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A New Comparison Theorem and the Solvability of a Third-Order Two-Point Boundary Value Problem

[1]YUQIANG FENG AND [2]GUANGJUN QU
School of Science, Wuhan University of Science and Technology,
Wuhan 430065, China
[1]fengyuqiang@wust.edu.cn, [2]quguangjun1229@163.com

Abstract. A new comparison theorem is proved and then used to investigate the solvability of a third-order two-point boundary value problem

$$\begin{cases} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \\ u(0) = u'(2\pi) = 0, \\ u''(0) = u''(2\pi). \end{cases}$$

Some existence results are established for this problem via upper and lower solutions method and fixed point theory.

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1. Introductions

The third-order differential equations arise in an important number of physical problems, such as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity-driven flows [8]. During the last three decades, third-order differential equations have attracted considerable attention, and many techniques for such problems have appeared (see [1–3, 9–14] and references therein).

Recently in [15], we established the following principle.

Theorem 1.1 (Comparison theorem). *Assume* $0 < M \leq 2$. *If* $q \in C^2[0,1]$ *satisfies*

$$q''(t) \geq M \int_0^t q(s)ds, t \in [0,1]; q(0) \leq 0, q(1) \leq 0$$

then $q(t) \leq 0, t \in [0,1]$.

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By the use of this result and the upper and lower solutions method, we show the existence of solution and positive solution for the following problem.

$$\begin{cases} u'''(t) + f(t, u(t)) = 0, 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases}$$

Then in [4], we get the existence results for the more generalized problem

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

by a new comparison theorem as follows.

Theorem 1.2 (Comparison theorem)**.** *Assume* $M \ge 0, N \ge 0, M + N \le 2$*. If* $q \in C^2[0, l]$ *satisfies*

$$q''(t) \ge M \int_0^t q(s)ds + Nq(t), t \in [0, 1]; q(0) \le 0, q(1) \le 0$$

then $q(t) \le 0, t \in [0, l]$*.*

Very recently in [5], we consider the boundary value problem of a semi-linear third order differential equation as follows:

$$\begin{cases} u'''(t) - a(t)u'(t) = f(t, u(t)), 0 \le t \le 1, \\ u'(0) = u(1) = u'(1) = 0, \end{cases}$$

where $a(t) \in C([0, 1], [0, \infty))$,$f : [0, 1] \times R \to R$ is continuous. The following comparison theorem is crucial to prove the existence result for the above problem.

Theorem 1.3 (Comparison theorem)**.** *If* $q(t) \in C^2[0, 1]$*satisfies*

$$q''(t) \ge b(t) \int_t^1 q(s)ds + a(t)q(t), (0 \le t \le 1), q(0) \le 0, q(1) \le 0$$

where $a(t),\ b(t)$ *satisfy*

$$0 < a(t) + (1 - t)b(t) \le 2,\ \forall t \in (0, 1),$$

then $q(t) \le 0,\ \forall t \in [0, 1]$*.*

In [6], we were concerned with the solvability of boundary value problems for third-order implicit equations. In [7], we considered the existence and multiplicity of positive periodic solutions for third-order equations.

In this paper, we are still concerned with the boundary value problem of third-order differential equation

$$\begin{cases} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \\ u(0) = u'(2\pi) = 0, \\ u''(0) = u''(2\pi), \end{cases} \tag{1.1}$$

where $f : [0, 2\pi] \times R^3 \to R$ is continuous. We prove a new comparison theorem, and then establish the existence of solutions for the above given problem via the use of the comparison theorem, fixed point theory and the upper and lower solutions method. By these methods, we can obtain the iterative scheme for this problem, which implies that the solutions are computable.

This paper is organized as follows. In Section 2, a new comparison theorem is proved. The existence results for problem (1.1) are established in Section 3. In the last section, we give the proof of the main result.

