

A New Characterization of $PGL(2, p)$ by its Noncommuting Graph

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Abstract. Let G be a finite non-abelian group. The noncommuting graph of G is denoted by $\nabla(G)$ and is defined as follows: the vertex set of $\nabla(G)$ is $G \setminus Z(G)$ and two vertices x and y are adjacent if and only if $xy \neq yx$. Let p be a prime number. In this paper, it is proved that the almost simple group $PGL(2, p)$ is uniquely determined by its noncommuting graph. As a consequence of our results the validity of a conjecture of Thompson and another conjecture of Shi and Bi for the group $PGL(2, p)$ are proved.

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1. Introduction

There is a close relation between group theory and graph theory. For studying some algebraic properties of finite groups, many authors assign appropriate graphs to groups and using the properties of these graphs, they have proved many interesting results in group theory. For example, Kegel and Gruenberg introduced the prime graph of a finite group G . The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. The prime graph of G is a graph whose vertex set is the set of all prime divisors of $|G|$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq (see [3]). We use $A \setminus B$ for the set of elements of A which are not in B .

The noncommuting graph of G is constructed as follows: the vertex set is $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if and only if $xy \neq yx$. This graph is denoted by $\nabla(G)$. The noncommuting graph of a group G was first considered

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by Paul Erdős, when he posed the following problem in 1975: Let G be a group whose noncommuting graph has no infinite complete subgraph. Is it true that there is a finite bound on the cardinalities of complete subgraphs of $\nabla(G)$? Neumann answered positively to this question (see [13]).

In [2] and [12], some graph theoretical properties of $\nabla(G)$ and the relations between some properties of $\nabla(G)$ and the structure of G were studied. For example it is proved that for every group G , the noncommuting graph $\nabla(G)$ is connected.

The noncommuting graphs of two finite groups G and H are said to be isomorphic (we write $\nabla(G) \cong \nabla(H)$) if there exists a bijective map $\phi : G \setminus Z(G) \rightarrow H \setminus Z(H)$ such that for every two distinct vertices x and y of $\nabla(G)$ we have $xy = yx$ if and only if $\phi(x)\phi(y) = \phi(y)\phi(x)$. In [2], the authors put forward the following conjecture:

Conjecture 1.1 (AAM's Conjecture). *Let S be a finite non-abelian simple group and G be a group such that $\nabla(G) \cong \nabla(S)$. Then $G \cong S$.*

This conjecture is known to hold for all simple groups with non-connected prime graphs (for more details see [7, 19, 20, 23–25, 28, 29]). Also it is proved that some finite simple groups with connected prime graphs, say A_{10} , $U_4(7)$, $L_4(8)$, $L_4(4)$ and $L_4(9)$, can be uniquely determined by their noncommuting graphs (see [22, 26, 27]). In [1], it is proved that $SL(2, q)$ is characterizable by its noncommuting graph.

In this paper as the main result we prove that if p is a prime number, then the projective general linear group $PGL(2, p)$, which is almost simple, is characterizable by its noncommuting graph. For the proof of this result, we prove that if $\nabla(G) \cong \nabla(PGL(2, p))$, then $|Z(G)| = 1$ and using this result we prove that $|G| = |PGL(2, p)|$ (Theorem 3.1). Then using Lemma 2.8 we conclude that $OC(G) = OC(PGL(2, p))$. Hence G has a normal series $1 \trianglelefteq H < K \trianglelefteq G$ and K/H is a simple group (Lemma 3.1). By the classification of finite simple groups, it follows that $K/H \cong A_1(q)$, for $q = 4$ or p (Theorem 3.3). As a consequence of our results we prove the validity of a conjecture of Thompson and another conjecture of Shi and Bi for the group $PGL(2, p)$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [6], for example. If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n . If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$.

