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Free-Surface Flow Under a Sluice Gate from Deep Water

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Abstract. Fully nonlinear solutions of free surface flow under a sluice gate are presented in this paper. The fluid upstream is assumed to be infinite in depth, and it flows under the gate forming a uniform stream far downstream. The problem is solved numerically by a boundary element method derived from the integral equation along the free surface. The numerical procedure is able to obtain solutions for upstream Froude number $F \geq 0.192$. A free surface with back flow near the edge of the gate is indicated for $F \leq 0.317$. As the limiting case, free surface flow with a stagnation point at the edge of the gate can be computed for $F = 0.192$.

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1. Introduction

We study the steady two-dimensional irrotational flow of an ideal fluid in a domain bounded by an infinite horizontal wall at the bottom, a semi-infinite vertical wall representing a sluice gate, and a free surface as illustrated in Figure 1(a). Physically, the infinite depth fluid flows through a slit under the vertical wall, and it forms a stream with a free surface as the boundary. Far downstream the stream is uniform. When the net volume flux of the fluid approaching the slit is Q and the width of the slit measured from the bottom wall is D, the free surface profile is observed, especially near the edge of the gate and far downstream.

Most sluice gate flows are observed for fluid of finite depth in upstream, such as in Frangmeier and Strelkoff [5], Loroch [8] and Chung [4]. For relatively new studies, see Asavanant and Vanden-Broeck [1], Vanden-Broeck [9] and Binder and

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Vanden-Broeck [2]. The solutions are characterized by uniform and supercritical flow far downstream, and the flow far upstream is supercritical or subcritical. A train of waves can be obtained when the upstream flow is subcritical. Binder and Vanden-Broeck [3] then developed a problem involving multiple disturbances on the bottom of the channel and the free surface; such as submerged obstacle, pressure distribution and sluice gate. They obtained the solutions with radiation condition, i.e. waves are formed near the gate and disappear, tend to uniform, far upstream. However, all types of solutions have uniform and supercritical flow far downstream. This character is also obtained in this paper, but it is caused from the infinite depth fluid in the upstream. The difference with the previous solutions is the existence of back flow near the edge of the gate, and it becomes a stagnation point. The free surface separates the vertical wall with angle $2\pi/3$, or $-5\pi/6$ to the vertical axis. This limiting case agrees with the result obtained by Vanden-Broeck and Tuck [10] who observed free surface flow locally near a vertical wall.

Most of the works mentioned above solves the problem numerically by the boundary element method. The method is also used to solve other free surface flows. Wiryanto and Tuck [11, 12] applied the method to free surface flow producing one jet and also two jets. For free surface flows caused by a line sink or source, we can read for example in Wiryanto [13] and Hocking and Forbes [6, 7]. The boundary element method is constructed from an integral equation of hodograph complex variable corresponding to particle velocity. In expressing the real part of the variable into the imaginary one, the Cauchy integral theorem is applied. The problems using the boundary element method usually have the hodograph variable Ω satisfying conditions of Cauchy integral theorem, i.e. analytic and $\Omega(\zeta) \to 0$ as $|\zeta| \to \infty$. ζ is an artificial complex variable as the result of conformal mapping of physical plane. However, in this study an infinite depth of the fluid is involved. This causes the second condition of Cauchy integral theorem to be not satisfied anymore, since the fluid velocity far upstream is uniform radially. Therefore, we need to construct the appropriate function for Cauchy integral theorem.

The construction of the analytic and bounded function is explained in Section 2. A similar problem has been studied for the case of zero gravity by Wiryanto [14], but the horizontal wall is terminated so that the flow becomes a waterfall, and analytical solutions are obtained. In Section 3, the numerical procedure in solving the integral equation is presented. The integration is approximated by trapezoidal method involving unknown variables. A system of nonlinear algebraic equation is then constructed from the integral equation, and it is solved by Newton iteration method. As a result, we present in Section 4 some plots of the surface profile and discuss the numerical observation.

2. Formulation

We consider the steady two-dimensional irrotational flow of an inviscid and incompressible fluid in a dam of infinite depth, bounded by a vertical wall as a sluice gate with width of slit D. We choose Cartesian coordinates with the x-axis along the bottom and y-axis directed vertically upward along the vertical wall. The net volume of the flux in the dam is Q per unit distance perpendicular to the plane of flow, and the flow is assumed to leave the edge of the gate tangentially, see Figure 1(a).

