nilpotent by Theorem 5.1. Thus we only need to consider the case when p is an odd prime. Suppose that the theorem is not true and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we can consider the quotient group $G/O_{p'}(G)$. By Lemma 2.6(3), it is easy to see that $G/O_{p'}(G)$ satisfies the hypotheses. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent. It follows that G is p-nilpotent, a contradiction.

(2) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

Since, clearly, $N_M(P) \leq N_G(P)$, $N_M(P)$ is p-nilpotent. By Lemma 2.6(4), we see that M satisfies the hypotheses. Hence by the choice of G, we have that M is p-nilpotent.

(3) $G = PQ$ is soluble, where Q is a Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson theorem [11, (10.4.1)], there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P with $H < K \leq P$. Obviously, $N_G(P) < N_G(H)$. Then, by (2), $N_G(H) = G$. This leads to $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P satisfying $O_p(G) < K \leq P$. Now, by Thompson theorem [11, (10.4.1)] again, we see that $G/O_p(G)$ is p-nilpotent and so G has the following $p'p$-series

$$1 < O_p(G) < O_{pp'}(G) < O_{pp'p}(G) = G.$$

By [3, Theorem 6.3.5], we see that there exists a Sylow q-subgroup Q of G such that $G_1 = PQ$ is a subgroup of G. If $G_1 < G$, then by (2) G_1 is p-nilpotent. This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [11, (9.3.1)]. This contradiction shows that (3) holds.

(4) Final contradiction.

By (1) and (3), $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. It is easy to see that G/N satisfies the hypotheses. Hence G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Thus, $O_p(G) = N$ is an elementary abelian p-group by Lemma 2.7 and there exists a maximal subgroup L of G such that $G = NL$ and $N \cap L = 1$. Let P^* be a Sylow p-subgroup of L. Then $P = NP^*$. If $P = N$, then $N_G(P) = N_G(N) = G$ is p-nilpotent, a contradiction. Thus $P \neq N$. Let P_1 is a maximal subgroup of P with $P^* \leq P_1$. By the hypotheses, there exists a normal subgroup K of G such that P_1K is a normal Hall subgroup of G and $(P_1 \cap K)(P_1)_G/(P_1)_G \leq Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. Obviously, $K \neq 1$. Since$N$ is the unique minimal normal subgroup of G and $N \nleq P_1$, $(P_1)_G = 1$ and $N \leq K$. Hence $P_1 \cap K \leq Z_\infty^{\mathfrak{U}}(G)$. If $P_1K < G$, then by (3), $P_1K = P \trianglelefteq G$. It follows that $G = N_G(P)$ is p-nilpotent, a contradiction. Hence $P_1K = G$. If $P_1 \cap K \neq 1$, then $Z_\infty^{\mathfrak{U}}(G) \neq 1$ and so $N \leq Z_\infty^{\mathfrak{U}}(G)$. It follows that $|N| = p$. Hence $Aut(N)$ is a cyclic group of order $p-1$. If $p < q$, then by [11, (10.1.9)], NQ is p-nilpotent and therefore $Q \leq C_G(N) = C_G(O_p(G))$, which contradicts $C_G(O_p(G)) \leq O_p(G)$. Thus we may assume that $q < p$. Since $C_G(N) = C_G(O_p(G)) = O_p(G) = N$ by Lemma 2.7, $L \simeq G/N = N_G(N)/C_G(N)$ is isomorphic with a subgroup of $Aut(N)$. Hence L and Q are cyclic groups. By using [11, (10.1.9)] again, G is q-nilpotent and thereby P is normal in G. This implies that $N_G(P) = G$ is p-nilpotent, a contradiction again. Hence $P_1 \cap K = 1$. Then since $P = P \cap G = P \cap P_1K = P_1(P \cap K)$ and $P_1 \cap (P \cap K) = 1$, $|P \cap K| = p$. It follows from $N \leq P \cap K$ that $|N| = p$. The same as above we have $N_G(P) = G$ is p-nilpotent. This contradiction completes the proof. ∎

Corollary 5.3. [5] *Let p be an odd prime dividing the order of a group G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

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