2. Preliminary results

Definition 2.1. *If G is a finite group and $\Gamma(G)$ is the prime graph of G , then the number of connected components of $\Gamma(G)$ is denoted by $t(G)$ and the vertex set of the connected components are denoted by $\pi_i(G)$, $i = 1, \dots, t(G)$. If $2 \in \pi(G)$, then we assume that $2 \in \pi_1(G)$. Now $|G|$ can be expressed as a product of coprime positive integers m_i , $i = 1, \dots, t(G)$, where $\pi(m_i) = \pi_i(G)$. These integers are called the order components of $|G|$ and the set of order components of $|G|$ is denoted by $OC(G)$; i.e.,*

$$OC(G) = \{m_i | i = 1, \dots, t(G)\}.$$

Lemma 2.1. [2, Lemma 3.1] *Let G be a finite non-abelian group. If H is a group such that $\nabla(G) \cong \nabla(H)$, then H is a finite non-abelian group such that $|Z(H)|$*

divides

$$\gcd(|G| - |Z(G)|, |G| - |C_G(x)|, |C_G(x)| - |Z(G)| : x \in G \setminus Z(G)).$$

Lemma 2.2. [2, Proposition 3.2] *Let G be a group such that $\nabla(G) \cong \nabla(S_3)$. Then $G \cong S_3$.*

Definition 2.2. [8] *A finite group G is called a 2-Frobenius group if it has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$, where K and G/H are Frobenius groups with kernels H and K/H , respectively.*

Lemma 2.3. *Let G be a Frobenius group of even order and H, K be Frobenius complement and Frobenius kernel of G , respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H), \pi(K)$ and G has one of the following structures:*

- (a) $2 \in \pi(K)$ and all Sylow subgroups of H are cyclic;
- (b) $2 \in \pi(H)$, K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the Sylow 2-subgroups of H are cyclic or generalized quaternion groups;
- (c) $2 \in \pi(H)$, K is an abelian group and there exists $H_0 \leq H$ such that $|H : H_0| \leq 2$, $H_0 = Z \times SL(2, 5)$, $\pi(Z) \cap \{2, 3, 5\} = \emptyset$ and the Sylow subgroups of Z are cyclic.

Also the next lemma follows from [8] and the properties of Frobenius groups (see [9]):

Lemma 2.4. *Let G be a 2-Frobenius group, i.e., G has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$, such that K and G/H are Frobenius groups with kernels H and K/H , respectively. Then*

- (a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;
- (b) *The quotient groups G/K and K/H are cyclic groups, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \text{Aut}(K/H)$;*
- (c) H is nilpotent and G is a solvable group.

Lemma 2.5. [5, Lemma 8] *Let G be a finite group with $t(G) \geq 2$ and let N be a normal subgroup of G . If N is a π_i -group for some prime graph component of G and m_1, m_2, \dots, m_r are some order components of G but not π_i -numbers, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.*

Lemma 2.6. [4, Lemma 1.4] *Suppose G and M are two finite groups satisfying $t(M) \geq 2$, $N(G) = N(M)$, where $N(G) = \{n \mid G \text{ has a conjugacy class of size } n\}$, and $Z(G) = 1$. Then $|G| = |M|$.*

Lemma 2.7. [4, Lemma 1.5] *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(G_1) = t(G_2)$ and $OC(G_1) = OC(G_2)$.*

Lemma 2.8. *Let G be a group such that $\nabla(G) \cong \nabla(M)$, where M is a finite group such that $|G| = |M|$. Then $OC(G) = OC(M)$.*

Proof. Since $\nabla(G) \cong \nabla(M)$, the set of vertex degrees of two graphs are the same. Thus

$$\{|G| - |C_G(x)| \mid x \in G\} = \{|M| - |C_M(y)| \mid y \in M\}.$$

Since $|G| = |M|$, it follows that $N(G) = N(M)$. Now Lemma 2.7 implies that $OC(G) = OC(M)$. ■

Lemma 2.9. [11, Lemma 2.8] *Let G be a finite group and M be a finite group with $t(M) = 2$ satisfying $OC(G) = OC(M)$. Let $OC(M) = \{m_1, m_2\}$. Then one of the following holds:*

- (a) G is a Frobenius or 2-Frobenius group;
- (b) G has a normal series $1 \trianglelefteq H < K \trianglelefteq G$ such that G/K is a π_1 -group, H is a nilpotent π_1 -group, and K/H is a non-abelian simple group. Moreover $OC(K/H) = \{m'_1, m'_2, \dots, m'_s, m_2\}$, where $m'_1 m'_2 \dots m'_s \mid m_1$. Also $G/K \leq Out(K/H)$.