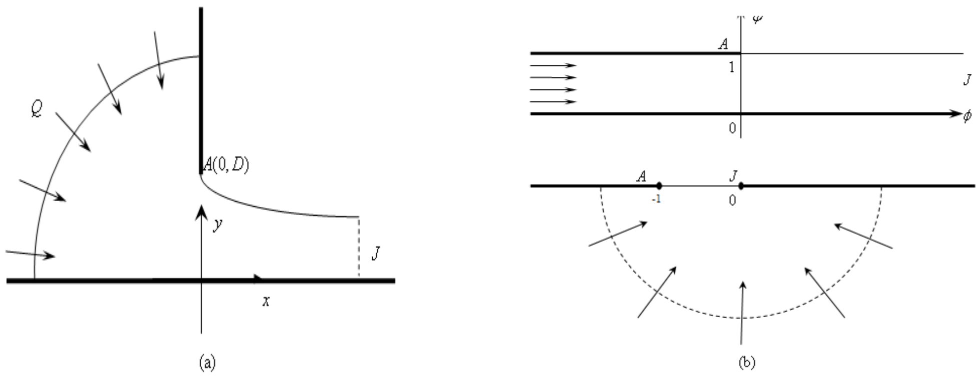

Figure 1. Sketch of the flow under a sluice gate (a) in physical z-plane, (b) in f-plane and artificial ζ-plane.

From the assumption of the fluid and the flow, we present the stream in a complex potential $f = \phi + i\psi$ corresponding to the complex velocity $df/dz = u - iv$, where $z = x + iy$. For convenience, we work in non-dimensional variables by taking Q as the unit flux and D as the unit length, and we define $\phi = 0$, $\psi = 0$ at the center coordinates of physical plane z. Therefore, the f-plane is a strip with width 1 which is also the non-dimensional width of the slit. Now, our task is to solve the boundary value problem

$$\nabla^2 \phi = 0$$

in the flow domain. The dynamic condition is expressed by Bernoulli equation

$$\frac{1}{2} F^2 \left(\phi_x^2 + \phi_y^2\right) + y = \text{constant} \tag{2.1}$$

along the free boundary representing hydrostatics pressure. F is Froude number defined as

$$F = \frac{Q}{\sqrt{gD^3}}, \tag{2.2}$$

where g is acceleration of gravity. The other condition is the kinematic condition along the solid and free boundaries, satisfying

$$\frac{\partial \phi}{\partial \bar{n}} = 0 \tag{2.3}$$

where $\bar{n}$ is a normal vector of the boundaries. Physically, this condition states that fluid particles on the boundary remain on it.

In determining ϕ, we first introduce a hodograph variable $\Omega = \tau - i\theta$ having the following relationship to the velocity vector

$$\frac{df}{dz} = e^{\Omega}. \tag{2.4}$$

Meanwhile, the flow domain in f-plane is mapped into a half lower artificial plane $\zeta = \xi + i\eta$ by

$$f = -\frac{1}{\pi}\log\zeta. \tag{2.5}$$

The downstream is mapped to $\zeta = 0$ and the edge of the gate A is mapped to $\zeta = -1$. The schematic diagram of the flow is shown in Figure 1(b). The bold lines correspond to the solid boundary, and the thick line corresponds to the free boundary.

Instead of determining ϕ, we solve the hodograph variable Ω with respect to the artificial variable ζ, satisfying

$$\nabla^2\Omega = 0$$

subject to (the dynamic condition (2.1) becomes)

$$\frac{1}{2}F^2e^{2\tau} + y = c, \qquad -1 < \xi < 0 \tag{2.6}$$

where c is an unknown constant; and the kinematic condition (2.3) becomes

$$\theta = \begin{cases} -\pi/2, & -\infty < \xi < -1 \\ 0, & 0 < \xi < \infty \end{cases} \tag{2.7}$$

Here θ is unknown for $-1 < \xi < 0$.