2. Comparison theorem

The following comparison theorem is crucial for the paper:

Theorem 2.1 (Comparison theorem). *If $m(t)$ is differentiable on $[0, 2\pi]$ and satisfies*

$$m'(t) \le -\lambda_1 \int_0^{2\pi} G(t,s)m(s)ds - \lambda_2 \int_t^{2\pi} m(s)ds - \lambda_3 m(t), m(0) \le m(2\pi),$$

where $\lambda_1, \lambda_2, \lambda_3$ are positive numbers satisfying $\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3 \le \frac{1}{2}$

$$G(t,s) = \begin{cases} s, & 0 \le s \le t \le 2\pi \\ t, & 0 \le t \le s \le 2\pi. \end{cases}$$

Then $m(t) \le 0, \forall t \in [0, 2\pi]$.

Proof. Let $J = [0, 2\pi]$. On the contrary, suppose there exists $t_0 \in (0, 2\pi]$ such that

$$m_0 = m(t_0) = \max_{t \in J} m(t) > 0,$$

we shall get a contradiction by the several steps.

Claim 1. *There exists $t_1 \in [0, 2\pi)$ such that $m_1 = m(t_1) = \min_{t \in J} m(t) < 0$.*

Otherwise, we have $m(t) \ge 0, t \in J$. Noting that $G(t,s) > 0, (t,s) \in (0,2\pi) \times (0, 2\pi)$ and $m(t)$ is positive on some subset $[\tau_1, \tau_2]$ of positive measure, we get that for arbitrary $\tau \in (0, 2\pi)$,

$$\begin{aligned} m'(\tau) &\le -\lambda_1 \int_0^{2\pi} G(\tau,s)m(s)ds - \lambda_2 \int_\tau^{2\pi} m(s)ds - \lambda_3 m(\tau) \\ &\le -\lambda_1 \int_0^{2\pi} G(\tau,s)m(s)ds < 0, \end{aligned}$$

which means that $m(t)$ is strictly decreasing on $[0, 2\pi]$, so we have $m(0) > m(2\pi)$, which contradicts $m(0) \le m(2\pi)$. Hence, Claim 1 is valid.

Claim 2. $m(2\pi) < 0$.

In fact, if

$$\delta(t) = \int_t^{2\pi} \int_0^{2\pi} G(\tau,s)dsd\tau,$$

then $\delta(t)$is nonincreasing on $[0, 2\pi]$, and

$$\begin{aligned} m(2\pi) - m_1 &= \int_{t_1}^{2\pi} m'(t)dt \\ &\le \int_{t_1}^{2\pi} [-\lambda_1 \int_0^{2\pi} G(t,s)m(s)ds - \lambda_2 \int_t^{2\pi} m(s)ds - \lambda_3 m(t)]dt \\ &\le -\lambda_1 \delta(t_1)m_1 - \lambda_2 \frac{(2\pi - t_1)^2}{2} m_1 - \lambda_3(2\pi - t_1)m_1 \end{aligned}$$

$$\begin{aligned}&\leq -\lambda_1\delta(0)m_1 - \lambda_2 2\pi^2 m_1 - \lambda_3 2\pi m_1\\ &= -\left(\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3\right)m_1.\end{aligned}$$

Thus we have

$$m(2\pi) \leq \left(1 - \frac{8\pi^3}{3}\lambda_1 - 2\pi^2\lambda_2 - 2\pi\lambda_3\right)m_1 < 0$$

and

$$m_1 \geq \frac{m(2\pi)}{1 - \frac{8\pi^3}{3}\lambda_1 - 2\pi^2\lambda_2 - 2\pi\lambda_3}.$$

Hence, Claim 2 is valid.

Finally, we will get a contradiction.

