

Matrix Summability Methods on The Approximation of Multivariate q -MKZ Operators

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Abstract. In this paper, a q -based generalization of Meyer-König and Zeller (MKZ) operators in several variables are introduced. A Korovkin-type approximation theorem via A -statistical convergence is obtained and their various A -statistical approximation properties are investigated when A is any non-negative regular summability matrix.

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1. Introduction

The idea of approximating a function by means of a sequence of positive linear operators was first noticed by Korovkin [12] (see also [2]). He investigated this problem, the so-called Korovkin-type approximation theory, when the function is continuous in the algebraic case, and it is continuous with period 2π in the trigonometric case. Later many mathematicians studied and improved this theory not only by defining positive linear operators on various functions spaces but also by using summability methods instead of the classical convergence method. Especially, in recent years, considering the concept of A -statistical convergence in the Korovkin-type approximation theory settings, where A is a non-negative regular summability matrix, many powerful results have been obtained (see, for instance, [4, 6, 7]). Our primary interest of the present paper are to introduce the q -based generalization of the classical

Meyer-König and Zeller (MKZ) operators, which is one of the well-known approximating operators, and to investigate their approximation properties via A -statistical convergence.

In order to get some monotonicity properties Cheney and Sharma [3] gave a slight modification of the classical MKZ operators. Later, Trif [16] and Dođru and Duman [4] introduced some generalizations of this modification of MKZ operators by using the q -integers instead of the natural numbers. Furthermore, some other generalizations and various properties of these operators may be found in the papers [1, 5, 10, 11, 14, 15].

As usual, for a fixed real number $q > 0$, the q -integers, q -binomial coefficients and q -factorials are defined as follows:

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & \text{if } q \neq 1 \\ k, & \text{if } q = 1, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}, \quad (k = 0, 1, \dots, n),$$

$$[k]_q! := \begin{cases} [1]_q [2]_q \dots [k]_q, & \text{if } k \geq 1 \\ 1, & \text{if } k = 0. \end{cases}$$

Now we recall that, for an infinite non-negative regular summability matrix $A = (a_{jn})$, a sequence $x := (x_n)$ is called A -statistically convergent to a number L if, for every $\varepsilon > 0$, $\lim_{j \rightarrow \infty} \sum_{n: |x_n - L| \geq \varepsilon} a_{jn} = 0$ holds. This limit is denoted by $st_A - \lim x = L$ [9]. In the case $A = C_1$, the Cesáro matrix, C_1 -statistical convergence coincides with statistical convergence (see [8]). Moreover if $A = I$, the identity matrix, then it reduces to ordinary convergence. Also, every convergent sequence is A -statistically convergent, but the converse is not always true.

2. Construction of the operators

The aim of this section is to introduce a q -based generalization of MKZ operators in several variables. Let $B(I^r)$ be the space of all real valued bounded functions defined on the set $I^r = [0, B_1] \times \dots \times [0, B_r]$, ($0 < B_i < 1$; $i = 1, 2, \dots, r$), endowed with the usual supremum norm

$$\|f\| := \|f\|_{B(I^r)} = \sup_{(x_1, \dots, x_r) \in I^r} |f(x_1, \dots, x_r)|.$$

Let w be a function of the type of modulus of continuity. So, throughout the paper we assume the following conditions on the function w :

- (a) w is a non-negative increasing function on $[0, \infty)$,
- (b) $w(\delta_1 + \delta_2) \leq w(\delta_1) + w(\delta_2)$ for any $\delta_1, \delta_2 > 0$,
- (c) $\lim_{\delta \rightarrow 0} w(\delta) = 0$.

Then, by $H_w(I^r)$ we denote the space of all real-valued functions f defined on I^r such that

$$|f(u_1, \dots, u_r) - f(x_1, \dots, x_r)| \leq w \left(\sqrt{\sum_{v=1}^r \left(\frac{u_v}{1-u_v} - \frac{x_v}{1-x_v} \right)^2} \right)$$

holds for every $(u_1, \dots, u_r), (x_1, \dots, x_r) \in I^r$. It is clear that if $f \in H_w(I^r)$, then

$$|f(u_1, \dots, u_r)| \leq |f(0, \dots, 0)| + w \left(\sqrt{\sum_{v=1}^r \left(\frac{B_v}{1 - B_v} \right)^2} \right).$$

Therefore, f is bounded whenever $f \in H_w(I^r)$.

