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Generalized Intuitionistic Fuzzy Ideals of Ordered Semigroups

[1]YUNQIANG YIN, [2]JIANMING ZHAN AND [3]DEHUA XU
[1,3]School of Sciences, East China Institute of Technology,
Fuzhou, Jiangxi 344000, China
[2]Department of Mathematics, Hubei Institute for Nationalities,
Enshi, Hubei Province 445000, China
[1]yunqiangyin@gmail.com, [2]zhanjianming@hotmail.com,
[3]schedulingresearch@gmail.com

Abstract. The notion of intuitionistic fuzzy ideals with thresholds of an ordered semigroup is considered and some of its properties are given. An embedding theorem of the set of all intuitionistic fuzzy ideals with thresholds is presented and the homomorphic images and inverse images of intuitionistic fuzzy ideals with thresholds are studied too.

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1. Introduction

The concept of intuitionistic fuzzy sets was first introduced by Atanassov in 1986 [3]. Then Biswas [7] applied this concept to develop the theory of intuitionistic fuzzy subgroups of a quasigroup. The concept of intuitionistic fuzzy ideals of a semigroup was due to Kim and Jun [16]. Jun [12] further studied intuitionistic fuzzy bi-ideals of ordered semigroups. Dudek *et al.* [11] and Davvaz *et al.* [8] considered the intuitionistic fuzzy hyperquasigroups and intuitionistic fuzzy H_v-submodules, respectively. Akram and Dudek [1] described the structure of intuitionistic fuzzy left k-ideals of semirings and also see Dudek [10]. On the other hand, Akram and Shum [2] considered the bifuzzy ideals of nearrings, and the intuitionistic (T, S)-fuzzy ideals of nearrings were studied by Shum and Akram in [19].

Using the notion "belongingness $(\in)$" and "quasi-coincidence (q)" of a fuzzy point with a fuzzy set introduced by Pu and Liu [17], the concept of (α, β)-fuzzy subgroups where α, β are any two of $\{\in, q, \in\vee q, \in\wedge q\}$ with $\alpha \neq \in\wedge q$ was introduced by Bhakat and Das [6] in 1992, in which the $(\in, \in\vee q)$-fuzzy subgroup is an important and useful

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generalization of Rosenfeld's fuzzy subgroup [18]. The detailed study with $(\in, \in \vee q)$-fuzzy subgroup has been considered in Bhakat [5]. As a generalization of Rosenfeld's fuzzy subgroup [18] and Bhakat and Das's fuzzy subgroup [6], the fuzzy subgroups with thresholds were studied in Yuan *et al.* [21]. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. With this objective in mind, we first introduce the concept of intuitionistic fuzzy ideals with thresholds (α, β) of ordered semigroups and investigate some related properties. Then we give an embedding theorem of the set of all intuitionistic fuzzy ideals with thresholds of order semigroups. The homomorphism between such intuitionistic fuzzy ideals are also considered.

For notations, terminologies and applications, the reader is refereed to [3,4,9,13–15,20].

2. Intuitionistic fuzzy sets and ordered semigroups

Let X be a non-empty set. A mapping $\mu : X \to [0,1]$ is called a *fuzzy set* in X. The complement of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. For any $P \subseteq X$ and $r \in [0,1]$, define a fuzzy set r_P in X by $r_P(x) = r$ if $x \in P$ and 0 otherwise for all $x \in X$. In particular, if $r = 1$, then 1_P is said to be the *characteristic function* of P, and we shall use the symbol κ_P for 1_P.

For any fuzzy set μ in X and $r \in [0,1]$, define two sets

$$U(\mu; r) = \{x \in X | \mu(x) > r\} \quad \text{and} \quad L(\mu; r) = \{x \in X | \mu(x) < r\},$$

which are called an *upper and lower r-strong level cut* of μ, respectively.

As an important generalization of the notion of fuzzy sets, Atanassov introduced the concept of an intuitionistic fuzzy set as follows.

Definition 2.1. [3] *An intuitionistic fuzzy set A in a non-empty set X is an object having the form*

$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$.

Definition 2.2. [4,9] *Let $A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$ and $B = \{(x, \mu_B(x), \lambda_B(x)) | x \in X\}$ be intuitionistic fuzzy sets in X and let $r, t \in [0,1]$. Then*

(1) $A \subseteq B$ *iff* $\mu_A(x) \le \mu_B(x)$ *and* $\lambda_A(x) \ge \lambda_B(x)$ *for all* $x \in X$,
(2) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \lambda_A(x) \vee \lambda_B(x)) | x \in X\}$,
(3) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \lambda_A(x) \wedge \lambda_B(x)) | x \in X\}$,
(4) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) | x \in X\}$,
(5) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) | x \in X\}$,
(6) $A^n = \{(x, [\mu_A(x)]^n, 1 - [1 - \lambda_A(x)]^n) | x \in X\}$,
(7) $nA = \{(x, 1 - [1 - \mu_A(x)]^n, [\lambda_A(x)]^n) | x \in X\}$,
(8) $P_{r,t}(A) = \{(x, r \vee \mu_A(x), t \wedge \lambda_A(x)) | x \in X\}$ *for* $r + t \le 1$,
(9) $Q_{r,t}(A) = \{(x, r \wedge \mu_A(x), t \vee \lambda_A(x)) | x \in X\}$ *for* $r + t \le 1$,
(10) $F_{r,t}(A) = \{(x, \mu_A(x) + r \cdot \pi_A(x), \lambda_A(x) + t \cdot \pi_A(x)) | x \in X\}$ *for* $r + t \le 1$,

(11) $H_{r,t}(A) = \{(x, r \cdot \mu_A(x), \lambda_A(x) + t \cdot \pi_A(x)) | x \in X\}$,
(12) $J_{r,t}(A) = \{(x, \mu_A(x) + r \cdot \pi_A(x), t \cdot \lambda_A(x)) | x \in X\}$,
where $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x)$.

For the sake of simplicity, we use $A = (\mu_A, \lambda_A)$ to denote the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$. For any $r, t \in [0, 1]$, denote $A^{(r,t)} = \{x \in X | \mu_A(x) > r \text{ and } \lambda_A(x) < t\}$, which is called the (r, t)-*strong level cut* of A. It is clear that $A^{(r,t)} = U(\mu_A; r) \cap L(\lambda_A; t)$ for all $r, t \in [0, 1]$.

In the sequel, unless otherwise stated, S always denotes an ordered semigroup. A fuzzy set μ in S is called a *fuzzy ideal* of S if $\mu(xy) \geq \mu(x) \vee \mu(y)$ and $x \leq y$ implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$ [15].

For two intuitionistic fuzzy sets A and B in S, define the product of A and B, denoted by $A \circ B$, by

$$A \circ B = \{(x, (\mu_A \check{\circ} \mu_B)(x), (\lambda_A \hat{\circ} \lambda_B)(x)) | x \in S\},$$

in which

$$(\mu_A \check{\circ} \mu_B)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} \mu_A(y) \wedge \mu_B(z) & \text{if } A_x \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$(\lambda_A \hat{\circ} \lambda_B)(x) = \begin{cases} \bigwedge\limits_{(y,z) \in A_x} \lambda_A(y) \vee \lambda_B(z) & \text{if } A_x \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

where $A_x = \{(y, z) \in S \times S | x \leq yz\}$.