In fact, $m(t_1) < 0$ and $m(t_0) > 0$ implies there exists $t_2 \in (0, 2\pi)$ such that $m(t_2) = 0$, and then

$$\begin{aligned}m(0) = m(0) - m(t_2) &= \int_0^{t_2} -m'(t)dt\\ &\geq \int_0^{t_2}\left[\lambda_1\int_0^{2\pi} G(t,s)m(s)ds + \lambda_2\int_t^{2\pi} m(s)ds + \lambda_3 m(t)\right]dt\\ &\geq \lambda_1(\delta(0) - \delta(t_2))m_1 + \lambda_2\left(2\pi^2 - \frac{(2\pi - t_2)^2}{2}\right)m_1 + \lambda_3 t_2 m_1\\ &> \lambda_1\delta(0)m_1 + \lambda_2 2\pi^2 m_1 + \lambda_3 2\pi m_1\\ &= \left(\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3\right)m_1.\end{aligned}$$

Furthermore, we get

$$0 > m(2\pi) \geq m(0) > \left(\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3\right)m_1, \tag{2.1}$$

$$m_1 \geq \frac{m(2\pi)}{1 - \frac{8\pi^3}{3}\lambda_1 - 2\pi^2\lambda_2 - 2\pi\lambda_3}. \tag{2.2}$$

and (2.1) and (2.2) imply

$$\frac{\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3}{1 - \frac{8\pi^3}{3}\lambda_1 - 2\pi^2\lambda_2 - 2\pi\lambda_3} > 1. \tag{2.3}$$

It is easy to compute that (2.3) holds if and only if $\frac{1}{2} < \frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3 < 1$, which contradicts the assumption that $\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3 \leq \frac{1}{2}$. ∎

3. Existence results and applications

$\alpha(t), \beta(t) \in C^3[0, 2\pi]$ are called the lower solution and the upper solution of problem (1.1), respectively, if

$$\begin{cases}\alpha'''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geq 0,\\ \alpha(0) = \alpha'(2\pi) = 0, \alpha''(0) \geq \alpha''(2\pi),\end{cases}$$

$$\begin{cases} \beta'''(t) + f(t, \beta(t), \beta'(t), \beta''(t)) \leq 0, \\ \beta(0) = \beta'(2\pi) = 0, \beta''(0) \leq \beta''(2\pi). \end{cases}$$

Throughout this section, we assume that $f : [0, 2\pi] \times R^3 \to R$ is continuous and there exist positive numbers $\lambda_1, \lambda_2, \lambda_3$ such that

(H_1) for $t \in [0, 2\pi], z \in R, x_1 \geq x_2, y_1 \geq y_2$

$$f(t, x_1, y_1, z) - f(t, x_2, y_2, z) \geq -\lambda_1(x_1 - x_2) - \lambda_2(y_1 - y_2)$$

(H_2) for $t \in [0, 2\pi], x, y \in R, z_1 \geq z_2$

$$f(t, x, y, z_1) - f(t, x, y, z_2) \leq \lambda_3(z_1 - z_2).$$

The main result reads as follows.

Theorem 3.1. *Assume* (1.1) *have a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ such that*

$$\alpha''(t) \geq \beta''(t), \forall t \in [0, 2\pi].$$

If $\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3 \leq 1/2$ and $\lambda_3 > 1/2(\sqrt{(1+4\pi)^2 + 16\pi} + 1 + 4\pi)$, then problem (1.1) *has a solution $u^*(t) \in C^3[0, 2\pi]$ satisfying $\alpha(t) \leq u^*(t) \leq \beta(t)$.*

Corollary 3.1. *Assume f satisfies $(H_1 - H_2)$ and $\lambda_1, \lambda_2, \lambda_3$ satisfy $\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3 \leq 1/2$ and$\lambda_3 > 1/2(\sqrt{(1+4\pi)^2 + 16\pi} + 1 + 4\pi)$.*

(1) *If $\min_{0 \leq t \leq 2\pi} f(t, 0, 0, 0) \geq 0$ and there exists $c > 0$ such that*

$$\max\{f(t, x, y, -c) \mid (t, x, y) \in [0, 2\pi] \times [0, 2\pi^2 c] \times [0, 2\pi c]\} \leq 0,$$

then (1.1) *has a solution u^* satisfying $0 \leq u^*(t) \leq c(2\pi t - \frac{t^2}{2})$.*

