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Investigation of Some Conditions on $N(k)$-Quasi Einstein Manifolds

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Abstract. We consider $N(k)$-quasi Einstein manifolds satisfying the conditions $R(\xi, X) \cdot H = 0$, $H(\xi, X) \cdot S = 0$, $P(\xi, X) \cdot H = 0$, $R(\xi, X) \cdot \bar{P} = 0$ and $\bar{P}(\xi, X) \cdot S = 0$ where H, P and $\bar{P}$ denote the conharmonic curvature tensor, the projective curvature tensor and pseudo projective curvature tensor, respectively.

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1. Introduction

The notion of a quasi Einstein manifold was introduced by Chaki in [1]. A non flat n-dimensional Riemannian manifold (M, g) is said to be a quasi Einstein manifold if its Ricci tensor S satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \ \forall X, Y \in TM \tag{1.1}$$

for some smooth functions a and $b \neq 0$, where η is a non zero 1-form such that

$$g(X, \xi) = \eta(X), \ g(\xi, \xi) = \eta(\xi) = 1 \tag{1.2}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If $b = 0$ then the manifold reduces to an Einstein manifold. For more details about quasi Einstein manifolds see also [2, 6].

In [15], it was shown that a conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(k)$-quasi Einstein manifold. The derivation conditions $R(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot S = 0$ were also studied in [15], where R and S denote the curvature and Ricci tensor, respectively. In [10], derivation conditions $R(\xi, X) \cdot \rho = 0$, $\rho(\xi, X) \cdot S = 0$ and $\rho(\xi, X) \cdot \rho = 0$ were studied where ρ is the projective curvature tensor, also

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physical examples of $N(k)$-quasi Einstein manifolds were given. The derivation conditions $R(\xi,X)\cdot C=0$, $R(\xi,X)\cdot\tilde{C}=0$ studied in [11], where C and $\tilde{C}$ denote the conformal curvature tensor and quasi conformal curvature tensor, respectively. In this paper, we consider $N(k)$-quasi Einstein manifolds satisfying the conditions $R(\xi,X)\cdot H=0$, $H(\xi,X)\cdot S=0$, $P(\xi,X)\cdot H=0$, $R(\xi,X)\cdot\bar{P}=0$ and $\bar{P}(\xi,X)\cdot S=0$, where H, P and $\bar{P}$ denote the conharmonic curvature tensor, the projective curvature tensor and the pseudo projective curvature tensor, respectively.

2. $N(k)$-quasi Einstein manifolds

From (1.1) and (1.2) we obtain

$$S(X,\xi)=(a+b)\eta(X), \tag{2.1}$$

$$r=na+b \tag{2.2}$$

where r is the scalar curvature of M.

The Ricci operator Q of a Riemannian manifold (M,g) is defined by

$$S(X,Y)=g(QX,Y).$$

If (M,g) is a quasi Einstein manifold [1], its Ricci operator satisfies

$$Q=aI+b\eta\otimes\xi.$$

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ [14] of a Riemannian manifold defined by

$$N(k):p\longrightarrow N_p(k)=\{Z\in T_pM\mid R(X,Y)Z=k\{g(Y,Z)X-g(X,Z)Y\}\}$$

for all X , $Y\in TM^n$, where k is some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then is said to be an $N(k)$-quasi Einstein manifold [15].

Lemma 2.1. [12] *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k=\frac{a+b}{n-1}. \tag{2.3}$$

Let (M^n,g) be an $N(k)$-quasi Einstein manifold. Then, we have [12]

$$R(Y,Z)\xi=\frac{a+b}{n-1}\{\eta(Z)Y-\eta(Y)Z\}. \tag{2.4}$$

The equation (2.4) is equivalent to

$$R(\xi,Y)Z=\frac{a+b}{n-1}\{g(Y,Z)\xi-\eta(Z)Y\}=-R(Y,\xi)Z. \tag{2.5}$$

In [10], we view the following physical examples of $N(k)$-quasi Einstein manifolds.

