Some Jensen’s Type Inequalities for Log-Convex Functions of Selfadjoint Operators in Hilbert Spaces

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Abstract. Some Jensen’s type inequalities for Log-Convex functions of self-adjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

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1. Introduction

Let $A$ be a selfadjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism $\Phi$ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of $A$, denoted $Sp(A)$, and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows (see for instance [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(i) $\Phi (\alpha f + \beta g) = \alpha \Phi (f) + \beta \Phi (g)$;
(ii) $\Phi (fg) = \Phi (f) \Phi (g)$ and $\Phi (f) = \Phi (f^*)$;
(iii) $\|\Phi (f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi (f_0) = 1_H$ and $\Phi (f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi (f) \quad \text{for all } f \in C(Sp(A))$$

and we call it the continuous functional calculus for a selfadjoint operator $A$. 

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If $A$ is a selfadjoint operator and $f$ is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on $H$. Moreover, if both $f$ and $g$ are real valued functions on $Sp(A)$ then the following important property holds:

\[ f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \]

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [6] and the references therein. For other results, see [7, 9, 11, 12]. For recent results, see [1,3,4].

2. Some Jensen's type inequalities for log-convex functions

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [10] (see also [6, p. 5]).

**Theorem 2.1.** [10] Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then

\[ f(\langle Ax, x \rangle) \leq \langle f(A) x, x \rangle \]

for each $x \in H$ with $\|x\| = 1$.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of log-convex functions, namely functions $f : I \rightarrow (0, \infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

\[ f((1-t)a + tb) \leq \ln [f(a)]^{1-t} [f(b)]^t \]

for any $a, b \in I$ and $t \in [0, 1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex function. However, obviously, there are functions that are convex but not log-convex.

As an immediate consequence of the Mond-Pečarić inequality above we can provide the following result.

**Theorem 2.2.** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $g : [m, M] \rightarrow (0, \infty)$ is log-convex, then

\[ g(\langle Ax, x \rangle) \leq \exp(\ln g(A) x, x) \leq \langle g(A) x, x \rangle \]

for each $x \in H$ with $\|x\| = 1$.

**Proof.** Consider the function $f := \ln g$, which is convex on $[m, M]$. Writing (2.1) for $f$ we get $\ln[g(\langle Ax, x \rangle)] \leq \langle \ln g(A) x, x \rangle$, for each $x \in H$ with $\|x\| = 1$, which, by taking the exponential, produces the first inequality in (2.2).

If we also use (2.1) for the exponential function, we get

\[ \exp(\ln g(A) x, x) \leq \langle \exp(\ln g(A)) x, x \rangle = \langle g(A) x, x \rangle \]

for each $x \in H$ with $\|x\| = 1$ and the proof is complete. \[ \square \]
It is also important to observe that, as a special case of Theorem 2.1 we have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality.

**Theorem 2.3.** [8] Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. Then

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
(iii) If $A$ is invertible, then $\langle A^{-r} x, x \rangle \geq \langle Ax, x \rangle^{-r}$ for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

Since the function $g(t) = t^r$ for $r > 0$ is log-convex, we can improve the Hölder-McCarthy inequality as follows.

**Proposition 2.1.** Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible, then

$$\langle f(A)x, x \rangle \leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot f(M)$$

for each $x \in H$ with $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [6, p. 57].

**Theorem 2.4.** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $f$ is a convex function on $[m, M]$, then

$$\langle f(A)x, x \rangle \leq M - \langle Ax, x \rangle \cdot f(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot f(M)$$

for each $x \in H$ with $\|x\| = 1$.

This result can be improved for log-convex functions as follows.