(2) *If $\max_{0 \leq t \leq 2\pi} f(t, 0, 0, 0) \leq 0$ and there exists $c > 0$ such that*

$$\min\{f(t, x, y, c) \mid (t, u, v) \in [0, 2\pi] \times [-2\pi^2 c, 0] \times [-2\pi c, 0]\} \geq 0,$$

then (1.1) *has a solution u^* satisfying $0 \geq u^*(t) \geq -c(2\pi t - \frac{t^2}{2})$.*

(3) *If there exists $c > 0$ such that*

$$\max\{f(t, x, y, -c) \mid (t, u, v) \in [0, 2\pi] \times [0, 2\pi^2 c] \times [0, 2\pi c]\} \leq 0,$$

$$\min\{f(t, x, y, c) \mid (t, u, v) \in [0, 2\pi] \times [-2\pi^2 c, 0] \times [-2\pi c, 0]\} \geq 0,$$

then (1.1) *has a solution u^* satisfying $-c(2\pi t - \frac{t^2}{2}) \leq u^*(t) \leq c(2\pi t - \frac{t^2}{2})$.*

Example 3.1. Consider the following third-order two-point boundary value problem:

$$\begin{cases} u'''(t) + \frac{1}{12\pi}(u''(t) + e^{-u''(t) - 4\pi}) + \frac{1}{12\pi^2}\cos u'(t) + \frac{1}{32\pi^3}\ln(1 + (u(t))^2) = 0, \\ \\ u(0) = u'(2\pi) = 0, u''(0) = u''(2\pi). \end{cases}$$

Let

$$f(t, x, y, z) = f(x, y, z) = \frac{1}{32\pi^3}\ln(1 + x^2) + \frac{1}{12\pi^2}\cos y + \frac{1}{12\pi}(z + e^{-z-4\pi}).$$

Then $f(0, 0, 0) > 0$ and f satisfies $(H_1 - H_2)$ with $\lambda_1 = \frac{1}{32\pi^3}, \lambda_2 = \frac{1}{12\pi^2}, \lambda_3 = \frac{1}{12\pi}$. It is obvious that

$$\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2 + 2\pi\lambda_3 \leq \frac{1}{2},\ \lambda_3 > \frac{1}{2(\sqrt{(1+4\pi)^2 + 16\pi} + 1 + 4\pi)}.$$

Furthermore, let $c = 4\pi$, we have $\max\{f(x, y, -4\pi) \mid (x, y) \in [0, 8\pi^3] \times [0, 8\pi^2]\} < 0$.

Then Corollary 3.1(1) assures the above problem has a solution between $0, 4\pi(2\pi t - \frac{t^2}{2})$.

4. Proof of the existence results

Let $X = C[0, 2\pi]$, the norm on X is $\|.\| : \|x\| = \max_{0 \le t \le 2\pi} |x(t)|$ for $x \in X$. Let $K = \{x \in X \mid x(t) \ge 0, 0 \le t \le 2\pi\}$, the partial order "$\le$" on X is induced by K: for $x, y \in X, y \le x \Leftrightarrow x - y \in K$, then (X, K) is an ordered Banach space.

Proof of Theorem 3.1. Let $v(t) = -u^{''}(t)$, then (1.1) is equivalent to the following integro-differential equation:

$$\begin{cases} v^{'}(t) = f(t, \int_0^{2\pi} G(t,s)v(s)ds, \int_t^{2\pi} v(s)ds, -v(t)), \\ v(0) = v(2\pi), \end{cases} \tag{4.1}$$

where $G(t, s) = \begin{cases} s, & 0 \le s \le t \le 2\pi, \\ t, & 0 \le t \le s \le 2\pi. \end{cases}$

Defining two operators $L : D \subset X \to X, N : X \to X$ as follows:

$$\begin{aligned} Lv &= v^{'}(t) + \lambda_1 \int_0^{2\pi} G(t,s)v(s)ds + \lambda_2 \int_t^{2\pi} v(s)ds + \lambda_3 v(t), \\ Nv &= f\left(t, \int_0^{2\pi} G(t,s)v(s)ds, \int_t^{2\pi} v(s)ds, -v(t)\right) \\ &\quad + \lambda_1 \int_0^{2\pi} G(t,s)v(s)ds + \lambda_2 \int_t^{2\pi} v(s)ds + \lambda_3 v(t), \end{aligned}$$

where $D = \{v \in X \mid v' \in X, v(0) = v(2\pi)\}$.