In [15], Tripathi and Kim proved that an n-dimensional conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold. Now we consider a conformally flat perfect fluid spacetime (M^4,g) satisfying Einstein's equation without cosmological constant. Further, let ξ be the unit time-like velocity vector of the fluid. It is known [9] that Einstein's equation without cosmological constant can be written as

$$S(X,Y)-\frac{1}{2}rg(X,Y)=\kappa T(X,Y), \tag{2.6}$$

where κ is the gravitational constant and T is the energy momentum tensor of type (0, 2). In the present case (2.6) can be written as follows:

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa[(\sigma + p)\eta(X)\eta(Y) + pg(X,Y)],$$

where σ is the energy density and p is the isotropic pressure of the fluid. Then we have

$$S(X,Y) = \left(\kappa p + \frac{1}{2}r\right) g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{2.7}$$

Since the space-time is conformally flat, by [15], it is $N(k)$-quasi Einstein. From (2.7), by a contraction we get

$$r = \kappa(\sigma - 3p).$$

Hence the equation (2.7) can be written as

$$S(X,Y) = \left(\frac{\kappa}{2}(\sigma - p)\right) g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y).$$

So from (1.1) we have

$$a = \frac{\kappa}{2}(\sigma - p)$$

and

$$b = \kappa(\sigma + p).$$

Hence we can state the following example.

Example 2.1. [10] A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation without cosmological constant is an $N(\kappa(3\sigma + p)/6)$-quasi Einstein manifold.

Now we consider a conformally flat perfect fluid spacetime (M^4, g) satisfying Einsteins equation with cosmological constant. Further, let ξ be the unit time-like velocity vector of the fluid. The Einstein's equation can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \lambda g(X,Y) = \kappa[(\sigma + p)\eta(X)\eta(Y) + pg(X,Y)],$$

which gives us

$$S(X,Y) = \left(\kappa p + \frac{1}{2}r - \lambda\right) g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{2.8}$$

So from (2.8), by a contraction, we get

$$r = 4\lambda + \kappa(\sigma - 3p).$$

Hence the equation (2.8) turns into

$$S(X,Y) = \left(\lambda + \frac{\kappa}{2}(\sigma - p)\right) g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y).$$

So from (1.1) we have

$$a = \lambda + \frac{\kappa}{2}(\sigma - p)$$

and

$$b = \kappa(\sigma + p).$$

Since $k = (a + b)/(n - 1)$ we obtain

$$k = \frac{\lambda}{3} + \frac{\kappa(3\sigma + p)}{6}.$$

So as a generalization of Example 2.1, we obtain the following example.

Example 2.2. [10] A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation with cosmological constant is an $N((\lambda/3)+(\kappa(3\sigma + p)/6))$-quasi Einstein manifold.

3. Conharmonic curvature tensor of an $N(k)$-quasi Einstein manifold

Let (M^n, g) be a Riemannian manifold. The conharmonic curvature tensor [7] is defined by

$$\begin{aligned} H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{&S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\}, \end{aligned} \tag{3.1}$$

where Q is the Ricci operator.

Also $R \cdot H$ is defined by

$$\begin{aligned} (R(\xi,X) \cdot H)(Y,Z,W) = {} & R(\xi,X)H(Y,Z)W - H(R(\xi,X)Y,Z)W \\ & - H(Y,R(\xi,X)Z)W) - H(Y,Z)R(\xi,X)W, \end{aligned} \tag{3.2}$$

where R denote the Riemannian curvature tensor of a Riemannian manifold M [8].

Now, we prove the following theorem.

Theorem 3.1. *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $R(\xi,X) \cdot H = 0$ if and only if $a + b = 0$ or*

$$\acute{H}(Y,Z,W,X) = -\frac{na+b}{(n-1)(n-2)}\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\},$$

where $\acute{H}(Y,Z,W,X) = g(H(Y,Z)W,X)$.

