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Oscillation Results for Third Order Nonlinear Delay Dynamic Equations on Time Scales

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Abstract. In this paper, we consider the third order nonlinear delay dynamic equations

$$(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta} + f(t, x(\tau(t))) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is a quotient of odd positive integers, a and r are positive rd-continuous functions on $\mathbb{T}$, and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$, $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(t, u) > 0$, for $u \neq 0$ and there exists a positive rd-continuous function p on $\mathbb{T}$ such that $f(t, u)/u^{\gamma} \geq p(t)$, for $u \neq 0$.

We establish some new results. Some examples are considered to illustrate the main results.

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1. Introduction

A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or often modeled by continuous dynamic systems), die out, say in winter, while their eggs are incubating or dormant, and then in season again, hatching gives rise to a nonoverlapping population. Not only does the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to the

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so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0 \quad \text{for} \quad q > 1\}$ (which has important applications in quantum theory) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{N}, \mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers.

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis (see Hilger [17]). Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal *et al.* [1] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [3] summarizes and organizes much of the time scale calculus. We refer also to the last book by Bohner and Peterson [4] for advances in dynamic equation on time scales. For the notations used below we refer to the next section that provides some basic facts on time scales extracted from Bohner and Peterson [3].

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various second order dynamic equations on time scales (we refer the reader to the articles [2, 5–7, 11–14, 18–24]).

To the best of our knowledge, there is very little known about the oscillatory behavior of third order dynamic equations. Erbe *et al.* [8–10] considered the third order dynamic equations

$$(a(t)[r(t)x^{\Delta}(t)]^{\Delta})^{\Delta} + p(t)f(x(t)) = 0,$$

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0,$$

and

$$(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta} + f(t, x(t)) = 0,$$

respectively and established some sufficient conditions for oscillation.

Recently, Hassan [15] considered the third order delay dynamic equations

$$(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta} + f(t, x(\tau(t))) = 0, \tag{1.1}$$

on a time scale $\mathbb{T}$, where $\gamma \geq 1$ is a quotient of odd positive integers, a and r are positive rd-continuous functions on $\mathbb{T}$, and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t, \tau^{\Delta}(t) \geq 0$, for $t \in \mathbb{T}$ and $\tau(t) \to \infty$ as $t \to \infty$, $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(t, u) > 0$, for $u \neq 0$ and there exists a positive rd-continuous function p on $\mathbb{T}$ such that $f(t, u)/u^{\gamma} \geq p(t)$, for $u \neq 0$. The author established some sufficient conditions for oscillation of (1.1), when the condition $\tau(\sigma(t)) = \sigma(\tau(t))$ holds.

The restriction $\tau(\sigma(t)) = \sigma(\tau(t))$ depends on time scale, so by suitable choosing for $\tau(t)$, we can find that, for example, in general, we can choose $\tau = \rho^k, k \in \mathbb{Z}^+$, where ρ is the backward jump operator, for any isolated time scale.

This paper considers equation (1.1) where $\gamma > 0$ is a quotient of odd positive integers, a and r are positive rd-continuous functions on $\mathbb{T}$, and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$, $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(t, u) > 0$, for $u \neq 0$ and there exists a positive rd-continuous function p on $\mathbb{T}$ such that $f(t, u)/u^{\gamma} \geq p(t)$, for $u \neq 0$.

As we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$

by $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty)\cap\mathbb{T}$. A solution $x(t)$ is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

We establish new oscillation criteria that can be applied on any time scale $\mathbb{T}$ and we complement the results in [15].

2. Main results

In this section we give some new oscillation criteria for (1.1). Throughout this paper, we let

$$d_+(t) := \max\{0, d(t)\}, \quad d_-(t) := \max\{0, -d(t)\},$$

and

$$\beta(t) := b(t), \quad 0 < \gamma \le 1, \quad \beta(t) := b^{\gamma}(t), \quad \gamma > 1,$$

$$b(t) = \frac{t}{\sigma(t)}, \quad \delta(t,T_1) := \int_{T_1}^{t} \frac{\Delta s}{a^{\frac{1}{\gamma}}(s)}.$$

In order to prove our main results, we will use the formula

$$((x(t))^{\gamma})^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1} x^{\Delta}(t)\mathrm{d}h, \tag{2.1}$$

where $x(t)$ is delta differentiable and eventually positive or eventually negative, which is a simple consequence of Keller's chain rule (see Bohner and Peterson [3, Theorem 1.90]). Also, we need the following lemmas which will play an important role in the proof of main results.