By the definition of L and N, (4.1) is equivalent to the following operator equation:

$$Lv = Nv \tag{4.2}$$

We shall show that the above operator equation is solvable. The proof will be given in several steps.

Step 1. *$L : D \subset X \to X$ is invertible.*

We will prove that for $\forall \eta \in X$, there exists $h \in X$ such that $Lh = \eta$.

Consider the following boundary value problem:

$$\begin{cases} v'(t) + \lambda_3 v(t) = \eta(t) - \lambda_1 \int_0^{2\pi} G(t,s)v(s)ds - \lambda_2 \int_t^{2\pi} v(s)ds, \\ v(0) = v(2\pi). \end{cases}$$

It is known by [12] that h is the solution of above problem if and only if h is the fixed point of the operator $A_\eta : X \to X$, where

$$\begin{aligned} A_\eta v(t) = e^{-\lambda_3 t} \Bigg[&\frac{1}{e^{2\pi\lambda_3} - 1} \int_0^{2\pi} \left(\eta(s) - \lambda_1 \int_0^{2\pi} G(s,\tau)v(\tau)d\tau - \lambda_2 \int_s^{2\pi} v(\tau)d\tau\right) e^{\lambda_3 s} ds \\ &+ \int_0^t \left(\eta(s) - \lambda_1 \int_0^{2\pi} G(s,\tau)v(\tau)d\tau - \lambda_2 \int_s^{2\pi} v(\tau)d\tau\right) e^{\lambda_3 s} ds \Bigg]. \end{aligned}$$

Noting that for all $t \in [0, 2\pi], u, v \in X$,

$$\begin{aligned}
|A_\eta u(t) - A_\eta v(t)| &\leq \frac{e^{-\lambda_3 t}}{\lambda_3}\left(\lambda_1 \int_0^{2\pi}\int_0^{2\pi} G(s,\tau)d\tau ds\|u-v\| + 2\pi^2\lambda_2\|u-v\|\right) \\
&\quad + \left(\lambda_1 \int_0^{2\pi}\int_0^{2\pi} G(s,\tau)d\tau ds\|u-v\| + 2\pi^2\lambda_2\|u-v\|\right) \\
&\leq \left(\frac{1}{\lambda_3}+1\right)\left(\lambda_1 \int_0^{2\pi}\int_0^{2\pi} G(s,\tau)d\tau ds + 2\pi^2\lambda_2\right)\|u-v\| \\
&= \left(\frac{1}{\lambda_3}+1\right)\left(\frac{8\pi^3}{3}\lambda_1 + 2\pi^2\lambda_2\right)\|u-v\| \\
&\leq \left(\frac{1}{\lambda_3}+1\right)\left(\frac{1}{2} - 2\pi\lambda_3\right)\|u-v\|.
\end{aligned}$$

Let $\rho = \left(\frac{1}{\lambda_3}+1\right)\left(\frac{1}{2}-2\pi\lambda_3\right)$, due to the monotonicity of $f(x) = \left(\frac{1}{x}+1\right)\left(\frac{1}{2}-2\pi x\right)$, we know that

$$\frac{1}{4\pi} > \lambda_3 > \frac{1}{2(\sqrt{(1+4\pi)^2+16\pi}+1+4\pi)}$$

implies $0 < \rho < 1$. Then $A_\eta : X \to X$ is a contractive mapping. The completeness of X means there exists a unique $h \in X$, such that $A_\eta h = h$, which implies $Lh = \eta$. In fact $h \in D$. Hence $L : D \subset X \to X$ is invertible.