3. Main results

Throughout this section let p be an odd prime number.

Theorem 3.1. *Let G be a group such that $\nabla(G) \cong \nabla(M)$, where $M = PGL(2, p)$. Then $|G| = |M|$.*

Proof. By Lemma 2.1, G is a finite non-abelian group. Since $\nabla(G) \cong \nabla(M)$, it follows that $|G| - |Z(G)| = |M| - |Z(M)|$. We know that $|Z(M)| = 1$, so it is sufficient to prove that $|Z(G)| = 1$.

Let P be a Sylow p -subgroup of M and $x \in P$. Note that $|P| = p$ and so P is abelian. Therefore $P \leq C_M(x)$, which implies that $|C_M(x)| = kp$, for some $k > 0$. We claim that $k = 1$. Otherwise let $p' \neq p$ be a prime divisor of k . So there exists $y \in C_M(x)$ such that $o(y) = p'$. Therefore $o(xy) = pp'$, which implies that p is adjacent to p' in $\Gamma(M)$. But we note that p is an isolated vertex in $\Gamma(M)$ and so we get a contradiction. Therefore $k = 1$ and so $|C_M(x)| = p$. By Lemma 2.1, we know that $|Z(G)|$ is a divisor of $|M| - |Z(M)| = p(p^2 - 1) - 1$ and $|C_M(x)| - |Z(M)| = p - 1$, which implies that $|Z(G)| = 1$, as desired. ■

Theorem 3.2. *Let G be a finite group and $OC(G) = OC(PGL(2, p))$. If $p > 3$, then G is neither a Frobenius group nor a 2-Frobenius group. If $p = 3$ and G is a 2-Frobenius group, then $G \cong S_4$.*

Proof. Clearly, $OC(G) = OC(PGL(2, p)) = \{p, p^2 - 1\}$. If G is a Frobenius group, then by Lemma 2.3, $OC(G) = \{|H|, |K|\}$, where K and H are Frobenius kernel and Frobenius complement of G , respectively. Therefore $\{|H|, |K|\} = \{p, p^2 - 1\}$. Since $|H| \mid (|K| - 1)$, it follows that $|H| < |K|$ and so $|H| = p$ and $|K| = p^2 - 1$. Thus $p \mid (p^2 - 2)$, which implies that $p = 2$, a contradiction. Therefore G is not a Frobenius group.

Let G be a 2-Frobenius group. Hence $G = ABC$, where A and AB are normal subgroups of G ; AB and BC are Frobenius groups with kernels A , B and complements B , C , respectively. By Lemma 2.4, we have $|B| = p$ and $|A||C| = p^2 - 1$. Since $|B| \mid (|A| - 1)$, we may assume that $|A| = pt + 1$, for some $t > 0$. On the other hand, since $|A| \mid (p^2 - 1)$, it follows that $p^2 - 1 = k(pt + 1)$, for some $k > 0$. Therefore $p \mid (k + 1)$ and so $p - 1 \leq k$. If $t > 1$, then $p^2 - 1 = k(pt + 1) \geq (p - 1)(pt + 1) > (p - 1)(p + 1)$,

which is a contradiction. Hence $t = 1$, which implies that $|A| = p + 1$ and so $|C| = p - 1$.

If there exists an odd prime q such that $q \mid (p+1)$, then let Q be a Sylow q -subgroup of A . Since A is a nilpotent group, it follows that Q is a normal subgroup of G . Now Lemma 2.5 implies that $p \mid (|Q| - 1)$ and $|Q| \mid (p+1)/2$, which is a contradiction. Therefore $p + 1 = 2^\alpha$, for some $\alpha > 1$. Since $Z(A) \neq 1$ is a characteristic subgroup of A , it follows that A is abelian. Let $X = \{x \in A \mid o(x) = 2\} \cup \{1\}$. Then X is a non-identity characteristic subgroup of A . Therefore A is an elementary abelian 2-subgroup of G and $|A| = 2^\alpha = p + 1$. Let $F = GF(2^\alpha)$ and so A is the additive group of F . Also $|B| = p = 2^\alpha - 1$ and so B is the multiplicative group of F . Now C acts by conjugation on A and similarly C acts by conjugation on B and this action is faithful. Therefore C keeps the structure of the field F and so C is isomorphic to a subgroup of the automorphism group of F . Hence $|C| = 2^\alpha - 2 \leq |Aut(F)| = \alpha$. Therefore $\alpha \leq 2$. Thus $\alpha = 2$, and so $G = S_4$, the symmetric group on 4 letters. ■