Proposition 2.1. *Let A and B be intuitionistic fuzzy sets in S. Then so is $A \circ B$.*

Proof. It is straightforward. ∎

3. Fuzzy ideals with thresholds (α, β) of ordered semigroups

Definition 3.1. *Let $\alpha, \beta \in [0, 1]$, $\alpha < \beta$ and μ be a fuzzy set in S. Then μ is called a fuzzy ideal with thresholds (α, β) of S if it satisfies:*

(i) $\mu(xy) \vee \alpha \geq (\mu(x) \vee \mu(y)) \wedge \beta$,
(ii) *if $x \leq y$, then $\mu(x) \vee \alpha \geq \mu(y) \wedge \beta$,*

for all $x, y \in S$.

Note that any fuzzy ideal of S according to [15] is a fuzzy ideal with thresholds (α, β) of S. The following example illustrates that a fuzzy ideal with thresholds (α, β) is not necessarily a fuzzy ideal.

Example 3.1. Let $S = \{a, b, c\}$ be an ordered semigroup defined by $xx = x$ and $xy = z$ if $x \neq y$ for $x, y, z \in S$ with linear order $a \leq b \leq c$, and μ a fuzzy set in S such that

$$\mu(a) = 0.5, \quad \mu(b) = 0.6, \quad \mu(c) = 0.6.$$

Put $0 \leq \alpha < \beta \leq 0.5$ or $0.6 \leq \alpha < \beta \leq 1$. It can be easily seen that μ is a fuzzy ideal with thresholds (α, β) of S, but is not a fuzzy ideal of S, since $\mu(bc) = \mu(a) = 0.5 < 0.6 = \mu(b) \vee \mu(c)$.

Proposition 3.1. *Let μ be any non-empty fuzzy set in S. Then $\mu(xy) \vee \alpha \geq (\mu(x) \vee \mu(y)) \wedge \beta$ for all $x, y \in S$ if and only if $(\kappa_S \check{\circ} \mu \cup \mu \check{\circ} \kappa_S) \cap \beta_S \subseteq \mu \cup \alpha_S$.*

Proof. Assume that $\mu(xy) \vee \alpha \geq (\mu(x) \vee \mu(y)) \wedge \beta$ for all $x, y \in S$. Let $x \in S$. If $A_x = \emptyset$, then it is clear that

$$0 = ((\kappa_S \check{\circ} \mu \cup \mu \check{\circ} \kappa_S) \cap \beta_S)(x) \leq (\mu \cup \alpha_S)(x). \tag{3.1}$$

If $\mu(x) \geq \beta$, then

$$((\kappa_S \check{\circ} \mu \cup \mu \check{\circ} \kappa_S) \cap \beta_S)(x) = (\kappa_S \check{\circ} \mu \cup \mu \check{\circ} \kappa_S)(x) \wedge \beta \leq \beta \leq \mu(x) \vee \alpha = (\mu \cup \alpha_S)(x).$$

Otherwise, we have

$$\begin{aligned}
&((\kappa_S \check{\circ} \mu \cup \mu \check{\circ} \kappa_S) \cap \beta_S)(x) \\
&= ((\kappa_S \check{\circ} \mu)(x) \vee (\mu \check{\circ} \kappa_S)(x)) \wedge \beta = \left(\left(\bigvee_{(a,b) \in A_x} \mu(b) \right) \vee \left(\bigvee_{(c,d) \in A_x} \mu(c) \right) \right) \wedge \beta \\
&\leq \left(\bigvee_{(y,z) \in A_x} \mu(y) \vee \mu(z) \right) \wedge \beta = \bigvee_{(y,z) \in A_x} (\mu(y) \vee \mu(z)) \wedge \beta \\
&\leq \bigvee_{(y,z) \in A_x} \mu(yz) \vee \alpha \leq \bigvee_{(y,z) \in A_x} \mu(x) \vee \alpha \\
&\qquad (\text{since } x \leq yz, \text{ we have } \beta > \mu(x) \vee \alpha \geq \mu(yz) \wedge \beta = \mu(yz)) \\
&= \mu(x) \vee \alpha = (\mu \cup \alpha_S)(x).
\end{aligned}$$

Summarizing the above arguments, we obtain $(\kappa_S \check{\circ} \mu \cup \mu \check{\circ} \kappa_S) \cap \beta_S \subseteq \mu \cup \alpha_S$.

Conversely, suppose if possible, let $x, y \in S$ and $\mu(xy) \vee \alpha < (\mu(x) \vee \mu(y)) \wedge \beta$. Then

$$\begin{aligned}
&((\kappa_S \check{\circ} \mu \cup \mu \check{\circ} \kappa_S) \cap \beta_S)(xy) \\
&= ((\kappa_S \check{\circ} \mu)(xy) \vee (\mu \check{\circ} \kappa_S)(xy)) \wedge \beta = \left(\left(\bigvee_{(a,b) \in A_{xy}} \mu(b) \right) \vee \left(\bigvee_{(c,d) \in A_{xy}} \mu(c) \right) \right) \wedge \beta \\
&\geq (\mu(y) \vee \mu(x)) \wedge \beta > \mu(xy) \vee \alpha = (\mu \cup \alpha_S)(xy),
\end{aligned}$$

a contradiction. Therefore, $\mu(xy) \vee \alpha \geq (\mu(x) \vee \mu(y)) \wedge \beta$. ∎

Theorem 3.1. *Let μ be any non-empty fuzzy set in S. Then μ is a fuzzy ideal with thresholds (α, β) of S if and only if $U(\mu; r)$ $(U(\mu; r) \neq \emptyset)$ is an ideal of S for all $r \in [\alpha, \beta)$.*

Proof. It is straightforward. ∎

As a consequence of Theorem 3.1, we have the following result.

Theorem 3.2. *A non-empty set P in S is an ideal of S if and only if κ_P is a fuzzy ideal with thresholds (α, β) of S.*

Naturally, a corresponding result should be considered when $U(\mu; r)$ is an ideal of S for all $r \in [1 - \beta, 1 - \alpha)$.

Theorem 3.3. *Let μ be any non-empty fuzzy set in S. Then $\mu(xy) \vee (1-\beta) \geq (\mu(x) \vee \mu(y)) \wedge (1-\alpha)$ and $x \leq y$ implies $\mu(x) \vee (1-\beta) \geq \mu(y) \wedge (1-\alpha)$ for all $x, y \in S$ if and only if $U(\mu;r)$ ($U(\mu;r) \neq \emptyset$) is an ideal of S for all $r \in [1-\beta, 1-\alpha)$.*

Proof. The proof is analogous to that of Theorem 3.1. ∎

4. Intuitionistic fuzzy ideals with thresholds (α, β) of ordered semigroups

Definition 4.1. *An intuitionistic fuzzy set A in S is called an intuitionistic fuzzy ideal with thresholds (α, β) of S if it satisfies:*

(i) $\mu_A(xy) \vee \alpha \geq (\mu_A(x) \vee \mu_A(y)) \wedge \beta$,
(ii) $\lambda_A(xy) \wedge (1-\alpha) \leq (\lambda_A(x) \wedge \lambda_A(y)) \vee (1-\beta)$,
(iii) *if $x \leq y$, then $\mu_A(x) \vee \alpha \geq \mu_A(y) \wedge \beta$ and $\lambda_A(x) \wedge (1-\alpha) \leq \lambda_A(y) \vee (1-\beta)$,*

for all $x, y \in S$.