Now we introduce the following q -type generalization of MKZ operators in several variables.

$$\begin{aligned} M_n(f; q_1, \dots, q_r; x_1, \dots, x_r) &= \left\{ \prod_{i=1}^r u_{n, q_i}(x) \right\} \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} f \left(\frac{q_1^n [k_1]_{q_1}}{[n + k_1]_{q_1}}, \dots, \frac{q_r^n [k_r]_{q_r}}{[n + k_r]_{q_r}} \right) \\ (2.1) \quad &\times \prod_{m=1}^r \begin{bmatrix} n + k_m \\ k_m \end{bmatrix}_{q_m} x_m^{k_m} \end{aligned}$$

where

$$(x_1, \dots, x_r) \in I^r, \quad f \in H_w(I^r),$$

and for each $i = 1, \dots, r$

$$(2.2) \quad u_{n, q_i}(x) = \prod_{s=0}^n (1 - q_i^s x_i) \quad \text{and} \quad 0 < q_i \leq 1.$$

One can easily see that the operators given by (2.1) are positive and linear. In the case of $r = 1$ and $q = 1$, they reduce to the Cheney-Sharma operators in [3]. Taking $r = 1$ and $r = 2$, we have the operators considered in [4] and [5], respectively. Also observe that taking $r = 1$ and $f \left(\frac{[k]_q}{[k+n]_q} \right)$ in place of $f \left(\frac{q^n [k]_q}{[k+n]_q} \right)$ in (2.1), we get Trif's generalization in [16].

3. Approximation properties of the operators

In the present section, we obtain a statistical Korovkin-type approximation theorem for the operators (2.1) with the help of the test functions

$$\begin{aligned} f_0(u_1, \dots, u_r) &= 1, \\ f_v(u_1, \dots, u_r) &= \frac{u_v}{1 - u_v}, \quad (v = 1, \dots, r), \\ f_{r+1}(u_1, \dots, u_r) &= \sum_{v=1}^r f_v^2(u_1, \dots, u_r) = \sum_{v=1}^r \left(\frac{u_v}{1 - u_v} \right)^2. \end{aligned}$$

Now let w be a function of the type of modulus of continuity satisfying the conditions (a) – (c) for which all the test functions mentioned above belong to $H_w(I^r)$. Then we first need the following statistical Korovkin theorem using the above test functions.

Theorem 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of positive linear operators from $B(I^r)$ into $B(I^r)$. Let w be a fixed function of the type of modulus of continuity satisfying (a) – (c). If, for each $v = 0, 1, \dots, r + 1$,*

$$(3.1) \quad st_A - \lim_n \|L_n(f_v) - f_v\| = 0$$

holds, then, for all $f \in H_w(I^r)$, we have

$$(3.2) \quad \text{st}_A - \lim_n \|L_n(f) - f\| = 0.$$

Proof. Let w be a fixed function of the type of modulus of continuity satisfying (a) – (c). Assume that (3.1) holds and $f \in H_w(I^r)$. Let $(x_1, \dots, x_r) \in I^r$ be fixed. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that the inequality

$$(3.3) \quad |f(u_1, \dots, u_r) - f(x_1, \dots, x_r)| \leq \varepsilon + \frac{2M}{\delta^2} \sum_{v=1}^r \left(\frac{u_v}{1-u_v} - \frac{x_v}{1-x_v} \right)^2.$$

holds. Now, the linearity and positivity of the operators give that

$$\begin{aligned} & |L_n(f; x_1, \dots, x_r) - f(x_1, \dots, x_r)| \\ & \leq \varepsilon + \left\{ \varepsilon + M + \frac{2rMB}{\delta^2} \right\} |L_n(f_0; x_1, \dots, x_r) - f_0(x_1, \dots, x_r)| \\ & \quad + \frac{4MB}{\delta^2} \sum_{v=1}^r |L_n(f_v; x_1, \dots, x_r) - f_v(x_1, \dots, x_r)| \\ & \quad + \frac{2M}{\delta^2} |L_n(f_{r+1}; x_1, \dots, x_r) - f_{r+1}(x_1, \dots, x_r)|, \end{aligned}$$

where $B := \max \left\{ \frac{B_1}{1-B_1}, \dots, \frac{B_r}{1-B_r}, \left(\frac{B_1}{1-B_1} \right)^2, \dots, \left(\frac{B_r}{1-B_r} \right)^2 \right\}$. Now setting

$$C := \varepsilon + M + \frac{2M}{\delta^2} \{Br + 2B + 1\}$$

we conclude that

$$(3.4) \quad \|L_n(f) - f\| \leq \varepsilon + C \sum_{v=0}^{r+1} \|L_n(f_v) - f_v\|.$$