A relation between θ and τ is then required in reducing the unknown variables. This can be obtained by using Cauchy integral theorem. As the complex function, we define

$$\chi(\zeta) = \Omega + \frac{1}{2}\log\zeta \tag{2.8}$$

This function is analytic and $\chi \to 0$ for $|\zeta| \to \infty$, so that it can be applied to Cauchy integral theorem. We come up to (2.8) since the upstream flow far from the slit is uniform with velocity $df/dz \to 0$ and the streamlines bouncing by horizontal and vertical walls having angle θ as given in (2.7). The logarithm function is the one having the character described above, so that

$$\Omega \to -\frac{1}{2}\log\zeta, \qquad \text{for } |\zeta| \to \infty$$

and this is used to construct χ as written in (2.8).

In applying χ to Cauchy integral theorem along closed path covering the flow domain in ζ-plane, it is enough to consider along the real ξ-axis giving

$$\chi(\xi) = -\frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{\xi(s)}{s-\xi}ds \tag{2.9}$$

The function χ is then expressed in τ and θ for both sides in (2.9), and the real part gives

$$\tau(\xi) = -\frac{1}{2}\log|1+\xi| + \frac{1}{\pi}\mathrm{PV}\int_{-1}^{0}\frac{\theta}{s-\xi}ds \tag{2.10}$$

for $-1 < \xi < 0$. Here, PV is used to denote Cauchy principal-value for the integration.

The last step, which has to be carried out before obtaining the integral equation, is to determine y along the free surface. We use (2.4) and (2.5) to have

$$\frac{dz}{d\zeta} = -\frac{e^{-\Omega}}{\pi\zeta} \tag{2.11}$$

and the imaginary part along the free surface gives

$$\frac{dy}{d\xi} = -\frac{e^{-\tau}\sin\theta}{\pi\xi} \tag{2.12}$$

Therefore, the value y is obtained by integrating (2.12) giving

$$y(\xi) = 1 - \int_{-1}^{\xi} \frac{e^{-\tau(s)}}{\pi s}\sin\theta(s)ds \tag{2.13}$$

τ in (2.13) is evaluated from (2.10). The formulae y in (2.13) and τ in (2.10) are then substituted to (2.6) giving the integral equation which has to be solved.

3. Numerical procedure

The nonlinear integral equation (2.6) converts to a set of N algebraic equations in N unknowns, if we approximate the integration (2.10) by summation in a suitable manner. The interval of integration (0,1) is first discretized by defining the end-points of $N-1$ subintervals $\xi_0 = -1 < \xi_1 < \xi_2 \cdots < \xi_{N-1} = -\epsilon$, and then we let $\theta_j = \theta(\xi_j)$ for $j = 1, 2, \cdots, N-1$, be $N-1$ unknowns. $-\epsilon$ is a small value representing the position of free surface relatively far from the slit, and we need this number to truncate the integration (2.10), as it is impossible to know the end of the free surface. However we need the rest of the subinterval $(-\epsilon, 0)$.

In order to evaluate the Cauchy principle-value singular integral in (2.10), we approximate $\theta(\xi)$ as varying linearly on the interval (ξ_{j-1}, ξ_j), and evaluate the integral over each such interval exactly. For any $\xi_j^* \in (\xi_{j-1}, \xi_j)$, $\tau(\xi_j^*)$ is evaluated by

$$\begin{aligned}\tau(\xi_j^*) \approx& -\frac{1}{2}\log|1+\xi_j^*| + \sum_{l=1}^{N-1}(\theta_{l-1}-\theta_l)\\ &+\left(\theta_l + (\theta_{l-1}-\theta_l)\frac{\xi_j^*-\xi_l}{\xi_{l-1}-\xi_l}\right)\log\left|\frac{\xi_{l-1}-\xi_j^*}{\xi_l-\xi_j^*}\right|\end{aligned} \tag{3.1}$$

Similarly, the integral (2.13) determining the y-coordinate of the free surface can be evaluated by numerical approximation, such as the trapezoidal rule

$$\begin{aligned}y(\xi_j^*) \approx& \; y(\xi_{j-1}^*)\\ &-\frac{1}{2}\left(\frac{e^{-\tau(\xi_j^*)}}{\pi\xi_j^*}\sin\theta(\xi_j^*) + \frac{e^{-\tau(\xi_{j-1}^*)}}{\pi\xi_{j-1}^*}\sin\theta(\xi_{j-1}^*)\right)(\xi_j^*-\xi_{j-1}^*)\end{aligned} \tag{3.2}$$