Proof. Let M be an $N(k)$-quasi Einstein manifold and satisfies the condition $R(\xi,X) \cdot H = 0$, then from (3.2) we can write

$$\begin{aligned} 0 = {} & R(\xi,X)H(Y,Z)W - H(R(\xi,X)Y,Z)W \\ & - H(Y,R(\xi,X)Z)W - H(Y,Z)R(\xi,X)W \end{aligned} \tag{3.3}$$

for all vector fields X, Y, Z, W on M. So from (2.5) in (3.3) we obtain

$$\begin{aligned} 0 = \frac{a+b}{n-1}\{&\acute{H}(Y,Z,W,X)\xi - \eta(H(Y,Z)W)X \\ & - g(X,Y)H(\xi,Z)W + \eta(Y)H(X,Z)W \\ & - g(X,Z)H(Y,\xi)W + \eta(Z)H(Y,X)W \\ & - g(X,W)H(Y,Z)\xi + \eta(W)H(Y,Z)X\}, \end{aligned}$$

which implies either $a + b = 0$ or

$$\begin{aligned} 0 = {} & \acute{H}(Y,Z,W,X)\xi - \eta(H(Y,Z)W)X \\ & - g(X,Y)H(\xi,Z)W + \eta(Y)H(X,Z)W \end{aligned} \tag{3.4}$$

$$- g(X,Z)H(Y,\xi)W + \eta(Z)H(Y,X)W - g(X,W)H(Y,Z)\xi + \eta(W)H(Y,Z)X,$$

holds on M. Taking the inner product of both sides of (3.4) with ξ we obtain

$$\begin{aligned} 0 = \acute{H}(Y,Z,W,X) &- \eta(H(Y,Z)W)\eta(X) \\ &- g(X,Y)\eta(H(\xi,Z)W) + \eta(Y)\eta(H(X,Z)W) \\ &- g(X,Z)\eta(H(Y,\xi)W) + \eta(Z)\eta(H(Y,X)W) \\ &- g(X,W)\eta(H(Y,Z)\xi) + \eta(W)\eta(H(Y,Z)X). \end{aligned} \tag{3.5}$$

On the other hand, since H is conharmonic curvature tensor from (1.1), (3.1) and (2.5) we have

$$\eta(H(X,Y)Z) = -\frac{na+b}{(n-1)(n-2)}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{3.6}$$

for all vector fields X, Y, Z on M. So putting (3.6) in (3.5) we obtain

$$0 = \acute{H}(Y,Z,W,X) + \frac{na+b}{(n-1)(n-2)}\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\}.$$

Hence we have

$$\acute{H}(Y,Z,W,X) = -\frac{na+b}{(n-1)(n-2)}\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\}.$$

The converse statement is trivial. This completes the proof of the theorem. ∎

Next, we have the following theorem

Theorem 3.2. *Let M be an n-dimensional N(k)-quasi Einstein manifold. Then M satisfies the condition $H(\xi,X)\cdot S = 0$ if and only if $na+b=0$.*

Proof. Since $H(\xi,X)\cdot S = 0$, we have

$$S(H(\xi,X)Y,Z) + S(Y,H(\xi,X)Z) = 0. \tag{3.7}$$

In view of (1.1) in (3.7) we have

$$b[\eta(H(\xi,X)Y)\eta(Z) + \eta(Y)\eta(H(\xi,X)Z)] = 0. \tag{3.8}$$

Since $b \neq 0$, then from (3.8) we have

$$\eta(H(\xi,X)Y)\eta(Z) + \eta(Y)\eta(H(\xi,X)Z) = 0. \tag{3.9}$$

In view of (3.6) in (3.9) we have

$$\frac{na+b}{(n-1)(n-2)}\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\} = 0. \tag{3.10}$$

From (3.10) by a contraction, we obtain

$$\frac{na+b}{n-2} = 0,$$

which give us $na+b=0$. The converse statement is trivial. This completes the proof of the theorem. ∎