Lemma 2.1. [15, Lemma 2.1, p. 1574] *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{a^{\frac{1}{\gamma}}(t)} = \infty, \quad \int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty, \tag{2.2}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \int_{t}^{\infty} \left[\frac{1}{a(s)} \int_{s}^{\infty} p(u)\Delta u\right]^{\frac{1}{\gamma}} \Delta s \Delta t = \infty. \tag{2.3}$$

Furthermore, suppose that (1.1) *has a positive solution* x *on* $[t_0,\infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0,\infty)_{\mathbb{T}}$, *sufficiently large, so that*

$$(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta} < 0, \quad (r(t)x^{\Delta}(t))^{\Delta} > 0, \quad t \in [T,\infty)_{\mathbb{T}},$$

and either $x^{\Delta}(t) > 0$ *on* $[T,\infty)_{\mathbb{T}}$ *or* $\lim_{t\to\infty} x(t) = 0$.

Lemma 2.2. [15, Lemma 2.2, p. 1575] *Assume that* x *is a positive solution of equation* (1.1) *such that*

$$(r(t)x^{\Delta}(t))^{\Delta} > 0, \quad x^{\Delta}(t) > 0,$$

on $[t_*,\infty)_{\mathbb{T}}, t_* \ge t_0$. *Then*

$$x^{\Delta}(t) \ge \frac{\delta(t,t_*)}{r(t)} a^{\frac{1}{\gamma}}(t)(r(t)x^{\Delta}(t))^{\Delta}.$$

Lemma 2.3. *Assume that x is a positive solution of equation* (1.1) *such that*

$$(r(t)x^{\Delta}(t))^{\Delta} > 0, \quad x^{\Delta}(t) > 0,$$

on $[t_*, \infty)_{\mathbb{T}}, t_* \geq t_0$. *Furthermore,* $r^{\Delta}(t) \leq 0$,

$$\int_{t_0}^{\infty} p(t)\tau^{\gamma}(t)\Delta t = \infty. \tag{2.4}$$

Then there exists a $T \in [t_*, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

$$x(t) > tx^{\Delta}(t),$$

$x(t)/t$ *is strictly decreasing,* $t \in [T, \infty)_{\mathbb{T}}$.

Proof. In view of

$$(r(t)x^{\Delta}(t))^{\Delta} = r^{\Delta}(t)x^{\Delta}(t) + r^{\sigma}(t)x^{\Delta\Delta}(t) > 0,$$

so we have $x^{\Delta\Delta}(t) > 0, t \in [t_*, \infty)_{\mathbb{T}}$. Let

$$U(t) := x(t) - tx^{\Delta}(t).$$

Hence $U^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) < 0$. We claim there exists a $t_1 \in [t_*, \infty)_{\mathbb{T}}$ such that $U(t) > 0, x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not. Then $U(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{2.5}$$

which implies that $x(t)/t$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Pick $t_2 \in [t_1, \infty)_{\mathbb{T}}$ so that $\tau(t) \geq \tau(t_1)$, for $t \geq t_2$. Then

$$\frac{x(\tau(t))}{\tau(t)} \geq \frac{x(\tau(t_1))}{\tau(t_1)} := d > 0,$$

so that $x(\tau(t)) \geq d\tau(t)$, for $t \geq t_2$. By (1.1) we have

$$(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta} \leq -p(t)x^{\gamma}(\tau(t)) < 0, \quad t \geq t_2. \tag{2.6}$$

Now by integrating both sides of (2.6) from t_2 to t, we have

$$a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma} - a(t_2)\{[r(t_2)x^{\Delta}(t_2)]^{\Delta}\}^{\gamma} + \int_{t_2}^{t} p(s)x^{\gamma}(\tau(s))\Delta s \leq 0.$$

This implies that

$$a(t_2)\{[r(t_2)x^{\Delta}(t_2)]^{\Delta}\}^{\gamma} \geq \int_{t_2}^{t} p(s)x^{\gamma}(\tau(s))\Delta s \geq d^{\gamma}\int_{t_2}^{t} p(s)\tau^{\gamma}(s)\Delta s,$$

which contradicts (2.4). So $U(t) > 0$ on $t \in [t_1, \infty)_{\mathbb{T}}$ and consequently,

$$(\frac{x(t)}{t})^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{2.7}$$

and we have that $x(t)/t$ is strictly decreasing on $t \in [t_1, \infty)_{\mathbb{T}}$. The proof is now complete. ∎