In obtaining the N algebraic equations, we use N collocation points ξ_j^* as the mid-point in each subinterval (ξ_{j-1}, ξ_j), except $\xi_N = -\epsilon/2$ and also $\theta(\xi_j^*)$ defined linearly between θ_{j-1} and θ_j. For each point ξ_j^*, the integral equation (2.6) gives one algebraic equation, so that there are N equations for unknowns $\theta_1, \theta_2, \cdots, \theta_{N-1}$ and the constant c in (2.6). The parameter Froude number F is given, also define

$\theta_0 = -\pi/2$ at the edge of the gate. This closed form is then solved numerically by Newton method. When the iteration converges, N-point coordinates (x_j, y_j) of the free surface are determined from

$$\frac{dx}{d\xi} = -\frac{e^{-\tau}}{\pi\xi}\cos\theta \tag{3.3}$$

and (3.2) for y. Numerical integration is applied to (3.3) to get $x(\xi^*)$, using θ obtained in the previous process. We then plot the coordinates (x_j, y_j) to get the surface profile.

4. Results

Most calculations of the numerical procedure described above use $N = 250$ and $\epsilon = 0.000001$. Typical free surface for moderate Froude number is shown in Figure 2. The flow produces a stream with smooth free surface leaving the vertical wall, no wave on the free surface; and the stream tends to uniform far downstream, the fluid depth is less than the width of the slit. We computed the result in Figure 2 for $F = 0.8$, it gives $c = 2.161$ and $y \to 0.652$ for large x. For higher Froude numbers, we obtain stream with slightly deeper fluid and also a higher value c.

We suppose that the uniform stream far downstream has dimensional depth H and horizontal velocity U. The non-dimensional Froude number based on this stream is defined as

$$F_\infty = \frac{U}{\sqrt{gH}}$$

In relation to our calculation, the Froude number F_∞ satisfies

$$F_\infty = \sqrt{\frac{2(c - y_\infty)}{y_\infty}}$$

where y_∞ is the value y at $x \to \infty$, and we can approximate by $y_\infty \approx y(-\epsilon)$. We obtain the above relation from (2.1). The dimensional constant of Bernoulli equation is $U^2/2 + gH$. This is then non-dimensionalized into $F_\infty^2 y_\infty/2 + y_\infty$. Meanwhile, our numerical scheme gives c for that 'constant'. Therefore, the result corresponding to Figure 2 has $F_\infty = 2.151$. For $F = 0.3$ and 0.2, we show plot of the fee surface in Figure 3 and Figure 4, to give comparison to Figure 2. The calculations give $c = 1.062$, $y_\infty \approx 0.572$ and $F_\infty = 1.310$ for $F = 0.3$; and $c = 1.000$, $y_\infty \approx 0.610$ and $F_\infty = 1.130$ for $F = 0.2$.

For small Froude numbers F, we are interested in observing the values θ near the edge of the gate. We plot θ versus ξ for $F = 0.8$, corresponding to Figure 2, $F = 0.317$, 0.3, and 0.25, shown in Figure 5. Our calculations indicate that the back-flow occurs firstly at $F = 0.317$. This can be seen by the value $\theta < -\pi/2$ near $\xi = -1$. We found that θ-curve for $F < 0.317$ is clearly concave. The minimum value θ_{min} is less than $-\pi/2$, and this position is shifted to $\xi \to -1$ for smaller F. We show the plot of F versus θ_{min} in Figure 6. The numerical procedure fails for $F = 0.197$ since the position of θ_{min} is too close to $\xi_0 = -1$. However, if we extrapolate the curve in Figure 6 to $\theta_{min} = -5\pi/6$, it gives $F = 0.192$. This limiting case is free-surface flow with a stagnation point, which agrees to Vanden-Broeck and Tuck [10], shown in Figure 7.

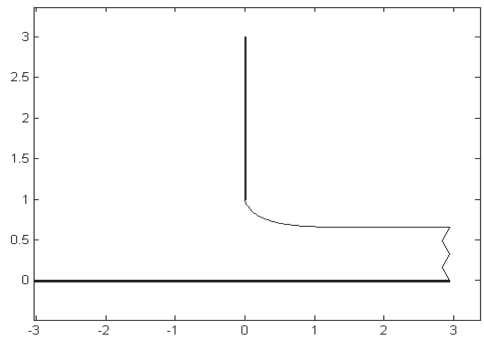

Figure 2. Free surface flow under a sluice gate for $F = 0.8$.