Now for a given $s > 0$ choose $\varepsilon > 0$ such that $\varepsilon < s$. Then define the following sets:

$$\begin{aligned} D & := \{n : \|L_n(f) - f\| \geq s\}, \\ D_v & := \left\{ n : \|L_n(f_v) - f_v\| \geq \frac{s - \varepsilon}{C(r + 2)} \right\}, \quad (v = 0, 1, \dots, r + 1). \end{aligned}$$

Inequality (3.4) immediately implies that $D \subseteq \bigcup_{v=0}^{r+1} D_v$. Then we have

$$(3.5) \quad \sum_{n \in D} a_{jn} \leq \sum_{v=0}^{r+1} \sum_{n \in D_v} a_{jn}.$$

Taking limit as $j \rightarrow \infty$ in (3.5) and applying (3.1) we easily see that

$$\lim_j \sum_{n \in D} a_{jn} = 0,$$

which completes the proof. ■

We now turn to our operators defined by (2.1). To get a Korovkin-type result for these operators we first need the following auxiliary results.

Lemma 3.1. *For all $n \in \mathbb{N}$, $(x_1, \dots, x_r) \in I^r$ and $0 < q_i \leq 1$ ($i = 1, 2, \dots, r$), the following conditions hold:*

- (i) $M_n(f_0; q_1, \dots, q_r; x_1, \dots, x_r) = 1,$
- (ii) $M_n(f_v; q_1, \dots, q_r; x_1, \dots, x_r) = \frac{[n+1]_{q_v} q_v^n x_v}{[n]_{q_v}^2 (1 - q_v^{n+1} x_v)}, \quad (v = 1, \dots, r),$
- (iii)

$$M_n(f_{r+1}; q_1, \dots, q_r; x_1, \dots, x_r) = \sum_{v=1}^r \left\{ \frac{[n+1]_{q_v} [n+2]_{q_v} q_v^{2n+1} x_v^2}{[n]_{q_v}^2 (1 - q_v^{n+1} x_v)(1 - q_v^{n+2} x_v)} + \frac{[n+1]_{q_v} q_v^{2n} x_v}{[n]_{q_v}^2 (1 - q_v^{n+1} x_v)} \right\}.$$

Proof. (i) It immediately follows from the definition of the operators.
 (ii) For $v = 1, \dots, r,$ observe that

$$(3.6) \quad [n+k_v]_{q_v} = [n]_{q_v} + q_v^n [k_v]_{q_v} \text{ and } q_v [k_v - 1]_{q_v} = [k_v]_{q_v} - 1.$$

Then, using the first equality in (3.6) we get, for each $v = 1, \dots, r,$ that

$$\begin{aligned} M_n(f_v; q_1, \dots, q_r; x_1, \dots, x_r) &= \left\{ \prod_{i=1}^r u_{n, q_i}(x) \right\} \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} \frac{q_v^n [k_v]_{q_v}}{[n]_{q_v}} \\ &\quad \times \prod_{m=1}^r \begin{bmatrix} n+k_m \\ k_m \end{bmatrix}_{q_m} x_m^{k_m} \\ &= \frac{q_v^n}{[n]_{q_v}} u_{n, q_v}(x) \sum_{k_v=1}^{\infty} [k_v]_{q_v} \begin{bmatrix} n+k_v \\ k_v \end{bmatrix}_{q_v} x_v^{k_v} \\ &= \frac{q_v^n [n+1]_{q_v} x_v}{[n]_{q_v} (1 - q_v^{n+1} x_v)}, \end{aligned}$$

which gives (ii).