We first give some characterizations of intuitionistic fuzzy ideals with thresholds (α, β) of S. Analogous to the proof of Proposition 3.1 and Theorem 3.1, it is not difficult to see that the following results are valid.

Theorem 4.1. *An intuitionistic fuzzy set A in S is an intuitionistic fuzzy ideal with thresholds (α, β) of S if and only if A satisfies:*

(i') $(\kappa_S \check{\circ} \mu_A \cup \mu_A \check{\circ} \kappa_S) \cap \beta_S \subseteq \mu_A \cup \alpha_S$,
(ii') $\lambda_A \cap (1-\alpha)_S \subseteq (\kappa_S^c \hat{\circ} \lambda_A \cap \lambda_A \hat{\circ} \kappa_S^c) \cup (1-\beta)_S$,
(iii') *if $x \leq y$, then $\mu_A(x) \vee \alpha \geq \mu_A(y) \wedge \beta$ and $\lambda_A(x) \wedge (1-\alpha) \leq \lambda_A(y) \vee (1-\beta)$,*

for all $x, y \in S$.

Theorem 4.2. *An intuitionistic fuzzy set A in S is an intuitionistic fuzzy ideal with thresholds (α, β) of S if and only if $A^{(r,t)}$ ($A^{(r,t)} \neq \emptyset$) is an ideal of S for all $r \in [\alpha, \beta)$ and $t \in (1-\beta, 1-\alpha]$.*

Next, let us consider the intuitionistic fuzzy ideals with thresholds induced by an intuitionistic fuzzy ideal with thresholds (α, β) of S.

Proposition 4.1. *If A is an intuitionistic fuzzy ideal with thresholds (α, β) of S, then so are $\Box A$, $\Diamond A$, $P_{r,t}(A)$ and $Q_{r,t}(A)$, where $r, t \in [0,1]$ and $r + t \leq 1$.*

Proof. It is straightforward. ∎

Proposition 4.2. *If A is an intuitionistic fuzzy ideal with thresholds (α, β) of S, then*

(1) *A^n is an intuitionistic fuzzy ideal with thresholds (α^n, β^n) of S,*
(2) *nA is an intuitionistic fuzzy ideal with thresholds $(1-(1-\alpha)^n, 1-(1-\beta)^n)$ of S.*

Proof. It is straightforward. ∎

Proposition 4.3. *If A is an intuitionistic fuzzy ideal with thresholds $(0,1)$ of S, then so are $F_{r,t}(A)$, $H_{r,t}(A)$ and $J_{r,t}(A)$, where $r, t \in [0,1]$ and $r + t \leq 1$.*

Proof. We only show that $F_{r,t}(A)$ is an intuitionistic fuzzy ideal with thresholds (α,β) of S. The cases for $H_{r,t}(A)$ and $J_{r,t}(A)$ can be similarly proved. Let $x,y\in S$. Since A is an intuitionistic fuzzy ideal with thresholds $(0,1)$ of S, we have

$$\mu_A(xy)\geq \mu_A(x)\vee\mu_A(y) \quad\text{and}\quad \lambda_A(xy)\leq\lambda_A(x)\wedge\lambda_A(y),$$

and so

$$\mu^c_A(xy)\leq \mu^c_A(x)\wedge\mu^c_A(y) \quad\text{and}\quad \lambda^c_A(xy)\geq\lambda^c_A(x)\vee\lambda^c_A(y).$$

∎

Hence

$$\begin{aligned}\mu_A(xy)+r\cdot\pi_A(xy)&=(1-r)\cdot\mu_A(xy)+r\cdot\lambda^c_A(xy)\\&\geq(1-r)\cdot(\mu_A(x)\vee\mu_A(y))+r\cdot(\lambda^c_A(x)\vee\lambda^c_A(y))\\&\geq((1-r)\cdot\mu_A(x)+r\cdot\lambda^c_A(x))\vee((1-r)\cdot\mu_A(y)+r\cdot\lambda^c_A(y))\\&=(\mu_A(x)+r\cdot\pi_A(x))\vee(\mu_A(y)+r\cdot\pi_A(y)).\end{aligned}$$

Similarly, $\lambda_A(xy)+t\cdot\pi_A(xy)\leq(\lambda_A(x)+t\cdot\pi_A(x))\wedge(\lambda_A(y)+t\cdot\pi_A(y))$.

Now, let $x\leq y$. Then

$$\mu_A(x)\geq\mu_A(y)\quad\text{and}\quad\lambda_A(x)\leq\lambda_A(y),$$

and so

$$\mu^c_A(x)\leq\mu^c_A(y)\quad\text{and}\quad\lambda^c_A(x)\geq\lambda^c_A(y).$$

Hence

$$\begin{aligned}\mu_A(x)+r\cdot\pi_A(x)&=(1-r)\cdot\mu_A(x)+r\cdot\lambda^c_A(x)\\&\geq(1-r)\cdot\mu_A(y)+r\cdot\lambda^c_A(y)\\&=\mu_A(y)+r\cdot\pi_A(y).\end{aligned}$$

Similarly, $\lambda_A(x)+t\cdot\pi_A(x)\leq\lambda_A(y)+t\cdot\pi_A(y)$. This completes the proof.

Note that Proposition 4.3 does not hold in general for the thresholds (α,β) as shown in the following example.

Example 4.1. Let S be as in Example 3.1. Put $\alpha=0.5,\beta=0.6,r=0.5$ and $t=0.5$. Define an intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in S as follows:

$$\mu_A(a)=0.4,\ \mu_A(b)=0.3,\ \mu_A(c)=0.3;\ \lambda_A(a)=0.4,\ \lambda_A(b)=0.2,\ \lambda_A(c)=0.4.$$

Then, it is obvious that A is an intuitionistic fuzzy ideal with thresholds $(0.5,0.6)$ of S. Also, we have

$$(\mu_A(ab)+0.5\cdot\pi_A(ab))\vee0.5=(\mu_A(c)+0.5\cdot\pi_A(c))\vee0.5=0.45\vee0.5=0.5,$$

$$((\mu_A(a)+0.5\cdot\pi_A(a))\vee(\mu_A(b)+0.5\cdot\pi_A(b)))\wedge0.6=(0.5\vee0.55)\wedge0.6=0.55,$$

$$(\lambda_A(ab)+0.5\cdot\pi_A(ab))\wedge(1-0.5)=(\lambda_A(c)+0.5\cdot\pi_A(c))\wedge0.5=0.55\wedge0.5=0.5$$

and

$$((\lambda_A(a)+0.5\cdot\pi_A(a))\wedge(\lambda_A(b)+0.5\cdot\pi_A(b)))\vee(1-0.6)=(0.5\wedge0.45)\vee0.4=0.45,$$

these give that

$$(\mu_A(ab)+0.5\cdot\pi_A(ab))\vee0.4<((\mu_A(a)+0.5\cdot\pi_A(a))\vee(\mu_A(b)+0.5\cdot\pi_A(b)))\wedge0.6$$

and

$$(\lambda_A(ab)+0.5\cdot\pi_A(ab))\wedge(1-0.5) > ((\lambda_A(a)+0.5\cdot\pi_A(a))\wedge(\lambda_A(b)+0.5\cdot\pi_A(b)))\vee(1-0.6).$$

Therefore, $F_{0.5,0.5}(A), H_{0.5,0.5}(A)$ and $J_{0.5,0.5}(A)$ are not intuitionistic fuzzy ideals with thresholds $(0.5, 0.6)$ of S.