**Theorem 2.5.** Let $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars $m, M$ with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then

$$\langle g(A)x, x \rangle \leq \left( \left[ g(m) \right]^{\frac{M_1 - A}{M - m}} \left[ g(M) \right]^{\frac{A - M_1}{M - m}} \right) x, x \right) \leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot g(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot g(M)$$

and

$$g(\langle Ax, x \rangle) \leq \left( \left[ g(m) \right]^{\frac{M - \langle Ax, x \rangle}{M - m}} \left[ g(M) \right]^{\frac{\langle Ax, x \rangle - m}{M - m}} \right) x, x \right) \leq \left( \left[ g(m) \right]^{\frac{M_1 - A}{M - m}} \left[ g(M) \right]^{\frac{A - M_1}{M - m}} \right) x, x \right) \leq \left( \left[ g(m) \right]^{\frac{M - \langle Ax, x \rangle}{M - m}} \left[ g(M) \right]^{\frac{\langle Ax, x \rangle - m}{M - m}} \right) x, x \right)$$

for each $x \in H$ with $\|x\| = 1$.

**Proof.** Observe that, by the log-convexity of $g$, we have

$$g(t) = g\left( \frac{M - t}{M - m} \cdot m + \frac{t - m}{M - m} \cdot M \right) \leq \left[ g(m) \right]^{\frac{M - t}{M - m}} \left[ g(M) \right]^{\frac{t - m}{M - m}}$$
for any $t \in [m, M]$.

Applying the property (1.1) for the operator $A$, we have that
\[ \langle g(A)x, x \rangle \leq \langle \Psi(A)x, x \rangle \]
for each $x \in H$ with $\|x\| = 1$, where $\Psi(t) := [g(m)]^{\frac{M-t}{M-m}}[g(M)]^{\frac{t-m}{M-m}}, \ t \in [m, M]$. This proves the first inequality in (2.5).

Now, observe that, by the weighted arithmetic mean-geometric mean inequality we have
\[ [g(m)]^{\frac{M-t}{M-m}}[g(M)]^{\frac{t-m}{M-m}} \leq \frac{M-t}{M-m} \cdot g(m) + \frac{t-m}{M-m} \cdot g(M) \]
for any $t \in [m, M]$.

Applying the property (1.1) for the operator $A$ we deduce the second inequality in (2.5).

Further on, if we use the inequality (2.7) for $t = \langle Ax, x \rangle \in [m, M]$ then we deduce the first part of (2.6).

Now, observe that the function $\Psi$ introduced above can be rearranged to read as
\[ \Psi(t) = g(m) \left[ \frac{g(M)}{g(m)} \right]^{\frac{t-m}{M-m}}, \ t \in [m, M] \]
showing that $\Psi$ is a convex function on $[m, M]$.

Applying Mond-Pečarić’s inequality for $\Psi$ we deduce the second part of (2.6) and the proof is complete.

The above result from Theorem 2.5 can be utilized to produce the following reverse inequality for negative powers of operators.

**Proposition 2.2.** Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $\text{Sp}(A) \subseteq [m, M] \ (0 < m < M)$, then
\[
\langle A^{-r}x, x \rangle \leq \left[ M - \frac{\langle Ax, x \rangle}{M - m} \right]^{-r} \langle x, x \rangle \leq M - \frac{\langle Ax, x \rangle}{M - m} \cdot m^{-r} + \frac{\langle Ax, x \rangle - m}{M - m} \cdot M^{-r}
\]
and
\[
\langle Ax, x \rangle^{-r} \leq \left[ g(m) \frac{M - \langle Ax, x \rangle}{M - m} \right]^{-r} \leq \left[ \frac{M^{1-H} - A}{M - m} \frac{A - m^{1-H}}{M - m} \right]^{-r} \langle x, x \rangle
\]
for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

**3. Jensen’s inequality for differentiable log-convex functions**

The following result provides a reverse for the Jensen type inequality (2.1).

**Theorem 3.1.** [5] Let $J$ be an interval and $f : J \to \mathbb{R}$ be a convex and differentiable function on $\bar{J}$ (the interior of $J$) whose derivative $f'$ is continuous on $\bar{J}$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq [m, M] \subset \bar{J}$, then
\[
(0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \langle f'(A)Ax, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle
\]
for any \( x \in H \) with \( \|x\| = 1 \).