Let (M^n, g) be a Riemannian manifold. The projective curvature tensor [16] is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}.$$

If P is a projective curvature tensor in an n-dimensional $N(k)$-quasi Einstein manifold, we have [10]

$$P(\xi, X)Y = \frac{b}{n-1}\{g(X,Y)\xi - \eta(X)\eta(Y)\xi\}. \tag{3.11}$$

Theorem 3.3. *Let that M is an n-dimensional $N(k)$-quasi Einstein manifold. If M satisfies the condition $P(\xi, X)\cdot H = 0$ then $k = 1/(n-1)$ or M is conharmonically flat.*

Proof. Assume that M, is $N(k)$-quasi Einstein manifold such that satisfies the condition $P(\xi, X)\cdot H = 0$. We can write

$$\begin{aligned}0 = {} & P(\xi,X)H(Y,Z)W - H(P(\xi,X)Y,Z)W \\ & - H(Y,P(\xi,X)Z)W - H(Y,Z)P(\xi,X)W\end{aligned} \tag{3.12}$$

for all vector fields X, Y, Z, W on M. So from (3.11) in (3.12) we obtain

$$\begin{aligned}0 = {} & \frac{b}{n-1}\{\acute{H}(Y,Z,W,X)\xi - \eta(H(Y,Z)W)\eta(X)\xi \\ & - g(X,Y)H(\xi,Z)W + \eta(X)\eta(Y)H(\xi,Z)W \\ & - g(X,Z)H(Y,\xi)W + \eta(Z)\eta(X)H(Y,\xi)W \\ & - g(X,W)H(Y,Z)\xi + \eta(X)\eta(W)H(Y,Z)\xi\}.\end{aligned}$$

Since $b \neq 0$ we have

$$\begin{aligned}0 = {} & \acute{H}(Y,Z,W,X)\xi - \eta(H(Y,Z)W)\eta(X)\xi \\ & - g(X,Y)H(\xi,Z)W + \eta(X)\eta(Y)H(\xi,Z)W \\ & - g(X,Z)H(Y,\xi)W + \eta(Z)\eta(X)H(Y,\xi)W \\ & - g(X,W)H(Y,Z)\xi + \eta(X)\eta(W)H(Y,Z)\xi.\end{aligned} \tag{3.13}$$

Taking the inner product of (3.13) by ξ, we obtain

$$\begin{aligned}0 = {} & \acute{H}(Y,Z,W,X) - \eta(H(Y,Z)W)\eta(X) \\ & - g(X,Y)\eta(H(\xi,Z)W) + \eta(X)\eta(Y)\eta(H(\xi,Z)W) \\ & - g(X,Z)\eta(H(Y,\xi)W) + \eta(Z)\eta(X)\eta(H(Y,\xi)W) \\ & - g(X,W)\eta(H(Y,Z)\xi) + \eta(X)\eta(W)\eta(H(Y,Z)\xi).\end{aligned} \tag{3.14}$$

From (3.6) in (3.14) we have

$$\begin{aligned}0 = {} & \acute{H}(Y,Z,W,X) - \frac{na+b}{(n-1)(n-2)}\{g(X,Y)g(Z,W) \\ & - g(X,Z)g(Y,W) + g(X,Z)\eta(Y)\eta(W) - S(X,Z)g(Y,W)\}.\end{aligned} \tag{3.15}$$

In view of (3.1) and (3.15) we have

$$0 = \acute{R}(Y,Z,W,X) - \frac{1}{n-2}\{S(Z,W)g(X,Y) \tag{3.16}$$

$$
\begin{aligned}
&- S(Y,W)g(X,Z) + g(Z,W)S(X,Y) \\
&- S(Z,X)g(W,Y)\} + \frac{na+b}{(n-1)(n-2)}\{g(X,Y)g(Z,W) \\
&- g(X,Z)g(Y,W) + g(X,Z)\eta(Y)\eta(W) \\
&- S(X,Z)g(Y,W)\}.
\end{aligned}
$$