From Proposition 4.1, it is easy to verify that the following result is valid.

Theorem 4.3. *An intuitionistic fuzzy set A in S is an intuitionistic fuzzy ideal with thresholds (α, β) of S if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy ideals with thresholds (α, β) of S.*

Corollary 4.1. *An intuitionistic fuzzy set A in S is an intuitionistic fuzzy ideal with thresholds (α, β) of S if and only if μ_A and λ_A^c are fuzzy ideals with thresholds (α, β) of S.*

Corollary 4.2. *$A = (\kappa_S, \kappa_S^c)$ is an intuitionistic fuzzy ideal with thresholds (α, β) of S.*

Combining Theorem 3.3 and Corollary 4.1, we have the following result.

Theorem 4.4. *A non-empty set P in S is an ideal of S if and only if $A = (\kappa_P, \kappa_P^c)$ is an intuitionistic fuzzy ideal with thresholds (α, β) of S.*

In the sequel, we consider the structural characteristics of the set of all intuitionistic fuzzy ideals with thresholds (α, β) of S, denoted by $\mathcal{IFI}(S)$.

Proposition 4.4. *Let $A, B \in \mathcal{IFI}(S)$. Then so is $A \circ B$.*

Proof. By proposition 2.1, we know that $A \circ B$ is an intuitionistic fuzzy set in S. Now, let $x, y \in S$. If $A_{xy} = \emptyset$, then so are A_x and A_y. In fact, if $A_x \neq \emptyset$, then there exist $a, b \in S$ such that $x \leq ab$, and so $xy \leq aby$. It follows that $A_{xy} \neq \emptyset$, a contradiction. Similarly, $A_y = \emptyset$. In this case, we have

$$\alpha = (\mu_A \check{\circ} \mu_B)(xy) \vee \alpha \geq ((\mu_A \check{\circ} \mu_B)(x) \vee (\mu_A \check{\circ} \mu_B)(y)) \wedge \beta = 0$$

and

$$1-\alpha = (\lambda_A \hat{\circ} \lambda_B)(xy) \wedge (1-\alpha) \leq ((\lambda_A \hat{\circ} \lambda_B)(x) \wedge (\lambda_A \hat{\circ} \lambda_B)(y)) \vee (1-\beta) = 1.$$

Otherwise, we have

$$\begin{aligned}(\mu_A \check{\circ} \mu_B)(x) \wedge \beta &= \left(\bigvee_{(a,b)\in A_x} \mu_A(a) \wedge \mu_B(b)\right) \wedge \beta = \bigvee_{(a,b)\in A_x} \mu_A(a) \wedge (\mu_B(b) \wedge \beta) \\ &\leq \bigvee_{(a,b)\in A_x} \mu_A(a) \wedge (\mu_B(by) \vee \alpha) \leq \bigvee_{(a,b)\in A_x} (\mu_A(a) \wedge \mu_B(by)) \vee \alpha \\ &= \left(\bigvee_{(a,b)\in A_x} \mu_A(a) \wedge \mu_B(by)\right) \vee \alpha \leq (\mu_A \check{\circ} \mu_B)(xy) \vee \alpha.\end{aligned}$$

In a similar way, we may prove that $(\mu_A \check{\circ} \mu_B)(y) \wedge \beta \leq (\mu_A \check{\circ} \mu_B)(xy) \vee \alpha$, and so

$$(\mu_A \check{\circ} \mu_B)(xy) \vee \alpha \geq ((\mu_A \check{\circ} \mu_B)(x) \vee (\mu_A \check{\circ} \mu_B)(y)) \wedge \beta.$$

On the other hand,

$$\begin{aligned}(\lambda_A \hat{\circ} \lambda_B)(x) \vee (1-\beta) &= \left(\bigwedge_{(a,b)\in A_x} \lambda_A(a) \vee \lambda_B(b) \right) \vee (1-\beta) \\ &= \bigwedge_{(a,b)\in A_x} \lambda_A(a) \vee (\lambda_B(b) \vee (1-\beta)) \\ &\geq \bigwedge_{(a,b)\in A_x} \lambda_A(a) \vee (\lambda_B(by) \wedge (1-\alpha)) \\ &\geq \bigwedge_{(a,b)\in A_x} (\lambda_A(a) \vee \lambda_B(by)) \wedge (1-\alpha) \\ &= \left(\bigwedge_{(a,b)\in A_x} \lambda_A(a) \vee \lambda_B(by) \right) \wedge (1-\alpha) \\ &\geq (\lambda_A \hat{\circ} \lambda_B)(xy) \wedge (1-\alpha).\end{aligned}$$

In a similar way, we may prove that $(\lambda_A \hat{\circ} \lambda_B)(y) \vee (1-\beta) \geq (\lambda_A \hat{\circ} \lambda_B)(xy) \wedge (1-\alpha)$, and so

$$(\lambda_A \hat{\circ} \lambda_B)(xy) \wedge (1-\alpha) \leq ((\lambda_A \hat{\circ} \lambda_B)(x) \wedge (\lambda_A \hat{\circ} \lambda_B)(y)) \vee (1-\beta).$$

Now, let $x \leq y$. Then $A_y \subseteq A_x$. If $A_x = \emptyset$, then so is A_y. Hence

$$\alpha = (\mu_A \check{\circ} \mu_B)(x) \vee \alpha \geq (\mu_A \check{\circ} \mu_B)(y) \wedge \beta = 0$$

and

$$1-\alpha = (\lambda_A \hat{\circ} \lambda_B)(x) \wedge (1-\alpha) \leq (\lambda_A \hat{\circ} \lambda_B)(y) \vee (1-\beta) = 1.$$

Otherwise, we have

$$\begin{aligned}(\mu_A \check{\circ} \mu_B)(y) \wedge \beta \leq (\mu_A \check{\circ} \mu_B)(y) \vee \alpha &= \left(\bigvee_{(a,b)\in A_y} \mu_A(a) \wedge \mu_B(b) \right) \vee \alpha \\ &\leq \left(\bigvee_{(a,b)\in A_x} \mu_A(a) \wedge \mu_B(b) \right) \vee \alpha = (\mu_A \check{\circ} \mu_B)(x) \vee \alpha\end{aligned}$$

and

$$\begin{aligned}(\lambda_A \hat{\circ} \lambda_B)(y) \vee (1-\beta) \geq (\lambda_A \hat{\circ} \lambda_B)(y) \wedge (1-\alpha) &= \left(\bigwedge_{(a,b)\in A_y} \lambda_A(a) \vee \lambda_B(b) \right) \wedge (1-\alpha) \\ &\geq \left(\bigwedge_{(a,b)\in A_x} \lambda_A(a) \vee \lambda_B(b) \right) \wedge (1-\alpha) = (\lambda_A \hat{\circ} \lambda_B)(x) \wedge (1-\alpha).\end{aligned}$$

This proves that $A \circ B$ is an intuitionistic fuzzy ideal with thresholds (α, β) of S. ∎

Note that Proposition 4.4 gives that $(\mathcal{IFI}(S), \circ)$ is a semigroup.