(iii) Now we first compute the value $M_n(f_v^2; q_1, \dots, q_r; x_1, \dots, x_r)$ for each $v = 1, \dots, r.$ Applying the second equality in (3.6), we have

$$\begin{aligned} M_n(f_v^2; q_1, \dots, q_r; x_1, \dots, x_r) &= \frac{q_v^{2n}}{[n]_{q_v}^2} u_{n, q_v}(x) \sum_{k_v=1}^{\infty} [k_v]_{q_v}^2 \begin{bmatrix} n+k_v \\ k_v \end{bmatrix}_{q_v} x_v^{k_v} \\ &= \frac{q_v^{2n}}{[n]_{q_v}^2} u_{n, q_v}(x) \sum_{k_v=1}^{\infty} \frac{[k_v]_{q_v} [n+k_v]_{q_v}!}{[n]_{q_v}! [k_v-1]_{q_v}!} x_v^{k_v} \\ &= \frac{q_v^{2n}}{[n]_{q_v}^2} u_{n, q_v}(x) \left\{ q_v x_v^2 \sum_{k_v=2}^{\infty} \frac{[n+k_v]_{q_v}!}{[n]_{q_v}! [k_v-2]_{q_v}!} x_v^{k_v-2} \right. \\ &\quad \left. + x_v \sum_{k_v=1}^{\infty} \frac{[n+k_v]_{q_v}!}{[n]_{q_v}! [k_v-1]_{q_v}!} x_v^{k_v-1} \right\}. \end{aligned}$$

So we may write that

$$M_n(f_v^2; q_1, \dots, q_r; x_1, \dots, x_r) = \frac{[n+1]_{q_v} [n+2]_{q_v} q_v^{2n+1} x_v^2}{[n]_{q_v}^2 (1 - q_v^{n+1} x_v)(1 - q_v^{n+2} x_v)}$$

$$(3.7) \quad + \frac{[n+1]_{q_v} q_v^{2n} x_v}{[n]_{q_v}^2 (1 - q_v^{n+1} x_v)}.$$

Since $f_{r+1} \equiv \sum_{v=1}^r f_v^2$, we obtain from (3.7) that

$$\begin{aligned} M_n(f_{r+1}; q_1, \dots, q_r; x_1, \dots, x_r) &= \sum_{v=1}^r M_n(f_v^2; q_1, \dots, q_r; x_1, \dots, x_r) \\ &= \sum_{v=1}^r \left\{ \frac{[n+1]_{q_v} [n+2]_{q_v} q_v^{2n+1} x_v^2}{[n]_{q_v}^2 (1 - q_v^{n+1} x_v)(1 - q_v^{n+2} x_v)} \right. \\ &\quad \left. + \frac{[n+1]_{q_v} q_v^{2n} x_v}{[n]_{q_v}^2 (1 - q_v^{n+1} x_v)} \right\}, \end{aligned}$$

whence the result. ■

Lemma 3.2. For all $n \in \mathbb{N}$, and $0 < q_i \leq 1$ ($i = 1, 2, \dots, r$), the following conditions hold:

- (i) $\|M_n(f_0; q_1, \dots, q_r; \cdot) - f_0\| = 0$,
- (ii) $\|M_n(f_v; q_1, \dots, q_r; \cdot) - f_v\| \leq B \left\{ \left| \frac{[n+1]_{q_v} q_v^n}{[n]_{q_v}} - 1 \right| + |1 - q_v^{n+1}| \right\}$,
 $(v = 1, \dots, r)$,
- (iii) $\|M_n(f_{r+1}; q_1, \dots, q_r; \cdot) - f_{r+1}\|$
 $\leq B \sum_{v=1}^r \left\{ \left| \frac{[n+1]_{q_v} [n+2]_{q_v} q_v^{2n+1}}{[n]_{q_v}^2} - 1 \right| \right.$
 $\left. + |1 - q_v^{2n+3}| + |2 - q_v^{n+1} - q_v^{n+2}| + \frac{[n+1]_{q_v} q_v^{2n}}{[n]_{q_v}^2} \right\}$,

where $B := \max \left\{ \frac{B_1}{1-B_1}, \dots, \frac{B_r}{1-B_r}, \left(\frac{B_1}{1-B_1} \right)^2, \dots, \left(\frac{B_r}{1-B_r} \right)^2 \right\}$.

Proof. (i) It follows from Lemma 3.1(i) and the definition of f_0 .