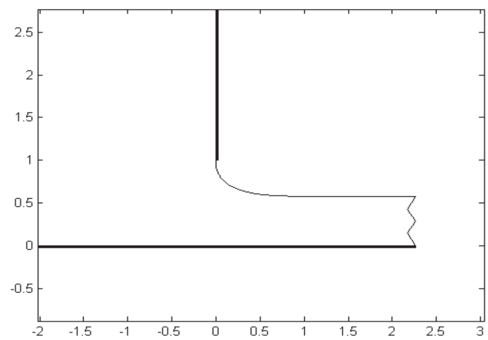

Figure 3. Free surface flow under a sluice gate for $F = 0.3$.

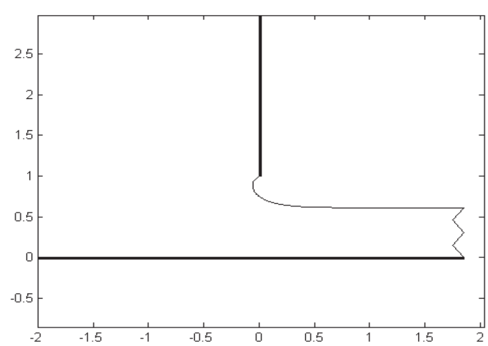

Figure 4. Free surface flow under a sluice gate for $F = 0.2$.

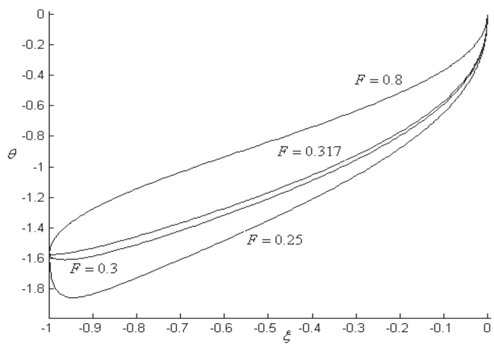

Figure 5. Plots of θ versus ξ for $F = 0.8$ (top), 0.317, 0.3, and 0.25 (bottom).

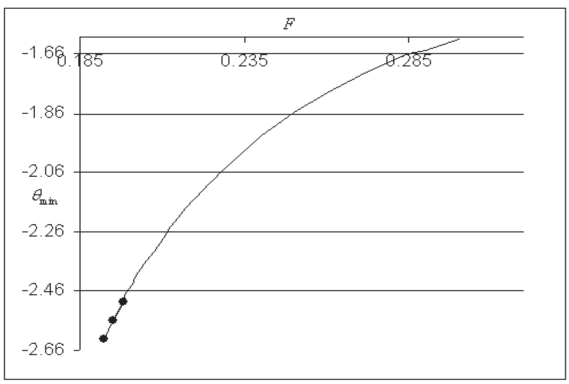

Figure 6. Plot of F versus θ_{min}, and extrapolated to $\theta_{min} = -5\pi/6$.

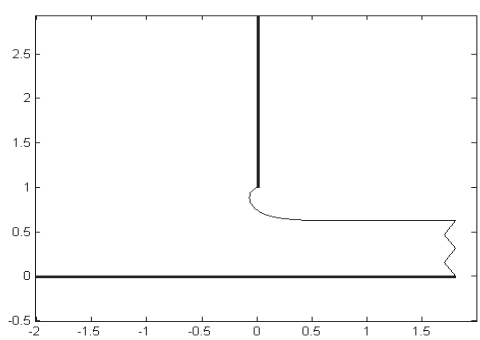

Figure 7. Plot of free surface flow with stagnation point.

5. Conclusions

We have solved numerically the free surface flow under a sluice gate from deep fluid by the boundary element method. The deep fluid at the upstream requires an analytical complex function constructed not directly from the hodograph variable, but it includes a term representing the character far from the slit of the gate. As a result, the free surface flows without waves, but smooth detachment at the edge of the vertical wall, exists for the upstream Froude number $F \geq 0.192$. Meanwhile solutions with back flow occur for $F < 0.317$, and its limiting flow has a stagnation point at the edge of the gate. All types of solutions are uniform and supercritical ($F_\infty > 1$) far downstream.

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