Step 2. *$L^{-1} : X \to D$ is continuous.*

Let $\eta \in X, \{\eta_n\} \subset X, \eta_n \to \eta, L^{-1}\eta = v, L^{-1}\eta_n = v_n$, then

$$\begin{aligned}
v_n(t) = e^{-\lambda_3 t}&\left[\frac{1}{e^{2\pi\lambda_3}-1}\int_0^{2\pi}\left(\eta_n(s) - \lambda_1\int_0^{2\pi} G(s,\tau)v_n(\tau)d\tau - \lambda_2\int_s^{2\pi} v_n(\tau)d\tau\right)e^{\lambda_3 s}ds\right. \\
&\left.+ \int_0^t \left(\eta_n(s) - \lambda_1\int_0^{2\pi} G(s,\tau)v_n(\tau)d\tau - \lambda_2\int_s^{2\pi} v_n(\tau)d\tau\right)e^{\lambda_3 s}ds\right], \\
v(t) = e^{-\lambda_3 t}&\left[\frac{1}{e^{2\pi\lambda_3}-1}\int_0^{2\pi}\left(\eta(s) - \lambda_1\int_0^{2\pi} G(s,\tau)v(\tau)d\tau - \lambda_2\int_s^{2\pi} v(\tau)d\tau\right)e^{\lambda_3 s}ds\right. \\
&\left.+ \int_0^t \left(\eta(s) - \lambda_1\int_0^{2\pi} G(s,\tau)v(\tau)d\tau - \lambda_2\int_s^{2\pi} v(\tau)d\tau\right)e^{\lambda_3 s}ds\right].
\end{aligned}$$

As a result,

$$\begin{aligned}
|v_n(t) - v(t)| = \Bigg| e^{-\lambda_3 t}&\left\{\frac{1}{e^{2\pi\lambda_3}-1}\int_0^{2\pi}\left[(\eta_n(s)-\eta(s)) - \lambda_1\int_0^{2\pi} G(s,\tau)(v_n(\tau)\right.\right. \\
&\left. -v(\tau))d\tau - \lambda_2\int_s^{2\pi}(v_n(\tau)-v(\tau))d\tau\right]e^{\lambda_3 s}ds + \int_0^t \left[(\eta_n(s)-\eta(s))\right. \\
&- \lambda_1\int_0^{2\pi} G(s,\tau)(v_n(\tau)-v(\tau))d\tau \\
&\left.\left. -\lambda_2\int_s^{2\pi}(v_n(\tau)-v(\tau))d\tau\right]e^{\lambda_3 s}ds\right\}\Bigg|
\end{aligned}$$

$$\leq 2\pi\left(\frac{1}{\lambda_3}+1\right)\|\eta_n-\eta\|+\left(\frac{1}{2}-2\pi\lambda_3\right)\left(\frac{1}{\lambda_3}+1\right)\|v_n-v\|.$$

Hence we have

$$\|v_n-v\|\leq\frac{1}{1-\rho}2\pi\left(\frac{1}{\lambda_3}+1\right)\|\eta_n-\eta\|,$$

where $\rho=(\frac{1}{\lambda_3}+1)(\frac{1}{2}-2\pi\lambda_3)$ has defined in Step 1 and $0<\rho<1$.

Consequently, $v_n\to v$ when $\eta_n\to\eta$. Therefore $L^{-1}:X\to D$ is continuous.

Step 3. $L^{-1}:X\to D$ *is a compact mapping.*

Let $S\subset X$ be a bounded subset, i.e., there exists a constant $M>0$ such that $\|\eta\|\leq M$ for any $\eta\in S$.