Lemma 3.1. *Let G be a finite group and $M = PGL(2, p)$, where $p > 3$ or $p = 3$ and G is not a 2-Frobenius group. If $OC(G) = OC(M)$, then G has a normal series $1 \trianglelefteq H < K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H is a simple group. Moreover the odd order component of M is equal to an odd order component of K/H . In particular, $t(K/H) \geq 2$. Also $|G/H|$ divides $|Aut(K/H)|$, and in fact $G/H \leq Aut(K/H)$.*

Proof. The first part of the lemma follows from Lemma 2.9 and Theorem 3.2, since the prime graph of M has two components. If K/H has an element of order pq , where p and q are primes, then G has an element of order pq . So by the definition of prime graph component, the odd order component of G is equal to an odd order component of K/H . Also $K/H \trianglelefteq G/H$ and $C_{G/H}(K/H) = 1$, which implies that

$$G/H = \frac{N_{G/H}(K/H)}{C_{G/H}(K/H)} \leq Aut(K/H). \quad \blacksquare$$

Theorem 3.3. *Let G be a finite group such that $OC(G) = OC(M)$, where $M = PGL(2, p)$. Then $G \cong PGL(2, p)$.*

Proof. If $p = 3$ and G is a 2-Frobenius group, then Theorem 3.2 implies that $G = S_4 \cong PGL(2, 3)$ as desired. Otherwise Lemma 3.1 implies that G has a normal series $1 \trianglelefteq H < K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H is a simple subgroup and $t(K/H) \geq 2$. So the possibilities for K/H are:

- (a) The alternating group A_n , where $n \geq 5$,
- (b) Sporadic simple groups,
- (c) Simple groups of Lie type.

Using the classification of finite simple groups and the results in Tables 1–3 in [10], we consider these cases.

Case (a). Let $K/H \cong A_n$, where $n \in \{p', p' + 1, p' + 2\}$, $p' \geq 5$ is a prime number. If n and $n - 2$ are not primes simultaneously, then $p' = p$. We note that $|A_n| \mid |G| = p(p^2 - 1)$. If $n > p$, then $|A_n| > (p + 1)p(p - 1)$, which is a contradiction.

So $n = p$ and $|A_p| = p!/2 \leq p(p^2 - 1)$. Hence $(p - 2)!/2 \leq p + 1$. Since $p - 2$ is not a prime, we have $p > 7$. So $(p - 2)(p - 3) < (p - 2)!/2 \leq p + 1$, which is a contradiction.

Let $K/H \cong A_{p'}$, where p' and $p' - 2$ are primes. Then $p = p'$ or $p' - 2$.

If $p = p' - 2$, then $p^2 - 1 = p'^2 - 4p' + 3$. Since $p' \mid |K/H|$ and $|K/H| \mid p(p^2 - 1)$ and $(p, p') = 1$, we have $p' \mid (p^2 - 1)$. So $p' \mid 3$, which implies that $p' = 3$ and hence $p = 1$, a contradiction.

If $p = p'$ and $p' \geq 7$, then we can get a contradiction similarly to the previous case. So $p = 5$ and $K/H \cong A_5 \cong PSL(2, 5)$. Since $K/H \leq G/H \leq Aut(K/H)$, we have $PSL(2, 5) \leq G/H \leq PGL(2, 5)$. Hence G/H is isomorphic to $PSL(2, 5)$ or $PGL(2, 5)$. If $G/H \cong PSL(2, 5)$, then $|H| = 2$. But $H \trianglelefteq G$, which implies that $H \subseteq Z(G)$ and so we get a contradiction. Therefore $G/H \cong PGL(2, 5)$, which implies that $H = 1$ and $G \cong PGL(2, 5)$.