Proposition 4.5. *Let $\{A_i = (\mu_{A_i}, \lambda_{A_i}) | i \in I\}$ be a family intuitionistic fuzzy ideals with thresholds (α, β) of S, then both $\bigcap_{i\in I} A_i = \left(\bigcap_{i\in I} \mu_{A_i}, \bigcup_{i\in I} \lambda_{A_i}\right)$ and $\bigcup_{i\in I} A_i = \left(\bigcup_{i\in I} \mu_{A_i}, \bigcap_{i\in I} \lambda_{A_i}\right)$ are intuitionistic fuzzy ideals with thresholds (α, β) of S.*

Proof. It is straightforward. ∎

From Proposition 4.5, it is easy to verify that the following result holds.

Theorem 4.5. *For an ordered semigroup S, $\mathcal{IFI}(S)$ equipped with intuitionistic fuzzy set inclusion relation " $\subseteq$ " constitutes a complete bounded lattice. And for any $A, B \in \mathcal{IFI}(S)$, $A \cap B$ and $A \cup B$ are the greatest lower bound and least upper bound of $\{A, B\}$, respectively. Its maximal element and minimal element are (κ_S, κ_S^c) and (κ_S^c, κ_S), respectively. Moreover, it is closed under intuitionistic fuzzy set intersection and union.*

Theorem 4.6. *Given any chain of ideals $S_0 \subset S_1 \subset \cdots \subset S_n = S$ of S, there exists an intuitionistic fuzzy ideal A with thresholds (α, β) of S whose non-empty strong level cut are precisely the members of the chain with $A^{(\beta, 1-\beta)} = S_0$.*

Proof. Let $\{r_i | r_i \in (\alpha, \beta), i = 1, 2, \cdots, n\}$ and $\{t_i | t_i \in (1-\beta, 1-\alpha), i = 1, 2, \cdots, n\}$ be such that $r_1 > r_2 > \cdots > r_n$, $t_1 < t_2 < \cdots < t_n$ and $r_i + t_i \leq 1$ for all $i = 1, 2, \cdots, n$. Let μ_A and λ_A be fuzzy sets in S such that

$$\mu_A(x) = \begin{cases} r_0 > \beta & if\ x \in S_0, \\ r_1 & if\ x \in S_1 - S_0, \\ \cdots & \\ r_n & if\ x \in S_n - S_{n-1}, \end{cases} \qquad \lambda_A(x) = \begin{cases} t_0 < 1-\beta & if\ x \in S_0, \\ t_1 & if\ x \in S_1 - S_0, \\ \cdots & \\ t_n & if\ x \in S_n - S_{n-1}, \end{cases}$$

for all $x \in S$. Then it is easy to see that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in S and

$$U(\mu_A; r) = \begin{cases} S_0 & if\ r \in [r_1, r_0), \\ S_1 & if\ r \in [r_2, r_1), \\ \cdots & \\ S_n & if\ r \in [0, r_n), \end{cases} \qquad L(\lambda_A; t) = \begin{cases} S_0 & if\ t \in (t_0, t_1], \\ S_1 & if\ t \in (t_1, t_2], \\ \cdots & \\ S_n & if\ t \in (t_n, 1]. \end{cases}$$

Hence, for any $r \in [0, r_0)$ and $t \in (t_0, 1]$, there exist $i, j \in \{1, 2, \cdots, n\}$ such that $r \in [r_i, r_{i-1})$ and $t \in (t_j, t_{j+1}]$. Without loss of generality, we may assume that $i \leq j$, then $U(\mu_A; r) = S_{i-1}, L(\lambda_A; t) = S_j$ and $A^{r,t} = U(\mu_A; r) \cap L(\lambda_A; t) = S_{i-1} \cap S_j = S_{i-1}$. Thus, our assumption and Theorem 4.2 give that A is an intuitionistic fuzzy ideal with thresholds (α, β) of S whose non-empty strong level cut are precisely the members of the chain. Clearly, $A^{(\beta, 1-\beta)} = S_0$. ∎

5. Embedding of $(\mathcal{IFI}(S), \circ)$

In this section, we investigate the embedding problem of $(\mathcal{IFI}(S), \circ)$. Denote by $\mathcal{IC}(S)$ the set $\{(f,g)|f : X \to \{0,1\}, g : X \to \{0,1\}\}$ where $(f,g)(x,y) = (f(x), g(y))$ for all $(f,g) \in \mathcal{IC}(S)$ and $x, y \in S$, and $\mathcal{IC}(S)^J$ the set $\{(f,g)|(f,g) : J \times J \to \mathcal{IC}(S)\}$ where $J = [0,1)$ and $(f,g)(r,t) = (f(r), g(t))$ for all $(f,g) \in \mathcal{IC}(S)^J$ and $r, t \in J$.

Definition 5.1. *Define the mapping $\mathcal{R} : \mathcal{IFI}(S) \to \mathcal{IC}(S)^J$ by*

$$\mathcal{R}(A) = (\mathcal{R}_{\mu_A}, \mathcal{R}_{\lambda_A}) \quad \forall A \in \mathcal{IFI}(S),$$

where

$$\mathcal{R}_{\mu_A}(r)(x) = \begin{cases} 0 & \text{if} \quad \mu_A(x) \le r, \\ 1 & \text{otherwise.} \end{cases}$$

and

$$\mathcal{R}_{\lambda_A}(t)(x) = \begin{cases} 0 & \text{if } \lambda_A(x) \ge t, \\ 1 & \text{otherwise.} \end{cases}$$

for all $r, t \in J$ and $x \in S$.

Lemma 5.1. *The mapping $\mathcal{R}$ is injective.*

Proof. Assume that $\mathcal{R}(A) = \mathcal{R}(B)$. Let $x \in S$. Since

$$\mathcal{R}(A)(\mu_B(x), \lambda_B(x))(x,x) = \mathcal{R}(B)(\mu_B(x), \lambda_B(x))(x,x) = (0,0),$$

the definition of $\mathcal{R}$ gives that $\mu_A(x) \le \mu_B(x)$ and $\lambda_A(x) \ge \lambda_B(x)$. Similarly, we have $\mu_B(x) \le \mu_A(x)$ and $\lambda_B(x) \ge \lambda_A(x)$. Hence $A = (\mu_A, \lambda_A) = (\mu_B, \lambda_B) = B$. ∎

Lemma 5.2. *Let $G = \{(f,g) \in \mathcal{IC}(S)^J | \forall r, t \in J, f(r) = \cup\{f(p)|p > r\}$ and $g(t) = \cap\{g(q)|q < t\}\}$ satisfy:*

(1) $0 \le \wedge\{r|f(r)(x) = 0\} + \vee\{t|g(t)(x) = 0\} \le 1$ *for all $x \in S$,*
(2) *both $\{x|f(r)(x) = 1\}$ and $\{y|g(t)(y) = 1\}$ are ideals of S for all $r \in [\alpha, \beta)$ and $t \in (1-\beta, 1-\alpha]$.*

Then $\mathrm{Im}(\mathcal{R}) = G$.

