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On Low-Dimensional Filiform Leibniz Algebras and Their Invariants

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Abstract. The paper deals with the complete classification of a subclass of complex filiform Leibniz algebras in dimensions 5 and 6. This subclass arises from the naturally graded filiform Lie algebras. We give a complete list of algebras. In parametric families cases, the corresponding orbit functions (invariants) are given. In discrete orbits case, we show a representative of the orbits.

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1. Introduction

Leibniz algebras were introduced by Loday [3]. A skew-symmetric Leibniz algebra is a Lie algebra. The main motivation of Loday to introduce this class of algebras was the search of an "obstruction" to the periodicity in algebraic K-theory. Besides this purely algebraic motivation, some relationships with classical geometry, noncommutative geometry, and physics have been recently discovered. The present paper deals with the low-dimensional case of a subclass of filiform Leibniz algebras.

The outline of the paper is as follows. Section 1 is an introduction to the subclass of Leibniz algebras that we are going to investigate. The main results of the paper consisting of a complete classification of a subclass of low dimensional filiform Leibniz algebras are in Section 2. Here, for 5- and 6-dimensional cases, we give complete classification. For parametric family cases, the corresponding invariant functions are presented. Since the proofs in 6-dimensional case are similar to those in 5-dimensional case, the detailed proofs are given for dimension 5 only.

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Definition 1.1. *An algebra L over a field K is called a Leibniz algebra, if its bilinear operation $[\cdot,\cdot]$ satisfies the following Leibniz identity:*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad x,y,z \in L.$$

Onward, all algebras are assumed to be over the fields of complex numbers $\mathbb{C}$. Let L be a Leibniz algebra. We put:

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \geq 1.$$

Definition 1.2. *A Leibniz algebra L is said to be nilpotent, if there exists $s \in \mathbb{N}$, such that*

$$L^1 \supset L^2 \supset ... \supset L^s = \{0\}.$$

Definition 1.3. *A Leibniz algebra L is said to be filiform, if $dimL^i = n - i$, where $n = dimL$ and $2 \leq i \leq n$.*

The set of n-dimensional filiform Leibniz algebras we denote by $Leib_n$.
The following theorem from [1] splits $Leib_{n+1}$ into three disjoint subset.

Theorem 1.1. *Any $(n+1)$-dimensional complex filiform Leibniz algebra admits a basis $\{e_0, e_1, ..., e_n\}$ called adapted, such that the table of multiplication of the algebra has one of the following forms, where undefined products are zero:*

$$FLeib_{n+1} = \begin{cases} [e_0,e_0] = e_2, & \\ [e_i,e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0,e_1] = \alpha_3 e_3 + \alpha_4 e_4 + ... + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j,e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + ... + \alpha_{n+1-j} e_n, & 1 \leq j \leq n-2, \end{cases}$$

$\alpha_3, \alpha_4, ..., \alpha_n, \theta \in \mathbb{C}$.

$$SLeib_{n+1} = \begin{cases} [e_0,e_0] = e_2, & \\ [e_i,e_0] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0,e_1] = \beta_3 e_3 + \beta_4 e_4 + ... + \beta_n e_n, & \\ [e_1,e_1] = \gamma e_n, & \\ [e_j,e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + ... + \beta_{n+1-j} e_n, & 2 \leq j \leq n-2, \end{cases}$$

$\beta_3, \beta_4, ..., \beta_n, \gamma \in \mathbb{C}$.

$$TLeib_{n+1} = \begin{cases} [e_i,e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0,e_i] = -e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0,e_0] = b_{0,0} e_n, & \\ [e_0,e_1] = -e_2 + b_{0,1} e_n, & \\ [e_1,e_1] = b_{1,1} e_n, & \\ [e_i,e_j] = a_{i,j}^1 e_{i+j+1} + \dots & \\ \qquad\qquad + a_{i,j}^{n-(i+j+1)} e_{n-1} + b_{i,j} e_n, & 1 \leq i < j \leq n-1, \\ [e_i,e_j] = -[e_j,e_i], & 1 \leq i < j \leq n-1, \\ [e_i,e_{n-i}] = -[e_{n-i},e_i] = (-1)^i b_{i,n-i} e_n, & \end{cases}$$

where $a_{i,j}^k, b_{i,j} \in \mathbb{C}$ and $b_{i,n-i} = b$ whenever $1 \leq i \leq n-1$, and $b = 0$ for even n.

According to this theorem, the isomorphism problem inside of each class can be studied separately. The classes $FLeib_n, SLeib_n$ in low dimensional cases have been considered in [6, 7]. The general methods of classification for $Leib_n$ has been given in [4, 5]. This paper deals with the classification problem of low-dimensional cases of $TLeib_n$.

Observe that the class of n-dimensional filiform Lie algebras is in $TLeib_n$.

Definition 1.4. *Let $\{e_0, e_1, ..., e_n\}$ be an adapted basis of $L \in TLeib_{n+1}$. Then a nonsingular linear transformation $f : L \to L$ is said to be adapted, if the basis $\{f(e_0), f(e_1), ..., f(e_n)\}$ is adapted.*

The set of all adapted elements of GL_{n+1} forms a subgroup and it is denoted by G_{ad}. The following proposition specifies elements of G_{ad}.