The following result may be stated.

**Proposition 3.1.** Let \( J \) be an interval and \( g : J \to \mathbb{R} \) be a differentiable log-convex function on \( J \) whose derivative \( g' \) is continuous on \( J \). If \( A \) is a selfadjoint operator on the Hilbert space \( H \) with \( \text{Sp}(A) \subseteq [m, M] \subset \hat{J} \), then

\[
1 \leq \frac{\exp \langle \ln g(A) x, x \rangle}{g(\langle Ax, x \rangle)} \leq \exp \left[ \langle g'(A) [g(A)]^{-1} Ax, x \rangle - \langle Ax, x \rangle \cdot \langle g'(A) [g(A)]^{-1} x, x \rangle \right]
\]

for each \( x \in H \) with \( \|x\| = 1 \).

**Proof.** It follows by the inequality (3.1) written for the convex function \( f = \ln g \) that

\[
\langle \ln g(A) x, x \rangle \leq \ln g(\langle Ax, x \rangle) + \langle g'(A) [g(A)]^{-1} Ax, x \rangle - \langle Ax, x \rangle \cdot \langle g'(A) [g(A)]^{-1} x, x \rangle
\]

for each \( x \in H \) with \( \|x\| = 1 \).

Now, taking the exponential and dividing by \( g(\langle Ax, x \rangle) > 0 \) for each \( x \in H \) with \( \|x\| = 1 \), we deduce the desired result (3.2). \( \square \)

**Remark 3.1.** Let \( A \) be a selfadjoint positive operator on a Hilbert space \( H \). If \( A \) is invertible, then

\[
1 \leq \langle Ax, x \rangle^r \exp \langle \ln (A^{-r}) x, x \rangle \leq \exp \left[ r \left( \langle Ax, x \rangle \cdot \langle A^{-1} x, x \rangle - 1 \right) \right]
\]

for all \( r > 0 \) and \( x \in H \) with \( \|x\| = 1 \), where \( 1_H \) denotes the identity operator on \( H \).

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen’s inequality can be stated as well.

**Theorem 3.2.** Let \( J \) be an interval and \( g : J \to \mathbb{R} \) be a log-convex differentiable function on \( \hat{J} \) whose derivative \( g' \) is continuous on \( \hat{J} \). If \( A \) is a selfadjoint operator on the Hilbert space \( H \) with \( \text{Sp}(A) \subseteq [m, M] \subset \hat{J} \), then

\[
1 \leq \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} \left( A - \langle Ax, x \rangle 1_H \right) \right] x, x \leq \exp \left[ \frac{g(A) x, x}{g(\langle Ax, x \rangle)} \right] \leq \exp \left[ \frac{g'(A) [g(A)]^{-1} (A - \langle Ax, x \rangle 1_H)}{g(A) x, x} \right]
\]

for each \( x \in H \) with \( \|x\| = 1 \), where \( 1_H \) denotes the identity operator on \( H \).

**Proof.** It is well known that if \( h : J \to \mathbb{R} \) is a convex differentiable function on \( \hat{J} \), then the following gradient inequality holds

\[ h(t) - h(s) \geq h'(s)(t - s) \]

for any \( t, s \in \hat{J} \).

Now, if we write this inequality for the convex function \( h = \ln g \), then we get

\[
\ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t - s)
\]
which is equivalent with
\[ g(t) \geq g(s) \exp \left[ \frac{g'(s)}{g(s)} (t - s) \right] \]
for any \( t, s \in \bar{J} \).

Further, if we take \( s := \langle Ax, x \rangle \in [m, M] \subset \bar{J} \), for a fixed \( x \in H \) with \( \| x \| = 1 \), in the inequality (3.6), then we get
\[ g(t) \geq g(\langle Ax, x \rangle) \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} (t - \langle Ax, x \rangle) \right] \]
for any \( t \in \bar{J} \).