Case (b). Let K/H be a sporadic simple group.

If $K/H \cong Ru$, then $p = 29$. On the other hand, $3^2 \mid |K/H|$ and so $3^2 \mid (p^2 - 1) = 840$, which is a contradiction.

If $K/H \cong J_3$, then $p = 17$ or 19 . Let $p = 17$. Since $5 \mid |K/H|$, we have $5 \mid (p^2 - 1) = 288$, which is impossible. So $p = 19$. Since $3^3 \mid |K/H|$, we conclude that $3^3 \mid (p^2 - 1) = 360$, which is a contradiction.

If $K/H \cong F_1 = M$, then $p = 41, 59$ or 71 . Let $p = 41$ or 59 . Since $3^2 \mid |K/H|$, it follows that $3^2 \mid (p^2 - 1)$, which is a contradiction. So $p = 71$. Since $11 \mid |K/H|$, we have $11 \mid (p^2 - 1) = 5040$, which is a contradiction.

For other sporadic simple groups we can get a contradiction similarly.

Case (c). Let K/H be a simple group of Lie type.

If K/H is isomorphic to one of the groups ${}^2A_3(2), {}^2F_4(2)', A_2(4), {}^2A_5(2), E_7(2), E_7(3), {}^2E_6(2)$, then we can get a contradiction similarly to Case (b).

If $K/H \cong A_{p'-1}(q)$, where $(p', q) \neq (3, 2), (3, 4)$, then $p = (q^{p'} - 1)/((q - 1)(p' - 1))$, which implies that $p \leq q^{p'} - 1 < q^{p'}$. So $p^2 - 1 < p^2 < q^{2p'}$. On the other hand, $q^{p'(p'-1)/2} \mid |K/H|$. So $q^{p'(p'-1)/2} < q^{2p'}$ and consequently $p'^2 - 5p' < 0$. Hence $p' = 3$ and $p = (q^2 + q + 1)/(3, q - 1)$. So $p < 2q^2$, which implies that $p^2 - 1 < 4q^4 - 1$. Also $q^3(q - 1)(q^2 - 1) \mid (p^2 - 1)$. If $q \geq 5$, then $p^2 - 1 < 4q^4 - 1 < q^3(q - 1)(q^2 - 1)$, which is impossible. So $q = 2, 3$ or 4 . Since $(p', q) \neq (3, 2), (3, 4)$, we have $q = 3$ and $p = 13$ and so $3^3(3^2 - 1)(3 - 1) \mid (p^2 - 1) = 168$, which is a contradiction.

If K/H is isomorphic to one of the following groups: $B_{p'}(3); C_{p'}(q)$, where $q = 2, 3; D_{p'}(q)$, where $q = 2, 3, 5$ and $p' \geq 5; D_{p'+1}(q)$, where $q = 2, 3$ and $A_{p'}(q)$, where $(q - 1) \mid (p' + 1)$, we can get a contradiction similarly. For example we consider the following case:

If $K/H \cong D_{p'}(q)$, where $q = 2, 3, 5$ and $p' \geq 5$, then $p = (q^{p'} - 1)/(q - 1)$. So $p \leq q^{p'} - 1 < q^{p'}$, which implies that $p^2 - 1 < p^2 < q^{2p'}$. Also $q^{p'(p'-1)} \mid (p^2 - 1)$ and hence $q^{p'(p'-1)} < q^{2p'}$. Therefore $p'^2 - 3p' < 0$, which is a contradiction.