Proposition 1.1. *Let $f \in G_{ad}$. Then f can be represented as follows:*

$$f(e_0) = e_0' = \sum_{i=0}^{n} A_i e_i,$$
$$f(e_1) = e_1' = \sum_{i=1}^{n} B_i e_i,$$
$$f(e_i) = e_i' = [f(e_{i-1}), f(e_0)], \quad 2 \le i \le n,$$

$A_0, A_i, B_j,\ (i, j = 1, ..., n)$ are complex numbers and $A_0\, B_1(A_0 + A_1 b) \neq 0$.

Proof. Since a filiform Leibniz algebra is 2-generated (see Theorem 1.1), it is sufficient to consider the adapted action of f on the generators e_0, e_1 :

$$f(e_0) = e_0' = \sum_{i=0}^{n} A_i e_i, \quad f(e_1) = e_1' = \sum_{i=0}^{n} B_i e_i.$$

Then

$$f(e_i) = [f(e_{i-1}), f(e_0)] = A_0^{i-2}(A_1 B_0 - A_0 B_1)e_i + \sum_{j=i+1}^{n} (*)e_j, \quad 2 \le i \le n.$$

Note that, $A_0 \neq 0$, and $A_1 B_0 - A_0 B_1 \neq 0$, otherwise $f(e_n) = 0$. The condition $A_0\, B_1(A_0 + A_1 b) \neq 0$ appears naturally, since f is not singular.

Let now consider $[f(e_1), f(e_2)] = B_0\,(A_1\, B_0 - A_0\, B_1)\, e_3 + \sum_{j=4}^{n}(*)e_j$. Then for the basis $\{f(e_0), f(e_1), ..., f(e_n)\}$ to be adapted $B_0(A_1 B_0 - A_0 B_1) = 0$. However, according to the observation above, $A_1 B_0 - A_0 B_1 \neq 0$. Therefore, $B_0 = 0$. ∎

2. The description of $TLeib_n$, $n = 5, 6$.

2.1. 5-dimensional case

In this section we deal with the class $TLeib_5$. By virtue of Theorem 1.1. we can represent $TLeib_5$ as follows:

$$TLeib_5 = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le 3, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le 3, \\ [e_0, e_0] = b_{0,0} e_4, & \\ [e_0, e_1] = -e_2 + b_{0,1} e_4, & \\ [e_1, e_1] = b_{1,1} e_4, & \\ [e_1, e_2] = -[e_2, e_1] = b_{1,2} e_4, & \end{cases}$$

$b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2} \in \mathbb{C}$. Further, the elements of $TLeib_5$ will be denoted by $L(\alpha) = L(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2})$.

Theorem 2.1 (Isomorphism criterion for $TLeib_5$). *Two algebras $L(\alpha)$ and $L(\alpha')$ from $TLeib_5$ are isomorphic, if and only if there exist complex numbers A_0, A_1 and $B_1 : A_0\, B_1 \neq 0$ and the following conditions hold:*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1}, \tag{2.1}$$

$$b'_{0,1} = \frac{A_0 b_{0,1} + 2\, A_1 b_{1,1}}{A_0^3}, \tag{2.2}$$

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^3}, \tag{2.3}$$

$$b'_{1,2} = \frac{B_1 b_{1,2}}{A_0^2}. \tag{2.4}$$

Proof. Part "if". Let L_1 and L_2 from $TLeib_5$ be isomorphic: $f : L_1 \cong L_2$. We choose the corresponding adapted bases $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{e'_0, e'_1, e'_2, e'_3, e'_4\}$ in L_1 and L_2, respectively. Then, in these bases the algebras will be presented as $L(\alpha)$ and $L(\alpha')$, where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2})$, and $\alpha' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2})$.

According to Proposition 1.1 one has:

$$\begin{aligned} e'_0 &= f(e_0) = A_0 e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3 + A_4 e_4, \\ e'_1 &= f(e_1) = B_1 e_1 + B_2 e_2 + B_3 e_3 + B_4 e_4. \end{aligned} \tag{2.5}$$

Then we get

$$\begin{aligned} e'_2 &= f(e_2) = [f(e_1), f(e_0)] \\ &= A_0 B_1 e_2 + A_0 B_2 e_3 + (A_0 B_3 + A_1 B_1 b_{1,1} + (A_2 B_1 - A_1 B_2) b_{1,2})\, e_4, \\ e'_3 &= f(e_3) = [f(e_2), f(e_0)] = A_0^2 B_1 e_3 + (A_0^2 B_2 - A_0 A_1 B_1 b_{1,2}) e_4, \\ e'_4 &= f(e_4) = [f(e_3), f(e_0)] = A_0^3 B_1 e_4. \end{aligned}$$

By using the table of multiplications one finds the relations between the coefficients $b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}$ and $b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}$. First, consider the equality $[f(e_0), f(e_0)] = b'_{0,0} f(e_4)$, we get equation (2.1) and from the equality $[f(e_1), f(e_0)] + [f(e_0), f(e_1)] = b'_{0,1} f(e_4)$ we have (2.2), and $[f(e_1), f(e_1)] = b'_{1,1}\, f(e_4)$ gives (2.3). Finally, the equality (2.4) comes out from $[f(e_1), f(e_2)] = b'_{1,2} f(e_4)$.