From (3.16) by a contraction, we obtain

$$\frac{na+b}{n-2}\{\eta(Z)\eta(W) - S(Z,W)\} = 0,$$

which gives us either $na+b=0$ or $\eta(Z)\eta(W) - S(Z,W) = 0$ (this means that $a=0$ and $b=1$). If $na+b=0$, then from (3.15) we have $\acute{H}(Y,Z,W,X)=0$. Also if $S(Z,W)=\eta(Z)\eta(W)$, then from Lemma 2.1 we have $k=1/(n-1)$. This completes the proof of the theorem. ∎

4. Pseudo-projective curvature tensor of an $N(k)$-quasi Einstein manifold

The Pseudo-projective curvature tensor $\bar{P}$ on a manifold M of dimension n is defined by [13]

$$
\begin{aligned}
P(X,Y)Z &= \alpha R(X,Y)Z + \beta\{S(Y,Z)X - S(X,Z)Y\} \\
&\quad - \frac{r}{n}\left[\frac{\alpha}{n-1} + \beta\right]\{g(Y,Z)X - g(X,Z)Y\},
\end{aligned}
\tag{4.1}
$$

where a and b are constants such that a, $b \neq 0$ and R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

Proposition 4.1. *In an n-dimensional N(k)-quasi Einstein manifold M, the Pseudo-projective curvature tensor $\bar{P}$ satisfies*

$$\bar{P}(X,Y)\xi = \left[\frac{((n-1)\beta+\alpha)b}{n}\right]\{\eta(Y)X - \eta(X)Y\} \tag{4.2}$$

$$\eta(\bar{P}(X,Y)Z) = \left[\frac{(\alpha-\beta)b}{n}\right]\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{4.3}$$

$$\bar{P}(\xi,X)Y = \left[\frac{(\alpha-\beta)b}{n}\right]\{g(X,Y)\xi - \eta(Y)X\} + \beta b\{\eta(X)\eta(Y)\xi - \eta(Y)X\} \tag{4.4}$$

for all vector fields X, Y, Z on M.

Proof. From (1.2), (2.1), (2.2), (2.4) and (4.1) the equations (4.2)–(4.4) follow easily. ∎

Theorem 4.1. *Let M be an n-dimensional N(k)-quasi Einstein manifold. Then M satisfies the condition $R(\xi,X)\cdot\bar{P}=0$ if and only if $a+b=0$.*

Proof. Assume that M is an n-dimensional $N(k)$-quasi Einstein manifold and satisfies the condition $R(\xi,X)\cdot\bar{P}=0$ we can write

$$
\begin{aligned}
0 &= R(\xi,X)\bar{P}(Y,Z)W - \bar{P}(R(\xi,X)Y,Z)W \\
&\quad - \bar{P}(Y,R(\xi,X)Z)W - \bar{P}(Y,Z)R(\xi.X)W
\end{aligned}
\tag{4.5}
$$

for all vector fields X, Y, Z, W on M.

Using (2.5), in (4.5) we find

$$\begin{aligned}0 = \frac{a+b}{n-1}\{&\bar{P}(Y,Z,W,X)\xi - \eta(\bar{P}(Y,Z)W)X \\ &- g(X,Y)\bar{P}(\xi,Z)W + \eta(Y)\bar{P}(X,Z)W \\ &- g(X,Z)\bar{P}(Y,\xi)W + \eta(Z)\bar{P}(Y,X)W \\ &- g(X,W)\bar{P}(Y,Z)\xi + \eta(W)\bar{P}(Y,Z)X\},\end{aligned}$$