(ii) By Lemma 3.1(ii), one can get, for each $v = 1, \dots, r$ and $(x_1, \dots, x_r) \in I^r$, that

$$\begin{aligned} &|M_n(f_v; q_1, \dots, q_r; x_1, \dots, x_r) - f_v(x_1, \dots, x_r)| \\ &\leq \left| \frac{[n+1]_{q_v} q_v^n x_v}{[n]_{q_v} (1 - q_v^{n+1} x_v)} - \frac{x_v}{1 - q_v^{n+1} x_v} \right| + \frac{x_v}{1 - q_v^{n+1} x_v} |1 - q_v^{n+1}| \frac{x_v}{1 - x_v} \\ &\leq \frac{x_v}{1 - x_v} \left\{ \left| \frac{[n+1]_{q_v} q_v^n}{[n]_{q_v}} - 1 \right| + |1 - q_v^{n+1}| \frac{x_v}{1 - x_v} \right\}. \end{aligned}$$

Now taking supremum over $(x_1, \dots, x_r) \in I^r$ on both sides of the above inequality, the proof of (ii) is completed.

(iii) Let $(x_1, \dots, x_r) \in I^r$. Then it follows from Lemma 3.1 (iii) that

$$\begin{aligned} &|M_n(f_{r+1}; q_1, \dots, q_r; x_1, \dots, x_r) - f_{r+1}(x_1, \dots, x_r)| \\ &\leq \sum_{v=1}^r \frac{x_v^2}{(1 - q_v^{n+1} x_v)(1 - q_v^{n+2} x_v)} \left| \frac{[n+1]_{q_v} [n+2]_{q_v} q_v^{2n+1}}{[n]_{q_v}^2} - 1 \right| \\ &\quad + \sum_{v=1}^r \left| \frac{x_v^2}{(1 - q_v^{n+1} x_v)(1 - q_v^{n+2} x_v)} - \left(\frac{x_v}{1 - x_v} \right)^2 \right| \end{aligned}$$

$$\begin{aligned}
 & + \sum_{v=1}^r \frac{x_v}{(1 - q_v^{n+1}x_v)} \frac{[n+1]_{q_v} q_v^{2n}}{[n]_{q_v}^2} \\
 \leq & \sum_{v=1}^r \frac{x_v^2}{(1 - q_v^{n+1}x_v)(1 - q_v^{n+2}x_v)} \left| \frac{[n+1]_{q_v} [n+2]_{q_v} q_v^{2n+1}}{[n]_{q_v}^2} - 1 \right| \\
 & + \sum_{v=1}^r \frac{x_v^4}{(1 - q_v^{n+1}x_v)(1 - q_v^{n+2}x_v)} |1 - q_v^{2n+3}| \\
 & + \sum_{v=1}^r \frac{x_v^3(1 - x_v)^2}{(1 - q_v^{n+1}x_v)(1 - q_v^{n+2}x_v)} |2 - q_v^{n+1} - q_v^{n+2}| \\
 & + \sum_{v=1}^r \frac{x_v(1 - x_v)^2}{(1 - q_v^{n+1}x_v)} \frac{[n+1]_{q_v} q_v^{2n}}{[n]_{q_v}^2},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & |M_n(f_{r+1}; q_1, \dots, q_r; x_1, \dots, x_r) - f_{r+1}(x_1, \dots, x_r)| \\
 \leq & B \sum_{v=1}^r \left\{ \left| \frac{[n+1]_{q_v} [n+2]_{q_v} q_v^{2n+1}}{[n]_{q_v}^2} - 1 \right| + |1 - q_v^{2n+3}| \right. \\
 & \left. + |2 - q_v^{n+1} - q_v^{n+2}| + \frac{[n+1]_{q_v} q_v^{2n}}{[n]_{q_v}^2} \right\}.
 \end{aligned}$$

So the proof is completed if we take supremum over $(x_1, \dots, x_r) \in I^r$ on both sides of the above inequality. ■

Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume, for each $v = 1, 2, \dots, r$, that $(q_{v,n})_{n \in \mathbb{N}}$ is a sequence from $(0, 1]$ such that

$$(3.8) \quad st_A - \lim_n q_{v,n}^n = 1.$$

In this case, since

$$\frac{1}{[n]_{q_{v,n}}} = \frac{1}{1 + q_{v,n} + \dots + q_{v,n}^{n-1}} \quad \text{and} \quad 0 \leq q_{v,n}^n \leq q_{v,n}^k \leq 1$$

for $k = 1, 2, \dots, n$, the condition (3.8) implies that

$$(3.9) \quad st_A - \lim_n \frac{1}{[n]_{q_{v,n}}} = 0.$$

Indeed, such a sequence $(q_{v,n})_{n \in \mathbb{N}}$ can be constructed in the following way. Take $A = C_1$, the Cesàro matrix of order one. Then we know that C_1 -statistical convergence coincides with the notion of statistical convergence (see [8]). In this case, we use the notation $st - \lim$ instead of $st_{C_1} - \lim$. Now, for each $v = 1, 2, \dots, r$, define

$$(3.10) \quad q_{v,n} := \begin{cases} \frac{1}{vn}, & \text{if } n = m^2 \ (m \in \mathbb{N}) \\ e^{-\frac{1}{n}} \left(1 + \frac{1}{n}\right), & \text{otherwise.} \end{cases}$$

Then observe that $q_{v,n} \in (0, 1]$ for each $n \in \mathbb{N}$ and $v = 1, 2, \dots, r$. Also, it is easy to see that

$$st - \lim_n q_{v,n}^n = 1 \quad \text{for } v = 1, 2, \dots, r.$$

On the other hand, if $n \neq m^2$ ($m \in \mathbb{N}$), then we have $q_{v,n}^k = e^{-\frac{k}{n}} (1 + \frac{1}{n})^k \geq e^{-\frac{k}{n}} \frac{k}{n} \geq e^{-1} \frac{k}{n}$ for any $v = 1, \dots, r$; $k = 1, \dots, n - 1$ and $n \in \mathbb{N}$. So we get

$$\begin{aligned} [n]_{q_{v,n}} &= 1 + q_{v,n} + q_{v,n}^2 + \dots + q_{v,n}^{n-1} \\ &\geq \frac{1}{e} \left(\frac{1}{n} + \dots + \frac{n-1}{n} \right) \\ &\geq \frac{n-1}{2e}, \end{aligned}$$

which implies that

$$st - \lim_n \frac{1}{[n]_{q_{v,n}}} = 0 \quad \text{for } v = 1, 2, \dots, r.$$

Therefore, the sequence $(q_{v,n})_{n \in \mathbb{N}}$ defined by (3.10) satisfies the conditions in (3.8) with the choice of $A = C_1$.

Now in the definition of the operators (2.1) replace the fixed number $q_v \in (0, 1]$ with a sequence $(q_{v,n})_{n \in \mathbb{N}}$, ($v = 1, 2, \dots, r$), satisfying (3.8). Then, by (3.8) and also by (3.9), observe that, for each $v = 1, 2, \dots, r$ and for any non-negative regular summability matrix $A = (a_{jn})$,

$$(3.11) \quad st_A - \lim_n \left| \frac{[n+1]_{q_{v,n}} q_{v,n}^n}{[n]_{q_{v,n}}} - 1 \right| = 0,$$

$$(3.12) \quad st_A - \lim_n |1 - q_{v,n}^{n+1}| = 0,$$

$$(3.13) \quad st_A - \lim_n \left| \frac{[n+1]_{q_{v,n}} [n+2]_{q_{v,n}} q_{v,n}^{2n+1}}{[n]_{q_{v,n}}^2} - 1 \right| = 0,$$

$$(3.14) \quad st_A - \lim_n |1 - q_{v,n}^{2n+3}| = 0,$$

$$(3.15) \quad st_A - \lim_n |2 - q_{v,n}^{n+1} - q_{v,n}^{n+2}| = 0,$$

$$(3.16) \quad st_A - \lim_n \frac{[n+1]_{q_{v,n}} q_{v,n}^{2n}}{[n]_{q_{v,n}}^2} = 0.$$

So, we get the following statistical approximation theorem.