Utilising the property (1.1) for the operator \( A \) and the Mond-Pečarić inequality for the exponential function, we can state the following inequality that is of interest in itself as well:
\[ \langle g(A) y, y \rangle \geq g(\langle Ax, x \rangle) \langle \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} (A - \langle Ax, x \rangle 1_H) \right] y, y \rangle \geq g(\langle Ax, x \rangle) \exp \left[ \frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)} (\langle Ay, y \rangle - \langle Ax, x \rangle) \right] \]
for each \( x, y \in H \) with \( \| x \| = \| y \| = 1 \).

Further, if we put \( y = x \) in (3.7), then we deduce the first and the second inequality in (3.4).

Now, if we replace \( s \) with \( t \) in (3.6) we can also write the inequality
\[ g(t) \exp \left[ \frac{g'(t)}{g(t)} (s - t) \right] \leq g(s) \]
which is equivalent with
\[ g(t) \leq g(s) \exp \left[ \frac{g'(t)}{g(t)} (t - s) \right] \]
for any \( t, s \in \bar{J} \).

Further, if we take \( s := \langle Ax, x \rangle \in [m, M] \subset \bar{J} \), for a fixed \( x \in H \) with \( \| x \| = 1 \), in the inequality (3.8), then we get
\[ g(t) \leq g(\langle Ax, x \rangle) \exp \left[ \frac{g'(t)}{g(t)} (t - \langle Ax, x \rangle) \right] \]
for any \( t \in \bar{J} \).

Utilising the property (1.1) for the operator \( A \), then we can state the following inequality that is of interest in itself as well:
\[ \langle g(A) y, y \rangle \leq g(\langle Ax, x \rangle) \left( \exp \left[ g'(A) g(A)^{-1} (A - \langle Ax, x \rangle 1_H) \right] y, y \right) \]
for each \( x, y \in H \) with \( \| x \| = \| y \| = 1 \).

Finally, if we put \( y = x \) in (3.9), then we deduce the last inequality in (3.4).

**Remark 3.2.** Let \( A \) be a selfadjoint positive operator on a Hilbert space \( H \). If \( A \) is invertible, then
\[ 1 \leq \left( \exp \left[ r \left( 1_H - \langle Ax, x \rangle^{-1} A \right) \right] x, x \right) \]
Theorem 3.3. Let $J$ be an interval and $g : J \rightarrow \mathbb{R}$ be a log-convex differentiable function on $J$ whose derivative $g'$ is continuous on $J$. If $A$ is a selfadjoint operator on the Hilbert space $H$ with $Sp(A) \subseteq [m, M] \subset J$, then

$$
(3.11) \quad 1 \leq \frac{\langle g(M)^{\frac{A-m_1H}{M-m}} g(m)^{\frac{M_1H-A}{M-m}} x, x \rangle}{\langle g(A) x, x \rangle} 
$$

for each $x \in H$ with $\|x\| = 1$.

The following reverse inequality may be proven as well.

Proof. Utilising the inequality (3.5) we have successively

$$
(3.12) \quad \frac{g((1 - \lambda) t + \lambda s)}{g(s)} \geq \exp \left[ (1 - \lambda) \frac{g'(s)}{g(s)} (t - s) \right]
$$

and

$$
(3.13) \quad \frac{g((1 - \lambda) t + \lambda s)}{g(t)} \geq \exp \left[ -\lambda \frac{g'(t)}{g(t)} (t - s) \right]
$$

for any $t, s \in J$ and any $\lambda \in [0, 1]$.

Now, if we take the power $\lambda$ in the inequality (3.12) and the power $1 - \lambda$ in (3.13) and multiply the obtained inequalities, we deduce

$$
(3.14) \quad \frac{[g(t)]^{1-\lambda} [g(s)]^\lambda}{g((1 - \lambda) t + \lambda s)} \leq \exp \left[ (1 - \lambda) \lambda \left( \frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)} \right) (t - s) \right]
$$

for any $t, s \in J$ and any $\lambda \in [0, 1]$.