Let $\eta\in E, L^{-1}\eta=v$, then

$$v(t)=e^{-\lambda_3 t}\left[\frac{1}{e^{2\pi\lambda_3}-1}\int_0^{2\pi}\left(\eta(s)-\lambda_1\int_0^{2\pi}G(s,\tau)v(\tau)d\tau-\lambda_2\int_s^{2\pi}v(\tau)d\tau\right)e^{\lambda_3 s}ds\right.$$
$$\left.+\int_0^t\left(\eta(s)-\lambda_1\int_0^{2\pi}G(s,\tau)v(\tau)d\tau-\lambda_2\int_s^{2\pi}v(\tau)d\tau\right)e^{\lambda_3 s}ds\right].$$

As a result,

$$\begin{aligned}\|v\|&\leq 2\pi\left(\frac{1}{\lambda_3}+1\right)\|\eta\|+\left(\frac{1}{2}-2\pi\lambda_3\right)\left(\frac{1}{\lambda_3}+1\right)\|v\|\\&\leq\frac{1}{1-\rho}2\pi\left(\frac{1}{\lambda_3}+1\right)\|\eta\|\leq\frac{1}{1-\rho}2\pi\left(\frac{1}{\lambda_3}+1\right)M=T,\end{aligned}$$

which implies $L^{-1}(S)$ is bounded.

In addition, let $t_1,t_2\in[0,1], t_1<t_2, \delta(t)=\int_t^{2\pi}\int_0^{2\pi}G(\tau,s)d\tau ds$, then for any $v\in L^{-1}(S)$ there exists $\eta\in D$ such that $L^{-1}\eta=v$ and

$$\begin{aligned}|v(t_1)-v(t_2)|=&\mid A_\eta v(t_1)-A_\eta v(t_2)\mid\\ \leq&\frac{1}{\lambda_3}\mid(e^{-\lambda_3 t_1}-e^{-\lambda_3 t_2})(2\pi\|\eta\|+\lambda_1\delta(0)\|\ u\|+2\pi^2\lambda_2\|u\|)\mid\\&+\Big|\|\eta\|(t_2-t_1)+\lambda_1(\delta(t_1)-\delta(t_2))\|u\|\\&+\lambda_2\|u\|\left(\frac{(2\pi-t_1)^2}{2}-\frac{(2\pi-t_2)^2}{2}\right)\Big|\\ \leq&\frac{1}{\lambda_3}\left[2\pi M+(\frac{1}{2}-2\pi\lambda_3)T\right](e^{-\lambda_3 t_1}-e^{-\lambda_3 t_2})\\&+M(t_2-t_1)+\lambda_1 T(\delta(t_1)-\delta(t_2))+\lambda_2 T\left(\frac{(2\pi-t_1)^2}{2}-\frac{(2\pi-t_2)^2}{2}\right).\end{aligned}$$

Due to the uniformly continuous functions,

$$f_1(t)=e^{-\lambda_3 t}, f_2(t)=t, \delta(t)=\int_t^{2\pi}\int_0^{2\pi}G(\tau,s)d\tau ds, f_3(t)=\frac{(2\pi-t)^2}{2}$$

on $[0,2\pi]$, we know that for $\forall\epsilon>0$, there exists $\sigma>0$ such that $t_2-t_1<\sigma$ implies

$$|v(t_1)-v(t_2)|<\epsilon.$$

Hence $L^{-1}(S)$ is equi-continuous. Making use of *Arzela–Ascoli*'s theorem, we know that $L^{-1}: X \to D$ is a compact mapping.

Since f is continuous, then $N: X \to X$ is continuous.

Step 4. *$L^{-1}N: X \to D$ is increasing.*

Suppose $\eta_1, \eta_2 \in X, \eta_1 \leq \eta_2$, then assumptions $(H_1 - H_2)$ imply $N\eta_1 \leq N\eta_2$. Let $v_1 = L^{-1}N\eta_1, v_2 = L^{-1}N\eta_2$, then $Lv_1 = N\eta_1 \leq N\eta_2 = Lv_2$. By Theorem 2.1 we obtain $v_1 \leq v_2$. Hence $L^{-1}N: X \to D$ is increasing.

Step 5. *there exist $x, y \in D, x \leq y$ such that $Lx \leq Nx$ and $Ly \geq Ny$.*

In fact, by Step 1, we know that there exist $x, y \in D$ such that $Lx = N(-\alpha''), Ly = N(-\beta'')$. In the following, we will verify that

(1) $x \leq y$;

(2) $Lx \leq Nx$ and $Ly \geq Ny$.