Proof. Suppose that $(f,g) = \mathcal{R}(A)$ for some $A \in \mathcal{IFI}(S)$. Then we obtain:

(a) $f(r) = \cup\{f(p)|p > r\}$ and $g(t) = \cap\{g(q)|q < t\}$. The verification is as follows. Let $r, t \in J$ and $x, y \in S$. Then $(f,g)(r,t)(x,y) = \mathcal{R}(A)(r,t)(x,y) = (0,0)$ if and only if $\mu_A(x) \le r$ and $\lambda_A(y) \ge t$. However, $\mu_A(x) \le r$ and $\lambda_A(y) \ge t$ if and only if $\mu_A(x) \le p$ and $\lambda_A(y) \ge q$ for all $p > r$ and $q < t$, respectively. Therefore, $(f,g)(r,t)(x,y) = (0,0)$ if and only if $(f,g)(p,q)(x,y) = (0,0)$ for all $p > r$ and $q < t$. Thus $f(r)(x) = \vee\{f(p)(x)|p > r\}$ and $g(s)(y) = \wedge\{g(q)(y)|q < t\}$. Since these equations hold for all $x, y \in S$, we have $f(r) = \cup\{f(p)|p > r\}$ and $g(t) = \cap\{g(q)|q < t\}$.

(b) $0 \le \wedge\{r|f(r)(x) = 0\} + \vee\{t|g(t)(x) = 0\} \le 1$ for all $x \in S$. The verification is as follows. Let $r, t \in J$. Then $(f,g)(r,t)(x,x) = (0,0)$ if and only if $\mu_A(x) \le r$ and $\lambda_A(x) \ge t$, and so $\mu_A(x) = \wedge\{r|f(r)(x) = 0\}$ and $\lambda_A(x) = \vee\{t|g(t)(x) = 0\}$ for all $x \in S$. Hence $0 \le \wedge\{r|f(r)(x) = 0\} + \vee\{t|g(t)(x) = 0\} \le 1$ for all $x \in S$.

(c) Both $\{x|f(r)(x) = 1\}$ and $\{y|g(t)(y) = 1\}$ are ideals of S for all $r \in [\alpha, \beta)$ and $t \in (1-\beta, 1-\alpha]$. The verification is as follows. Let $r, t \in J$ and $x, y \in S$. Then $(f,g)(r,t)(x,y) = (1,1)$ if and only if $\mu_A(x) > r$ and $\lambda_A(y) < t$, and so $U(\mu_A; r) = \{x|f(r)(x) = 1\}$ and $L(\lambda_A; t) = \{y|g(t)(y) = 1\}$. Hence, by Theorem 4.2, both $\{x|f(r)(x) = 1\}$ and $\{y|g(t)(y) = 1\}$ are ideals of S for all $r \in [\alpha, \beta)$ and $t \in (1-\beta, 1-\alpha]$.

Summing up the above arguments, $\mathcal{R}(A) = (f, g) \in G$.

Conversely, for any $(f, g) \in G$, define fuzzy sets μ_A and λ_A in S as follows:

$$\mu_A(x) = \wedge\{r \in J|f(r)(x) = 0\} \text{ and } \lambda_A(x) = \vee\{t \in J|g(t)(x) = 0\}$$

for all $x \in S$. Then we have:

(a') By the definitions of μ_A and λ_A, it is clear that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in S$.

(b') $U(\mu_A; p) = \{x|f(p)(x) = 1\}$ and $L(\lambda_A; q) = \{x|g(q)(x) = 1\}$ for all $p, q \in J$. The verification is as follows. Let $y \in U(\mu_A; p)$. Then $\mu_A(y) = \wedge\{r \in J|f(r)(y) = 0\} > p$, this implies that $f(p)(y) \neq 0$ and so $f(p)(y) = 1$. Hence $y \in \{x|f(p)(x) = 1\}$, that is, $U(\mu_A; p) \subseteq \{x|f(p)(x) = 1\}$. Conversely, let $y \in \{x|f(p)(x) = 1\}$. Then for any $r \leq p$, $f(r)(y) = (\cup\{f(s)|s > r\})(y) = \vee\{f(s)(y)|s > r\} = 1$, hence $f(r)(y) = 0$ implies that $r > p$. Thus $\mu_A(y) = \wedge\{r \in J|f(r)(y) = 0\} \geq p$. In addition, if $\mu_A(y) = \wedge\{r \in J|f(r)(y) = 0\} = p$, then $\wedge\{r \in J|f(r)(y) = 0\} < p + \varepsilon$ for any $\varepsilon > 0$ and so $s < p + \varepsilon$ for some $s \in J$ with $f(s)(y) = 0$. Thus $0 = f(s)(y) \geq f(p+\varepsilon)(x)$, that is, $f(p+\varepsilon)(x) = 0$, it follows that $f(p)(y) = 0$ since ε is arbitrary, a contradiction. Hence $\mu_A(y) = \wedge\{r \in J|f(r)(y) = 0\} > p$ and so $y \in U(\mu_A; p)$, that is, $\{x|f(p)(x) = 1\} \subseteq U(\mu_A; p)$. Therefore, $U(\mu_A; p) = \{x|f(p)(x) = 1\}$ for all $p \in J$. Similarly, we may prove that $L(\lambda_A; q) = \{x|g(q)(x) = 1\}$ for all $q \in J$.

Therefore, by the assumption and Theorem 4.2, $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy ideal of S, that is, $A \in \mathcal{IFI}(S)$.

(c') $\mathcal{R}(A) = (f, g)$. The verification is as follows. Since for any $r, s \in J$ and $x, y \in S$, the following statements are equivalent:

(1) $\mathcal{R}_{\mu_A}(r)(x) = 0$ (resp. $\mathcal{R}_{\lambda_A}(t)(y) = 0$).
(2) $\mu_A(x) \leq r$ (resp. $\lambda_A(y) \geq t$).
(3) $\forall\, p > r$ (resp. $q < t$), $\mu_A(x) \leq p$ (resp. $\lambda_A(y) \geq q$).
(4) $\forall\, p > r$ (resp. $q < t$), $f(p)(x) = 0$ (resp. $g(q)(y) = 0$).
(5) $f(r)(x) = (\cup\{f(p)|p > r\})(x) = 0$ (resp. $g(t)(y) = (\cap\{g(q)|q < t\})(y) = 0$. ∎

Define a binary operation $\star$ on $\mathcal{IC}(S)^J$ as follows: $\forall (f_1, g_1), (f_2, g_2) \in \mathcal{IC}(S)^J$ and $\forall r, t \in J$

$$((f_1, g_1) \star (f_2, g_2))(r, t) = (f_1(r) \check{\circ} f_2(r), g_1(t) \hat{\circ} g_2(t)).$$

Lemma 5.3. *For any $A, B \in \mathcal{IFI}(S)$, $\mathcal{R}(A \circ B) = \mathcal{R}(A) \star \mathcal{R}(B)$.*

Proof. The desired result follows from the fact that for any $A, B \in \mathcal{IFI}(S)$, $r, t \in J$ and $x, y \in S$, the following statements are equivalent:

(1) $\mathcal{R}_{\mu_A \check{\circ} \mu_B}(r)(x) = 1$ (resp. $\mathcal{R}_{\lambda_A \hat{\circ} \lambda_B}(t)(y) = 1$).