Theorem 2.1. *Assume that* (2.2), (2.3) *and* (2.4) *hold.* $r^{\Delta}(t) \leq 0$. *Furthermore, assume that there exists a positive function* $\alpha \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, *for all sufficiently large* $T_1 \in [t_0,\infty)_{\mathbb{T}}$, *there is a* $T > T_1$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma} - \frac{r^{\gamma}(s)((\alpha^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}\right]\Delta s = \infty. \tag{2.8}$$

Then every solution of equation (1.1) *is either oscillatory or tends to zero.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}, t_1 \in [t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when $x(t)$ is eventually negative is similar. Therefore from Lemma 2.1, we get

$$(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta} < 0, \quad (r(t)x^{\Delta}(t))^{\Delta} > 0, \quad t \in [t_1,\infty)_{\mathbb{T}},$$

and either $x^{\Delta}(t) > 0$ for $t \geq t_2 \geq t_1$ or $\lim_{t\to\infty} x(t) = 0$. Let $x^{\Delta}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$\omega(t) = \alpha(t)\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}}{x^{\gamma}(t)}. \tag{2.9}$$

By the product rule and then the quotient rule

$$\begin{aligned}
\omega^{\Delta}(t) &= \alpha^{\Delta}(t)\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}}{x^{\gamma}(t)} + \alpha^{\sigma}(t)[\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}}{x^{\gamma}(t)}]^{\Delta} \\
&= \alpha^{\Delta}(t)\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}}{x^{\gamma}(t)} + \alpha^{\sigma}(t)\frac{(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta}}{x^{\gamma\sigma}(t)} \\
&\quad - \alpha^{\sigma}(t)\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma\sigma}(t)}.
\end{aligned}$$

From (1.1) and the definition of $\omega(t)$ and using the fact $x(t)/t$ is strictly decreasing, $t \in [t_3,\infty)_{\mathbb{T}}, t_3 \geq t_2$, we have that

$$\omega^{\Delta}(t) \leq -\alpha^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + \frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t) - \alpha^{\sigma}(t)\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma\sigma}(t)}. \tag{2.10}$$

If $0 < \gamma \leq 1$, by (2.1), we have

$$(x^{\gamma}(t))^{\Delta} \geq \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t),$$

in view of (2.10), Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
\omega^{\Delta}(t) &\leq -\alpha^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + \frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t) - \gamma\alpha^{\sigma}(t)\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}x(t)x^{\Delta}(t)}{x^{\gamma+1}(t)x^{\sigma}(t)} \\
&\leq -\alpha^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + \frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t) - \gamma\alpha^{\sigma}(t)\frac{t}{\sigma(t)}\frac{\delta(t,t_*)}{r(t)}\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\alpha^{\frac{\gamma+1}{\gamma}}(t)}.
\end{aligned} \tag{2.11}$$

If $\gamma > 1$, also by (2.1), we have

$$(x^{\gamma}(t))^{\Delta} \geq \gamma(x(t))^{\gamma-1}x^{\Delta}(t),$$

in view of (2.10), Lemma 2.2 and Lemma 2.3, we have

$$\omega^{\Delta}(t) \leq -\alpha^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + \frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t) - \gamma\alpha^{\sigma}(t)\frac{a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma}x^{\gamma}(t)x^{\Delta}(t)}{x^{\gamma+1}(t)x^{\gamma\sigma}(t)}$$

$$\leq -\alpha^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + \frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t) - \gamma\alpha^{\sigma}(t)(\frac{t}{\sigma(t)})^{\gamma}\frac{\delta(t,t_*)}{r(t)}\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\alpha^{\frac{\gamma+1}{\gamma}}(t)}. \tag{2.12}$$

By (2.11), (2.12) and the definition of b and β, we have, for $\gamma > 0$,

$$\omega^{\Delta}(t) \leq -\alpha^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + \frac{(\alpha^{\Delta}(t))_+}{\alpha(t)}\omega(t) - \gamma\alpha^{\sigma}(t)\beta(t)\frac{\delta(t,t_*)}{r(t)}\frac{w^{\lambda}(t)}{\alpha^{\lambda}(t)}, \tag{2.13}$$

where $\lambda := (\gamma+1)/\gamma$. Define $A \geq 0$ and $B \geq 0$ by

$$A^{\lambda} := \gamma\alpha^{\sigma}(t)\beta(t)\frac{\delta(t,t_*)}{r(t)}\frac{w^{\lambda}(t)}{\alpha^{\lambda}(t)}, \quad B^{\lambda-1} := \frac{(\alpha^{\Delta}(t))_+ r^{\frac{1}{\lambda}}(t)}{\lambda(\gamma\beta(t)\alpha^{\sigma}(t)\delta(t,t_*))^{\frac{1}{\lambda}}}.$$