Further on, if we choose in (3.14) $t = M, s = m$ and $\lambda = (M - u)/(M - m)$, then, from (3.14) we get the inequality

$$
(3.15) \quad \frac{g(M)^{\frac{u-m}{M-m}} g(m)^{\frac{M-n}{M-m}}}{g(u)} \leq \exp \left[ \frac{(M - u) (u - m)}{M - m} \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]
$$

which, together with the inequality

$$
\frac{(M - u) (u - m)}{M - m} \leq \frac{1}{4} (M - m)
$$

produce

$$
(3.16) \quad [g(M)]^{\frac{u-m}{M-m}} [g(m)]^{\frac{M-n}{M-m}} \leq g(u) \exp \left[ \frac{(M - u) (u - m)}{M - m} \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right]
$$
\[ g(u) \exp \left[ \frac{1}{4} (M - m) \left( \frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)} \right) \right] \]

for any \( u \in [m, M] \).

Remark 3.3. Let \( A \) be a selfadjoint positive operator on a Hilbert space \( H \). If \( A \) is invertible and \( \text{Sp}(A) \subseteq [m, M] \) \((0 < m < M)\), then

\[
(1 \leq) \quad \frac{\langle g(M) \rangle^{r(M^1H-A)M-m} \langle g(m) \rangle^{r(A-m1H)M-m} x,x \rangle}{\langle A^{-r}x,x \rangle} \leq \exp \left[ \frac{1}{4} r \frac{(M-m)^2}{mM} \right]
\]

4. Applications for Ky Fan’s inequality

Consider the function \( g : (0, 1) \to \mathbb{R}, g(t) = ((1 - t)/(t))^r, r > 0 \). Observe that for the new function \( f : (0, 1) \to \mathbb{R}, f(t) = \ln g(t) \) we have

\[
f'(t) = \frac{-r}{t(1-t)} \quad \text{and} \quad f''(t) = \frac{2r(\frac{1}{2} - t)}{t^2(1-t)^2} \quad \text{for} \quad t \in (0, 1)
\]

showing that the function \( g \) is log-convex on the interval \((0, 1/2)\).

If \( p_i > 0 \) for \( i \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{n} p_i = 1 \) and \( t_i \in (0, 1/2) \) for \( i \in \{1, \ldots, n\} \), then by applying the Jensen inequality for the convex function \( f \) (with \( r = 1 \)) on the interval \((0, 1/2)\) we get

\[
\frac{\sum_{i=1}^{n} p_i t_i}{1 - \sum_{i=1}^{n} p_i t_i} \geq \prod_{i=1}^{n} \left( \frac{t_i}{1 - t_i} \right)^{p_i},
\]

which is the weighted version of the celebrated Ky Fan’s inequality, see [2, p. 3].

This inequality is equivalent with

\[
\prod_{i=1}^{n} \left( \frac{1 - t_i}{t_i} \right)^{p_i} \geq \frac{1 - \sum_{i=1}^{n} p_i t_i}{\sum_{i=1}^{n} p_i t_i},
\]

where \( p_i > 0 \) for \( i \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{n} p_i = 1 \) and \( t_i \in (0, 1/2) \) for \( i \in \{1, \ldots, n\} \).