which implies either $a+b=0$ or

$$\begin{aligned}0 = \grave{\bar{P}}&(Y,Z,W,X)\xi - \eta(\bar{P}(Y,Z)W)X \\ &- g(X,Y)\bar{P}(\xi,Z)W + \eta(Y)\bar{P}(X,Z)W \\ &- g(X,Z)\bar{P}(Y,\xi)W + \eta(Z)\bar{P}(Y,X)W \\ &- g(X,W)\bar{P}(Y,Z)\xi + \eta(W)\bar{P}(Y,Z)X,\end{aligned} \tag{4.6}$$

where $\grave{\bar{P}}(Y,Z,W,X) = g(\bar{P}(Y,Z)W,X)$. Assume that $a+b \neq 0$. Taking the inner product of (4.6) with ξ we obtain

$$\begin{aligned}0 = \grave{\bar{P}}&(Y,Z,W,X) - \eta(\bar{P}(Y,Z)W)\eta(X) \\ &- g(X,Y)\eta(\bar{P}(\xi,Z)W) + \eta(Y)\eta(\bar{P}(X,Z)W) \\ &- g(X,Z)\eta(\bar{P}(Y,\xi)W) + \eta(Z)\eta(\bar{P}(Y,X)W) \\ &- g(X,W)\eta(\bar{P}(Y,Z)\xi) + \eta(W)\eta(\bar{P}(Y,Z)X).\end{aligned} \tag{4.7}$$

Hence in view of (4.2)–(4.4) the equation (4.7) is reduced to

$$0 = \grave{\bar{P}}(Y,Z,W,X) - \left[\frac{(\alpha-\beta)b}{n}\right]\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\}. \tag{4.8}$$

From (4.1) in (4.8) we obtain

$$\begin{aligned}0 = \alpha R(Y,Z,W,X) &+ \beta[S(Z,W)g(X,Y) - S(Y,W)g(X,Z)] \\ - \frac{r}{n}&\left[\frac{\alpha}{n-1} + \beta\right]\{g(Z,W)g(X,Y) - g(Y,W)g(X,Z)\} \\ - &\left[\frac{(\alpha-\beta)b}{n}\right]\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\}.\end{aligned} \tag{4.9}$$

So by a suitable contraction of (4.9) we get

$$[(n-1)\beta + \alpha]S(Z,W) = [a(\alpha + (n-1)\beta) + \alpha b]g(Z,W). \tag{4.10}$$

If $\beta = -\alpha/(n-1)$, then from (4.10) we have

$$\alpha b g(Z,W) = 0.$$

This contradicts to our assumption that M is an $N(k)$-quasi Einstein manifold.

Also if $\beta \neq -\alpha/(n-1)$, then from (4.10) we have

$$S(Z,W) = \left[a + \frac{\alpha b}{(n-1)\beta + \alpha}\right]g(Z,W).$$

Since M is not an Einstein manifold this is not possible. The converse statement is trivial. This completes the proof of the theorem. ∎

Next, we have the following theorem.

Theorem 4.2. *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $\bar{P}(\xi, X) \cdot S = 0$ if and only if $\alpha = (\frac{na+b}{b})\beta$.*

Proof. The condition $\bar{P}(\xi, X) \cdot S = 0$, implies that

$$S(\bar{P}(\xi, X)Y, Z) + S(Y, \bar{P}(\xi, X)Z) = 0. \tag{4.11}$$

In view of (4.4) in (4.11) we get

$$b\left[a\beta + \frac{b(\beta-\alpha)}{n}\right]\{g(X,Z)\eta(Y) + g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)\} = 0. \tag{4.12}$$

From (4.12), by a contraction, we get

$$(n-1)b\left[a\beta + \frac{b(\beta-\alpha)}{n}\right]\eta(Y) = 0. \tag{4.13}$$

Since $b \neq 0$, from (4.13) we have

$$a\beta + \frac{b(\beta-\alpha)}{n} = 0. \tag{4.14}$$

From (4.14) we get $\alpha = (na + b)\beta/b$. The converse statement is trivial. This completes the proof of the theorem. ∎

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