Then using the inequality [16]

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda}, \tag{2.14}$$

which yields

$$\frac{(\alpha^{\Delta}(t))_+}{\alpha(t)}\omega(t) - \gamma\alpha^{\sigma}(t)\beta(t)\frac{\delta(t,t_*)}{r(t)}\frac{w^{\lambda}(t)}{\alpha^{\lambda}(t)} \leq \frac{r^{\gamma}(t)((\alpha^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\alpha^{\sigma}(t)\delta(s,t_*))^{\gamma}}.$$

From this last inequality and (2.13), we find

$$\omega^{\Delta}(t) \leq -\alpha^{\sigma}(t)p(t)(\frac{\tau(t)}{\sigma(t)})^{\gamma} + \frac{r^{\gamma}(t)((\alpha^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\alpha^{\sigma}(t)\delta(s,t_*))^{\gamma}}.$$

Integrating both sides from t_3 to t, we get

$$\int_{t_3}^{t}\left[\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma} - \frac{r^{\gamma}(s)((\alpha^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,t_*))^{\gamma}}\right]\Delta s \leq \omega(t_3) - \omega(t) \leq \omega(t_3),$$

which contradicts assumption (2.8). This contradiction completes the proof. ∎

Remark 2.1. From Theorem 2.1, we can obtain different conditions for oscillation of equation (1.1) with different choices of $\alpha(t)$.

Remark 2.2. The conclusion of Theorem 2.1 remains intact if assumption (2.8) is replaced by the two conditions

$$\limsup_{t\to\infty}\int_{T}^{t}\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma}\Delta s = \infty,$$

$$\liminf_{t\to\infty}\int_{T}^{t}\frac{r^{\gamma}(s)((\alpha^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}\Delta s < \infty.$$

Corollary 2.1. *Assume that* (2.2), (2.3) *and* (2.4) *hold.* $r^{\Delta}(t) \leq 0$. *Furthermore, suppose that there exist functions* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$ *such that*

$$H(t,t) = 0, \quad t \geq t_0, \quad H(t,s) > 0, \quad t > s \geq t_0,$$

and H *has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies*

$$H^{\Delta_s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\frac{\alpha^{\Delta}(s)}{\alpha(s)} = -\frac{h(t,s)}{\alpha(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}},$$

and for all sufficiently large $T_1 \in [t_0, \infty)_{\mathbb{T}}$, there is a $T > T_1$ such that

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), T)} \int_T^{\sigma(t)} K(t, s)\Delta s = \infty, \tag{2.15}$$

where α is a positive $\Delta-$differentiable function and

$$K(t,s) = H(\sigma(t), \sigma(s))\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma} - \frac{r^{\gamma}(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}.$$

Then every solution of equation (1.1) *is either oscillatory or tends to zero.*

Remark 2.3. The conclusion of Corollary 2.1 remains intact if assumption (2.15) is replaced by the two conditions

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), T)} \int_T^{\sigma(t)} H(\sigma(t), \sigma(s))\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma}\Delta s = \infty,$$

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t), T)} \int_T^{\sigma(t)} \frac{r^{\gamma}(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}\Delta s < \infty.$$

Remark 2.4. Define ω as (2.9), we also get

$$\omega^{\Delta}(t) = (\frac{\alpha(t)}{x^{\gamma}(t)})^{\Delta}(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\sigma} + \frac{\alpha(t)}{x^{\gamma}(t)}(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta},$$

similar to the proofs of Theorem 2.1, we can obtain different results. We leave the details to the reader.

3. Applications and examples

In this section, we give some examples to illustrate our main results.

Example 3.1. Consider the third order delay dynamic equation

$$x^{\Delta\Delta\Delta}(t) + \frac{\beta}{t\tau(t)}x(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{3.1}$$

where β is a positive constant. We have

$$a(t) = r(t) = 1, \quad p(t) = \frac{\beta}{t\tau(t)}, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

It is clear that condition (2.2), (2.3) and (2.4) hold. Therefore, by Theorem 2.1, pick $\alpha(t) = t$, we have

$$\begin{aligned}&\limsup_{t\to\infty} \int_T^t \left[\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma} - \frac{r^{\gamma}(s)((\alpha^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}\right]\Delta s\\ &= \limsup_{t\to\infty} \int_T^t \left[\frac{\beta}{s} - \frac{1}{(\gamma+1)^{\gamma+1}(s(s-T_1))}\right]\Delta s = \infty.\end{aligned}$$

Hence, every solution of equation (3.1) is oscillatory or tends to zero if $\beta > 0$.