Since $\alpha''(t) \geq \beta''(t)$ and N is nondecreasing, it is easy to know

$$Lx = N(-\alpha'') \leq N(-\beta'') = Ly,$$

an application of Theorem 2.1 gives that $x \leq y$.

By the definition of x,we have

$$\left\{\begin{array}{l} x'(t) + \lambda_1 \int_0^{2\pi} G(t,s)x(s)ds + \lambda_2 \int_t^{2\pi} x(s)ds + \lambda_3 x(t) \\ = f(t, \alpha(t), \alpha'(t), \alpha''(t)) + \lambda_1\alpha(t) + \lambda_2\alpha'(t) - \lambda_3\alpha''(t), \\ x(0) = x(2\pi). \end{array}\right. \tag{4.3}$$

Let $\phi(t) = -\alpha''(t)$. α is the lower solution means

$$\left\{\begin{array}{l} \phi'(t) + \lambda_1 \int_0^{2\pi} G(t,s)\phi(s)ds + \lambda_2 \int_t^{2\pi} \phi(s)ds + \lambda_3 \phi(t) \\ \leq f(t, \alpha(t), \alpha'(t), \alpha''(t)) + \lambda_1\alpha(t) + \lambda_2\alpha'(t) - \lambda_3\alpha''(t), \\ \phi(0) \leq \phi(2\pi). \end{array}\right. \tag{4.4}$$

(4.3)–(4.4) together with (H_2) lead to

$$\left\{\begin{array}{l} (x(t) - \phi(t))' + \lambda_1 \int_0^{2\pi} G(t,s)(x(s) - \phi(s))ds \\ +\lambda_2 \int_t^{2\pi} (x(s) - \phi(s))ds + \lambda_3(x(t) - \phi(t)) \geq 0, \\ (x(0) - \phi(0)) \geq (x(2\pi) - \phi(2\pi)). \end{array}\right. \tag{4.5}$$

By virtue of Theorem 2.1, we have $x(t) - \phi(t) \geq 0$. The nondecreasing of N gives $Nx \geq N\phi$, hence $Lx = N\phi \leq Nx$.

$Ny \leq Ly$ can be verified similarly.

Step 4–Step 5 means the operator $L^{-1}N$ maps $[x,y] \bigcap D$ into $[x,y] \bigcap D$. Since $[x,y] \bigcap D$ is convex, closed and bounded and $L^{-1}N$ is completely continuous, an application of *Schauder* fixed-point theorem implies $Lv = Nv$ has a solution v^* in $[x,y]$. Let $u^*(t) = \int_0^{2\pi} G(t,s)v^*(s)ds$, then $u^*(t)$ is a solution of problem (1.1) and satisfies $\alpha(t) \leq u^*(t) \leq \beta(t)$. This completes the proof. ∎

Proof of Corollary 3.1. Under condition (1), let $\alpha(t) \equiv 0, \beta(t) = c(2\pi t - \frac{t^2}{2})$. Under condition (2), let $\alpha(t) = -c(2\pi t - \frac{t^2}{2}), \beta(t) \equiv 0$ and while condition (3) holds, let $\alpha(t) = -c(2\pi t - \frac{t^2}{2}), \beta(t) = c(2\pi t - \frac{t^2}{2})$. It is easy to check that α, β are the lower and upper solutions of (P) under condition (i), $i = 1, 2, 3$, respectively. Theorem 3.1 asserts the existence of solution u^* of (P) under condition (i), $i = 1, 2, 3$. ∎

5. Conclusion

In this paper, we established a new comparison theorem and then use it to investigate the solvability of a third-order two-point boundary value problem

$$\begin{cases} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \\ u(0) = u'(2\pi) = 0, \\ u''(0) = u''(2\pi). \end{cases}$$

We give some existence results for this problem via upper and lower solutions method and fixed point theory.

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