(2) $(\mu_A \check{\circ} \mu_B)(x) > r$ (resp. $(\lambda_A \hat{\circ} \lambda_B)(y) < t$).
(3) $\vee\{\mu_A(x_1) \wedge \mu_B(x_2) | (x_1, x_2) \in A_x\} > r$ (resp. $\wedge\{\lambda_A(y_1) \vee \lambda_B(y_2) | (y_1, y_2) \in A_y\} < t$).
(4) There exists $(x_1, x_2) \in A_x$ (resp. $(y_1, y_2) \in A_y$) such that $\mu_A(x_1) > r$ and $\mu_B(x_2) > r$ (resp. $\lambda_A(y_1) < t$ and $\lambda_B(y_2) < t$).
(5) There exists $(x_1, x_2) \in A_x$ (resp. $(y_1, y_2) \in A_y$) such that $\mathcal{R}_{\mu_A}(r)(x_1) = 1$ and $\mathcal{R}_{\mu_B}(r)(x_2) = 1$ (resp. $\mathcal{R}_{\lambda_A}(t)(y_1) = 1$ and $\mathcal{R}_{\lambda_B}(t)(y_2) = 1$).
(6) $\vee\{\mathcal{R}_{\mu_A}(r)(x_1) \wedge \mathcal{R}_{\mu_B}(r)(x_2) | (x_1, x_2) \in A_x\} = 1$ (resp. $\wedge\{\mathcal{R}_{\lambda_A}(t)(y_1) \vee \mathcal{R}_{\lambda_B}(t)(y_2) | (y_1, y_2) \in A_y\} = 1$).
(7) $(R_{\mu_A}(r) \check{\circ} R_{\mu_B}(r))(x) = 1$ (resp. $(\mathcal{R}_{\lambda_A}(t) \hat{\circ} \mathcal{R}_{\lambda_B}(t))(y) = 1$). ∎

Theorem 5.1. *$(\mathcal{IFI}(S), \circ)$ is embedded in $(\mathcal{IC}(S)^J, \star)$.*

Proof. From Lemmas 5.1 and 5.3, $\mathcal{R}$ is an injective homomorphism of $(\mathcal{IFI}(S), \circ)$ into $(\mathcal{IC}(S)^J, \star)$ that establishes an isomorphism of $\mathcal{IFI}(S)$ with its image $\mathrm{Im}(\mathcal{R})$. ∎

6. Homomorphism

Let $(S, \cdot, \leq)$ and $(T, *, \preceq)$ be ordered semigroups and f a mapping from S into T. f is called isotone if $x, y \in S, x \leq y$ implies $f(x) \preceq f(y)$. f is said to be inverse *isotone* if $x, y \in S, f(x) \preceq f(y)$ implies $x \leq y$ [each inverse isotone mapping is (1-1)]. f is called a *homomorphism* if it is isotone and satisfies $f(xy) = f(x) * f(y)$ for all $x, y \in S$. f is said to be *isomorphism* if it is onto, homomorphism and inverse isotone.

Proposition 6.1. *Let $(S, \cdot, \leq)$ and $(T, *, \preceq)$ be ordered semigroups and f a mapping from S into T, and let A and B be intuitionistic fuzzy sets in S and T, respectively. Then the image $f(A) = (\check{f}(\mu_A), \hat{f}(\lambda_A))$ of A is an intuitionistic fuzzy set in T defined by*

$$\check{f}(\mu_A) : T \to [0,1] | x' \to \begin{cases} \bigvee\limits_{x \in f^{-1}(x')} \mu_A(x) & \text{if } f^{-1}(x') \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\hat{f}(\lambda_A) : T \to [0,1] | x' \to \begin{cases} \bigwedge\limits_{x \in f^{-1}(x')} \lambda_A(x) & \text{if } f^{-1}(x') \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

And the inverse image $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\lambda_B))$ of B is an intuitionistic fuzzy set in S defined by

$$f^{-1}(\mu_B) : S \to [0,1] | x \to \mu_B(f(x)) \quad \textit{and} \quad f^{-1}(\lambda_B) : S \to [0,1] | x \to \lambda_B(f(x)).$$

Proof. It is straightforward. ∎

Theorem 6.1. *Let $(S, \cdot, \leq)$ and $(T, *, \preceq)$ be ordered semigroups and f a homomorphism from S onto T, and let A and B be intuitionistic fuzzy ideals with thresholds (α, β) of S and T, respectively. Then*

(1) *$f(A)$ is an intuitionistic fuzzy ideal with thresholds (α, β) of T, provided f is inverse isotone.*
(2) *$f^{-1}(B)$ is an intuitionistic fuzzy ideal with thresholds (α, β) of S.*

(3) *The mapping $A \to f(A)$ defines a one-to-one correspondence between the set of all intuitionistic fuzzy ideals with thresholds (α, β) of S and the set of all intuitionistic fuzzy ideals with thresholds (α, β) of T, provided f is inverse isotone.*

Proof. We only show (1). (2) can be similarly proved and (3) is the direct consequence of (1) and (2). For any $x', y' \in T$, since f is a homomorphism from S onto T, we have

$$\begin{aligned}
\check{f}(\mu_A)(x' * y') \vee \alpha &= \left(\bigvee_{z \in f^{-1}(x' * y')} \mu_A(z) \right) \vee \alpha \\
&\geq \bigvee_{x \in f^{-1}(x'), y \in f^{-1}(y')} \mu_A(xy) \vee \alpha \\
&\geq \bigvee_{x \in f^{-1}(x'), y \in f^{-1}(y')} (\mu_A(x) \vee \mu_A(y)) \wedge \beta \\
&= \left(\bigvee_{x \in f^{-1}(x')} \mu_A(x) \vee \bigvee_{y \in f^{-1}(y')} \mu_A(y) \right) \wedge \beta \\
&= (\check{f}(\mu_A)(x') \vee \check{f}(\mu_A)(y')) \wedge \beta,
\end{aligned}$$

$$\begin{aligned}
\hat{f}(\lambda_A)(x' * y') \wedge (1-\alpha) &= \left(\bigwedge_{z \in f^{-1}(x' * y')} \lambda_A(z) \right) \wedge (1-\alpha) \\
&\leq \bigwedge_{x \in f^{-1}(x'), y \in f^{-1}(y')} \lambda_A(xy) \wedge (1-\alpha) \\
&\leq \bigwedge_{x \in f^{-1}(x'), y \in f^{-1}(y')} (\lambda_A(x) \wedge \lambda_A(y)) \vee (1-\beta) \\
&= \left(\bigwedge_{x \in f^{-1}(x')} \lambda_A(x) \wedge \bigwedge_{y \in f^{-1}(y')} \lambda_A(y) \right) \vee (1-\beta) \\
&= (\hat{f}(\lambda_A)(x') \wedge \hat{f}(\lambda_A)(y')) \vee (1-\beta).
\end{aligned}$$