Example 3.2. Consider the third order delay dynamic equation

$$\left(t^{\gamma}(x^{\Delta\Delta}(t))^{\gamma}\right)^{\Delta}+\frac{\beta}{t\tau^{\gamma}(t)}x^{\gamma}(\tau(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{3.2}$$

where β is a positive constant, $\gamma>0$. We have

$$a(t)=t^{\gamma},\quad r(t)=1,\quad p(t)=\frac{\beta}{t\tau^{\gamma}(t)},\quad t\in[t_0,\infty)_{\mathbb{T}}.$$

The condition (2.2), (2.3) and (2.4) hold (similar to [15, Example 2.1]). Thus, we assume $\mathbb{T}$ is a time scale satisfying $\sigma(t)\le kt$, for some $k>0, t\ge T_k>t_*$.

When $\gamma\ge 1$, by Theorem 2.1, pick $\alpha(t)=t^{\gamma}$, by (2.1), we have that $\alpha^{\Delta}(t)(t^{\gamma})^{\Delta}\le\gamma\sigma^{\gamma-1}(t)$. Therefore

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{T_k}^{t}\left[\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma}-\frac{r^{\gamma}(s)((\alpha^{\Delta}(s))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}\right]\Delta s\\
&\ge\limsup_{t\to\infty}\int_{T_k}^{t}\left[\frac{\beta}{s}-(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{k^{\gamma^2-1}}{s}\right]\Delta s\\
&\ge\left(\beta-(\frac{\gamma}{\gamma+1})^{\gamma+1}k^{\gamma^2-1}\right)\limsup_{t\to\infty}\int_{T_k}^{t}\frac{\Delta s}{s}=\infty,
\end{aligned}$$

if $\beta>(\gamma/(\gamma+1))^{\gamma+1}k^{\gamma^2-1}$.

When $0<\gamma<1$, pick $\alpha(t)=t$, by Theorem 2.1, we have that

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{T_k}^{t}\left[\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma}-\frac{r^{\gamma}(s)((\alpha^{\Delta}(s))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}\right]\Delta s\\
&\ge\limsup_{t\to\infty}\int_{T_k}^{t}\left[\frac{\beta}{k^{\gamma-1}s^{\gamma}}-\frac{1}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s\\
&\ge\left(\frac{\beta}{k^{\gamma-1}}-\frac{1}{(\gamma+1)^{\gamma+1}}\right)\limsup_{t\to\infty}\int_{T_k}^{t}\frac{\Delta s}{s^{\gamma}}=\infty,
\end{aligned}$$

if $\beta>k^{\gamma-1}/(\gamma+1)^{\gamma+1}$.

Hence, every solution of equation (3.2) is oscillatory or tends to zero if

$$\beta>(\frac{\gamma}{\gamma+1})^{\gamma+1}k^{\gamma^2-1},\gamma\ge 1;\beta>\frac{k^{\gamma-1}}{(\gamma+1)^{\gamma+1}},0<\gamma<1.$$

Example 3.3. Consider the third order delay dynamic equation

$$\left((\frac{1}{t}x^{\Delta}(t))^{\Delta})^{\gamma}\right)^{\Delta}+\frac{\beta\sigma^{\gamma}(t)}{t\tau^{\gamma}(t)}x^{\gamma}(\tau(t))=0,\quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{3.3}$$

where β is a positive constant, $\gamma>0$. We have

$$a(t)=1,\quad r(t)=\frac{1}{t},\quad p(t)=\frac{\beta\sigma^{\gamma}(t)}{t\tau^{\gamma}(t)},\quad t\in[t_0,\infty)_{\mathbb{T}}.$$

It is clear that condition (2.2), (2.3) and (2.4) hold. Therefore, by Theorem 2.1, pick $\alpha(t)=1$, we have

$$\limsup_{t\to\infty}\int_{T}^{t}\left[\alpha^{\sigma}(s)p(s)(\frac{\tau(s)}{\sigma(s)})^{\gamma}-\frac{r^{\gamma}(s)((\alpha^{\Delta}(s))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\alpha^{\sigma}(s)\delta(s,T_1))^{\gamma}}\right]\Delta s$$

$$= \limsup_{t\to\infty} \int_T^t \frac{\beta}{s}\Delta s = \infty.$$

Hence, every solution of equation (3.3) is oscillatory or tends to zero if $\beta > 0$.

Remark 3.1. In the equations (3.1), (3.2) and (3.3), we don't need the condition $\tau(\sigma(t)) = \sigma(\tau(t))$. Therefore, our results complement and improve the results in [15].

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