By the weighted arithmetic mean-geometric mean inequality we also have that

\[
\sum_{i=1}^{n} p_i (1 - t_i) t_i^{-1} \geq \prod_{i=1}^{n} \left( \frac{1 - t_i}{t_i} \right)^{p_i},
\]

giving the double inequality

\[
\sum_{i=1}^{n} p_i (1 - t_i) t_i^{-1} \geq \prod_{i=1}^{n} \left( (1 - t_i) t_i^{-1} \right)^{p_i} \geq \sum_{i=1}^{n} p_i (1 - t_i) \left( \sum_{i=1}^{n} p_i t_i \right)^{-1}.
\]

The following operator inequalities generalizing (4.2) may be stated.
Proposition 4.1. Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $\text{Sp}(A) \subset (0, 1/2)$, then
\[
\left\langle (A^{-1} (1_H - A))^r x, x \right\rangle \geq \exp \left\langle \ln (A^{-1} (1_H - A))^r x, x \right\rangle \\
\geq \left( \langle 1_H - A \rangle x, x \right) \langle A^{-1} x, x \rangle^{-1}
\]
for each $x \in H$ with $\|x\| = 1$ and $r > 0$.
In particular,
\[
\left\langle A^{-1} (1_H - A) x, x \right\rangle \geq \exp \left\langle \ln (A^{-1} (1_H - A))^r x, x \right\rangle \\
\geq \langle 1_H - A \rangle x, x \right) \langle A^{-1} x, x \rangle^{-1}
\]
for each $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 2.2 applied for the log-convex function $g(t) = ((1 - t)(1/t))^r$, $r > 0$, $t \in (0, 1/2)$.

Proposition 4.2. Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $\text{Sp}(A) \subseteq [m, M] \subset (0, 1/2)$, then
\[
\left\langle (1_H - A)^{-1} A^{-1} x, x \right\rangle \leq \left\langle \left[ \left( \frac{1 - m}{m} \right)^{r((M1_H - A)/M - m)} \left( \frac{1 - M}{M} \right)^{r(A^{-1} m1_M)} \right] x, x \right\rangle \\
\leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot \left( \frac{1 - m}{m} \right)^r + \frac{\langle Ax, x \rangle - m}{M - m} \cdot \left( \frac{1 - M}{M} \right)^r
\]
and
\[
\left( \frac{1 - \langle Ax, x \rangle}{\langle Ax, x \rangle} \right)^r \leq \left( \frac{1 - m}{m} \right)^{r(M - \langle Ax, x \rangle)/M - m} \left( \frac{1 - M}{M} \right)^{r(A^{-1} m1_M)}
\leq \left\langle \left[ \left( \frac{1 - m}{m} \right)^{r((M1_H - A)/M - m)} \left( \frac{1 - M}{M} \right)^{r(A^{-1} m1_M)} \right] x, x \right\rangle
\]
for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

The proof follows by Theorem 2.5 applied for the log-convex function $g(t) = ((1 - t)(1/t))^r$, $r > 0$, $t \in (0, 1/2)$.

Finally we have the following.

Proposition 4.3. Let $A$ be a selfadjoint positive operator on a Hilbert space $H$. If $A$ is invertible and $\text{Sp}(A) \subset (0, 1/2)$, then
\[
(1 \leq \exp \left\langle \ln (1_H - A)^{-1} x, x \right\rangle \\
\left( \frac{1 - \langle Ax, x \rangle}{\langle Ax, x \rangle} \right)^{-1} \right) \\
\leq \exp \left[ r \left( \langle Ax, x \rangle \cdot \langle A^{-1} (1_H - A)^{-1} x, x \rangle - \langle (1_H - A)^{-1} x, x \rangle \right) \right]
\]
and

\begin{equation}
1 \leq \left\langle \exp \left[ r \left( 1 - \langle Ax, x \rangle \right)^{-1} \left( 1_H - \langle Ax, x \rangle^{-1} A \right) \right] x, x \right\rangle
\leq \frac{\left\langle \left( (1_H - A)^{A^1} \right)^r x, x \right\rangle}{\left\langle \left( 1 - \langle Ax, x \rangle \right) \langle Ax, x \rangle^{-1} \right\rangle^r}
\leq \left\langle \exp \left[ r \left( 1_H - A \right)^{-1} \left( \langle Ax, x \rangle A^{-1} - 1_H \right) \right] x, x \right\rangle
\end{equation}

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

References