Let $x' \preceq y'$. Since f is reverse isotone, there exist unique $x, y \in S$ such that $f(x) = x'$, $f(y) = y'$ and $x \leq y$. Thus, we have

$$\begin{aligned}
\check{f}(\mu_A)(x') \vee \alpha &= \left(\bigvee_{z \in f^{-1}(x')} \mu_A(z) \right) \vee \alpha = \mu_A(x) \vee \alpha \\
&\geq \mu_A(y) \wedge \beta = \left(\bigvee_{z \in f^{-1}(y')} \mu_A(z) \right) \wedge \beta \\
&= \check{f}(\mu_A)(y') \wedge \beta
\end{aligned}$$

and

$$\begin{aligned}\hat{f}(\lambda_A)(x') \wedge (1-\alpha) &= \left(\bigwedge_{z\in f^{-1}(x')} \lambda_A(z)\right) \wedge (1-\alpha)\\ &= \lambda_A(x) \wedge (1-\alpha) \leq \lambda_A(y) \vee (1-\beta)\\ &= \left(\bigwedge_{z\in f^{-1}(y')} \lambda_A(z)\right) \vee (1-\beta) = \hat{f}(\lambda_A)(y') \vee (1-\beta).\end{aligned}$$

Therefore, $f(A)$ is an intuitionistic fuzzy ideal with thresholds (α, β) of T. ∎

Note that Theorem 6.1(1) may not be true if f is not inverse isotone as shown in the following example.

Example 6.1. Let $S = \{x, y, z\}$ and $T = \{a, b\}$ be ordered semigroups with the following Cayley tables respectively:

$\cdot$	x	y	z
x	z	z	z
y	x	x	z
z	z	z	z

$$\text{“}\leq\text{”} = \{(x,x),(y,y),(z,z),(x,y)\}$$

and

$*$	a	b
a	a	a
b	a	a

$$\text{“}\preceq\text{”} = \{(a,a),(b,b),(a,b)\}.$$

Let f be the mapping from S into T such that $x \to a$, $y \to a$, $z \to b$. Routine verification gives that f is a homomorphism from S onto T. However, f is not inverse isotone, since $f(x) = a \preceq b = f(z)$, but $x \not\leq z$. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in S as follows:

$$\mu_A(x) = 0.4,\ \mu_A(y) = 0.4,\ \mu_A(z) = 0.5;\ \lambda_A(x) = 0.6,\ \lambda_A(y) = 0.6,\ \lambda_A(z) = 0.5.$$

Set $\alpha = 0.4$ and $\beta = 0.6$, then A is an intuitionistic fuzzy ideal with thresholds $(0.4, 0.6)$ of S. But $f(A)$ is not an intuitionistic fuzzy ideal with thresholds $(0.4, 0.6)$ of T, since $a \preceq b$, while $\check{f}(\mu_A)(a) \vee 0.4 = \mu_A(x) \vee \mu_A(y) \vee 0.4 = 0.4 < 0.5 = \mu_A(z) \wedge 0.6 = \check{f}(\mu_A)(b) \wedge 0.6$ and $\hat{f}(\lambda_A)(a) \wedge (1-0.4) = \lambda_A(x) \wedge \lambda_A(y) \wedge (1-0.4) = 0.6 > 0.5 = \lambda_A(z) \vee 0.4 = \hat{f}(\lambda_A)(b) \vee (1-0.6)$.

7. Conclusions

Fuzzifications of algebraic structures play an important role in mathematics with wide range of applications in many disciplines such as computer sciences, engineering and medical diagnosis. Thus we have introduced the concept of intuitionistic fuzzy ideals with thresholds (α, β) in this paper. The obtained results probably can be applied in various fields such as computer sciences, control engineering, coding theory, theoretical physics. In our future research, we will consider the characterization of regular ordered semigroups in terms of intuitionistic fuzzy ideals with thresholds (α, β).

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References

[1] M. Akram and W. A. Dudek, Intuitionistic fuzzy left k-ideals of semirings, *Soft Computing* **12** (2008), 881–890.

[2] M. Akram and K. P. Shum, Bifuzzy ideals of nearrings, *Algebras Groups Geom.* **24** (2007), no. 4, 389–405.

[3] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), no. 1, 87–96.

[4] K. T. Atanassov, Remarks on the intuitionistic fuzzy sets. III, *Fuzzy Sets and Systems* **75** (1995), no. 3, 401–402.

[5] S. K. Bhakat, $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups, *Fuzzy Sets and Systems* **112** (2000), no. 2, 299–312.

[6] S. K. Bhakat and P. Das, On the definition of a fuzzy subgroup, *Fuzzy Sets and Systems* **51** (1992), no. 2, 235–241.

[7] R. Biswas, Intuitionistic fuzzy subgroups. *Math. Forum* **10** (1989), 37–46.

[8] B. Davvaz, W. A. Dudek and Y. B. Jun, Intuitionistic fuzzy H_v-submodules, *Inform. Sci.* **176** (2006), no. 3, 285–300.

[9] S. K. De, R. Biswas and A. R. Roy, Some operations on intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **114** (2000), no. 3, 477–484.

[10] W. A. Dudek, Special types of intuitionistic fuzzy left h-ideals of hemirings. *Soft Computing* **12** (2008), 359–364.

[11] W. A. Dudek, B. Davvaz and Y. B. Jun, On intuitionistic fuzzy sub-hyperquasigroups of hyperquasigroups, *Inform. Sci.* **170** (2005), no. 2–4, 251–262.

[12] Y. B. Jun, Intuitionistic fuzzy bi-ideals of ordered semigroups, *Kyungpook Math. J.* **45** (2005), no. 4, 527–537.

[13] Y. B. Jun, A. Khan and M. Shabir, Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 391–408.

[14] N. Kehayopulu, On weakly prime ideals of ordered semigroups, *Math. Japon.* **35** (1990), no. 6, 1051–1056.

[15] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered groupoids, *Semigroup Forum* **65** (2002), no. 1, 128–132.

[16] K. H. Kim and Y. B. Jun, Intuitionistic fuzzy ideals of semigroups, *Indian J. Pure Appl. Math.* **33** (2002), no. 4, 443–449.

[17] P. M. Pu and Y. M. Liu, Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.* **76** (1980), no. 2, 571–599

[18] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* **35** (1971), 512–517.

[19] K. P. Shum and M. Akram, Intuitionistic (T, S)-fuzzy ideals of nearrings, *J. Algebra Discrete Struct.* **6** (2008), no. 1, 37–52.

[20] Y. Yin, H. Li, Note on "Generalized fuzzy interior ideals in semigroups", *Inform. Sci.* **177** (2007), 5798–5800.

[21] X. Yuan, C. Zhang and Y. Ren, Generalized fuzzy groups and many-valued implications, *Fuzzy Sets and Systems* **138** (2003), no. 1, 205–211.