If $K/H \cong {}^2A_{p'-1}(q)$, then $p = (q^{p'} + 1)/((q + 1)(p', q + 1))$. Therefore $p < q^{p'} + 1$, which implies that $p^2 - 1 < q^{2p'} + 2q^{p'} < 2q^{2p'} \leq q^{2p'+1}$. Also $q^{p'(p'-1)/2} \mid (p^2 - 1)$ and hence $p'(p' - 1)/2 < 2p' + 1$. So $p' = 3, 5$. Let $p' = 5$. Then $p^2 - 1 < q^{11}$ and $q^{10}(q + 1)(q^2 - 1)(q^3 + 1)(q^4 - 1) \mid (p^2 - 1)$, which implies that $q^{10}(q + 1)(q^2 - 1)(q^3 + 1)(q^4 - 1) < q^{11}$. Therefore $(q + 1)(q^2 - 1)(q^3 + 1)(q^4 - 1) < q$, which is impossible, since $q \geq 2$. So $p' = 3$ and $p^2 - 1 < q^6 + 2q^3$. Also $q^3(q + 1)(q^2 - 1) \mid (p^2 - 1)$, which implies that $q^2 - q - 3 < 0$. Hence $q = 2$ and so $p = 1$, which is impossible.

If K/H is isomorphic to one of the following groups: $B_n(q)$, where $n = 2^m \geq 4$ and q is odd; $C_n(q)$, where $n = 2^m \geq 2$; ${}^2D_n(q)$, where $n = 2^m \geq 4$; ${}^2D_n(2)$, where $n = 2^m + 1 \geq 5$; ${}^2D_{p'}(3)$, where $p' \neq 2^m + 1$ and $p' \geq 5$; ${}^2D_n(3)$, where $n = 2^m + 1 \neq p'$ and $m \geq 2$; ${}^2D_{p'}(3)$, where $p' = 2^n + 1 \geq 5$ and ${}^2A_{p'}(q)$, where $(q + 1) \mid (p' + 1)$, then we get a contradiction similarly. For convenience we omit the proof of these cases and as an example we consider the following case: If $K/H \cong C_n(q)$, where $n = 2^m \geq 2$, then $p = (q^n + 1)/(2, q - 1)$. Hence $p \leq q^n + 1$, which implies that $p^2 - 1 \leq q^{2n} + 2q^n < 2q^{2n} \leq q^{2n+1}$. Also $q^{n^2} \mid (p^2 - 1)$ and hence $n^2 < 2n + 1$. So $n = 2$ and $p^2 - 1 < 2q^4$ and $q^4(q^2 - 1)^2 \mid (p^2 - 1)$, which is a contradiction.

If K/H is isomorphic to one of the following groups: $G_2(q)$, where $q \equiv \pm 1 \pmod{3}$ and $q > 2$; ${}^3D_4(q)$; $F_4(q)$, where q is an odd number; $E_6(q)$; ${}^2E_6(q)$; $F_4(q)$, where $q = 2^n > 2$; $G_2(q)$, where $3 \mid q$ and $E_8(q)$, then we can get a contradiction similarly. For convenience we state the following case: If $K/H \cong E_8(q)$, where $q \equiv 2, 3 \pmod{5}$, then we have three subcases:

If $p = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$, then $p < 4q^8 \leq q^{10}$. So $p^2 - 1 < p^2 \leq q^{20}$. But we have $q^{120} \mid (p^2 - 1)$, which is a contradiction.

If $p = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$, then $p < 4q^8 \leq q^{10}$ and we get a contradiction similarly.

If $p = q^8 - q^4 + 1$, then $p < 2q^8 \leq q^9$. So $p^2 - 1 < p^2 \leq q^{18}$. But we have $q^{120} \mid (p^2 - 1)$, which is a contradiction.

If K/H is isomorphic to one of the following groups: ${}^2B_2(q)$, where $q = 2^{2n+1} > 2$; ${}^2F_4(q)$, where $q = 2^{2n+1} > 2$ and ${}^2G_2(q)$, where $q = 3^{2n+1}$, then we get a contradiction similarly. For example if $K/H \cong {}^2G_2(q)$, where $q = 3^{2n+1} \geq 3$, then $p = q + \sqrt{3q} + 1$ or $p = q - \sqrt{3q} + 1$. If $p = q - \sqrt{3q} + 1$, then $p < q + 1$. So $p^2 - 1 < q^2 + 2q < 2q^2$. Also $q^3 \mid (p^2 - 1)$, which implies that $q^3 < 2q^2$, and this is a contradiction. If $p = q + \sqrt{3q} + 1$, then $p \leq 2q + 1$. So $p^2 - 1 \leq 4q^2 + 4q < 8q^2$. Also $q^3 \mid (p^2 - 1)$, which implies that $q^3 < 8q^2$. Therefore $q = 3$ and $p = 7$ and so we have $3^3 \mid 48$, which is a contradiction.