Theorem 3.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that, for each $v = 1, 2, \dots, r$, $(q_{v,n})_{n \in \mathbb{N}}$ is a sequence satisfying (3.8). Let w be a fixed function of the type of modulus of continuity satisfying (a)–(c). Then, for all $f \in H_w(I^r)$, we have*

$$st_A - \lim_n \|M_n(f; q_{1,n}, \dots, q_{r,n}; \cdot) - f\| = 0.$$

Proof. By Lemma 3.2(i), it is clear that

$$(3.17) \quad st_A - \lim_n \|M_n(f_0; q_{1,n}, \dots, q_{r,n}; \cdot) - f_0\| = 0.$$

Also, considering (3.11), (3.12) and Lemma 3.2(ii), we immediately get, for each $v = 1, 2, \dots, r$, that

$$(3.18) \quad st_A - \lim_n \|M_n(f_v; q_{1,n}, \dots, q_{r,n}; \cdot) - f_v\| = 0.$$

Now, for a given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} U &:= \{n : \|M_n(f_{r+1}; q_{1,n}, \dots, q_{r,n}; \cdot) - f_{r+1}\| \geq \varepsilon\}, \\ U_{1,v} &:= \left\{n : \left| \frac{[n+1]_{q_{v,n}} [n+2]_{q_{v,n}} q_{v,n}^{2n+1}}{[n]_{q_{v,n}}^2} - 1 \right| \geq \frac{\varepsilon}{4Br} \right\}, \\ U_{2,v} &:= \left\{n : |1 - q_{v,n}^{2n+3}| \geq \frac{\varepsilon}{4Br} \right\}, \\ U_{3,v} &:= \left\{n : |2 - q_{v,n}^{n+1} - q_{v,n}^{n+2}| \geq \frac{\varepsilon}{4Br} \right\}, \\ U_{4,v} &:= \left\{n : \frac{[n+1]_{q_{v,n}} q_{v,n}^{2n}}{[n]_{q_{v,n}}^2} \geq \frac{\varepsilon}{4Br} \right\}, \end{aligned}$$

where $v = 1, 2, \dots, r$. Then it follows from Lemma 3.2(iii) that

$$U \subseteq \bigcup_{v=1}^r (U_{1,v} \cup U_{2,v} \cup U_{3,v} \cup U_{4,v}),$$

which gives

$$(3.19) \quad \sum_{n \in U} a_{jn} \leq \sum_{v=1}^r \left(\sum_{n \in U_{1,v}} a_{jn} + \sum_{n \in U_{2,v}} a_{jn} + \sum_{n \in U_{3,v}} a_{jn} + \sum_{n \in U_{4,v}} a_{jn} \right).$$

Letting $j \rightarrow \infty$ in (3.19) and using (3.13), (3.14), (3.15), (3.16) we have

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which guarantees that

$$(3.20) \quad st_A - \lim_n \|M_n(f_{r+1}; q_{1,n}, \dots, q_{r,n}; \cdot) - f_{r+1}\| = 0.$$

Then the proof is completed by applying Theorem 3.1 and considering (3.17), (3.18), (3.20). ■

Notice that according to Theorem 3.2 we obtained a more general Korovkin-type approximation result for the multivariate MKZ operators based on q -integers by using the following facts:

- The usual limit operator is replaced by the A -statistical limit operator, where A is any non-negative regular summability matrix.
- Functions being approximated are not required to be continuous but are required to satisfy an appropriate condition with respect to some function w of the type of modulus of continuity.
- The natural numbers are replaced by the q -integers.

Now, if we replace the matrix A by the identity matrix, then the next result immediately follows from Theorem 3.2.

Corollary 3.1. *Assume that, for each $v = 1, 2, \dots, r$, $(q_{v,n})_{n \in \mathbb{N}}$ is a sequence satisfying the following conditions $\lim_n q_{v,n}^n = 1$. Let w be a fixed function of the type of modulus of continuity satisfying (a) – (c). Then, for all $f \in H_w(I^r)$, we have*

$$\lim_n \|M_n(f; q_{1,n}, \dots, q_{r,n}; \cdot) - f\| = 0.$$

Observe that Corollary 3.1 gives us the classical approximation behavior of the multivariate MKZ operators based on q -integers defined by (2.1). Of course, also taking $q_{v,n} = 1$ ($v = 1, 2, \dots, r$) in Corollary 3.1, we get the approximation result of the classical multivariate MKZ operators.

Finally, we should remark that if we take the sequence $(q_{v,n})_{n \in \mathbb{N}}$ defined by (3.10), then our statistical approximation result (Theorem 3.2) works; however its classical approximation (Corollary 3.1) does not work since $(q_{v,n})_{n \in \mathbb{N}}$ is non-convergent in the usual sense.

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