If $K/H \cong A_1(q)$, where $4 \mid q$, then $p = q - 1$ or $q + 1$. If $p = q - 1$, then $p < q$ and so $p^2 - 1 < q^2$. But we have $q(q + 1) \mid (p^2 - 1)$, which is a contradiction. If $p = q + 1$, then $p^2 - 1 = q^2 + 2q$. Also $q(q - 1) \mid (p^2 - 1)$ and hence $(q^2 - q) \mid 3q$, which implies that $q^2 - 4q \leq 0$. So $q = 4$ and hence $K/H \cong A_1(4) \cong A_5$. Therefore $G \cong PGL(2, 5)$, as we showed in Case (a).

If $K/H \cong A_1(q)$, where $4 \mid (q + 1)$, then $p = q$ or $(q - 1)/2$. If $p = (q - 1)/2$, then $p < q - 1 < q$. So $p^2 - 1 < q^2$. Also $q(q + 1) \mid (p^2 - 1)$, which is a contradiction. If $p = q$, then $K/H \cong A_1(p) = PSL(2, p)$. On the other hand, $K/H \leq G/H \leq Aut(K/H)$, which implies that $PSL(2, p) \leq G/H \leq PGL(2, p)$. Since $|PGL(2, p)| = 2|PSL(2, p)|$, we conclude that G/H is isomorphic to $PSL(2, p)$ or $PGL(2, p)$. If $G/H \cong PSL(2, p)$, then $|H| = 2$ and since $H \trianglelefteq G$ we have $H \subseteq Z(G)$, which is a contradiction by $Z(G) = 1$. So $G/H \cong PGL(2, p)$. Since $|G| = |PGL(2, p)|$, we have $H = 1$ and $G \cong PGL(2, p)$, as required.

If $K/H \cong A_1(q)$, where $4 \mid (q - 1)$, then $p = q$ or $(q + 1)/2$. If $p = (q + 1)/2$, then $p^2 - 1 = (q^2 + 2q - 3)/4$. Also $q(q - 1) \mid (p^2 - 1)$, which implies that $q^2 - 2q + 1 \leq 0$ and this is a contradiction. If $p = q$, then $K/H \cong A_1(p) = PSL(2, p)$ and similarly to the previous case we have $G \cong PGL(2, p)$. ■

Theorem 3.4. *Let G be a group such that $\nabla(G) \cong \nabla(M)$, where $M = PGL(2, p)$ and p is a prime number. Then $G \cong M$.*

Proof. If $p = 2$, then $PGL(2, 2) \cong S_3$, and so the proof follows from Lemma 2.2. If p is an odd prime, then obviously the theorem follows from Theorems 3.1, 3.3 and Lemma 2.8. ■

Remark 3.1. It is a well known conjecture of J. G. Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We can give a positive answer to this conjecture for the group $PGL(2, p)$ (where p is a prime) by our characterization of this group.

Corollary 3.1. *Let G be a finite group with $Z(G) = 1$ and $M = PGL(2, p)$, where p is a prime. If $N(G) = N(M)$, then $G \cong M$.*

Proof. By Lemmas 2.6 and 2.7, if G and M are two finite groups satisfying the conditions of Corollary 3.1, then $OC(G) = OC(M)$. So by Theorem 3.3 we get the result. ■

Remark 3.2. W. Shi and J. Bi in [17] put forward the following conjecture:

Conjecture 3.1. *Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if*

- (i) $|G| = |M|$, and,
- (ii) $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of orders of elements in G .

This conjecture is valid for sporadic simple groups [14], alternating groups [18], and some simple groups of Lie type [15–17]. As a consequence of Theorem 3.3, we prove the validity of this conjecture for the almost simple group $PGL(2, p)$, where p is a prime.

Corollary 3.2. *Let G be a finite group and $M = PGL(2, p)$, where p is a prime. If $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.*

Proof. By assumption we have $OC(G) = OC(M)$. Thus the corollary follows from Theorem 3.3. ■

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