"Only if" part.

Let the equalities (2.1)–(2.4) hold. Then, the base change (2.5) above is adapted and it transforms $L(\alpha)$ into $L(\alpha')$. Indeed,

$$\begin{aligned} [e'_0, e'_0] &= \left[\sum_{i=0}^{4} A_i e_i, \sum_{i=0}^{4} A_i e_i\right] \\ &= A_0^2 [e_0, e_0] + A_0 A_1 [e_0, e_1] + A_0 A_1 [e_1, e_0] + A_1^2 [e_1, e_1] \\ &= \left(A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}\right) e_4 = b'_{0,0} A_0^3 B_1 e_4 = b'_{0,0} e'_4, \end{aligned}$$

and

$$\begin{aligned}
[e_0', e_1'] &= \left[\sum_{i=0}^{4} A_i e_i, \sum_{i=1}^{4} B_i e_i\right] \\
&= -(A_0 B_1 e_2 + A_0 B_2 e_3 + (A_1 B_1 b_{1,1} + A_2 B_1 b_{1,2} - A_1 B_2 b_{1,2} + A_0 B_3) e_4) \\
&\qquad + B_1 \left(b_{0,1} A_0 + 2\, A_1 B_{1,1}\right) e_4 \\
&= -e_2' + A_0^3 B_1 b_{0,1}' e_4 = -e_2' + b_{0,1}' e_4'.
\end{aligned}$$

By the same manner one can prove that $[e_1', e_1'] = b_{1,1}' e_4'$, $[e_1', e_2'] = b_{1,2}'\, e_4'$ and the other products are zero. ∎

For the purpose of simplicity, we establish the following notation: $\Delta = 4b_{0,0}b_{1,1} - b_{0,1}^2$. Now, we list the isomorphism classes of algebras from $TLeib_5$.

Represent $TLeib_5$ as a disjoint union of the following subsets: $TLeib_5 = \bigcup\limits_{i=1}^{9} U_5^i$, where

$$\begin{aligned}
U_5^1 &= \{L(\alpha) \in TLeib_5 : b_{1,1} \neq 0,\ b_{1,2} \neq 0\}; \\
U_5^2 &= \{L(\alpha) \in TLeib_5 : b_{1,1} \neq 0,\ b_{1,2} = 0,\ \Delta \neq 0\}; \\
U_5^3 &= \{L(\alpha) \in TLeib_5 : b_{1,1} \neq 0,\ b_{1,2} = \Delta = 0\}; \\
U_5^4 &= \{L(\alpha) \in TLeib_5 : b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,2} \neq 0\}; \\
U_5^5 &= \{L(\alpha) \in TLeib_5 : b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,2} = 0\}; \\
U_5^6 &= \{L(\alpha) \in TLeib_5 : b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,2} \neq 0\}; \\
U_5^7 &= \{L(\alpha) \in TLeib_5 : b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,2} = 0\}; \\
U_5^8 &= \{L(\alpha) \in TLeib_5 : b_{1,1} = b_{0,1} = b_{0,0} = 0,\ b_{1,2} \neq 0\}; \\
U_5^9 &= \{L(\alpha) \in TLeib_5 : b_{1,1} = b_{0,1} = b_{0,0} = b_{1,2} = 0\}.
\end{aligned}$$

Proposition 2.1.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_5^1 are isomorphic, if and only if*

$$\left(\frac{b_{1,2}'}{b_{1,1}'}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta.$$

(2) *For any λ from $\mathbb{C}$, there exists $L(\alpha) \in U_5^1 : \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \lambda$.*

Proof. (1) ($\Longrightarrow$) Let $L(\alpha)$ and $L(\alpha')$ be isomorphic. Then, due to Theorem 2.1 it is easy to see that

$$\left(\frac{b_{1,2}'}{b_{1,1}'}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta.$$

($\Longleftarrow$) Let the equality

$$\left(\frac{b_{1,2}'}{b_{1,1}'}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta$$

hold. Consider the base change (2.5) above with $A_0 = \frac{b_{1,1}}{b_{1,2}}$, $A_1 = -\frac{b_{0,1}}{2b_{1,2}}$ and $B_1 = \frac{b_{1,1}^2}{b_{1,2}^3}$. This changing leads $L(\alpha)$ into

$$L\left(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta, 0, 1, 1\right).$$

An analogous base change transforms $L(\alpha')$ into

$$L\left(\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta', 0, 1, 1\right).$$

Since

$$\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta,$$

then $L(\alpha)$ is isomorphic to $L(\alpha')$.
(2) Obvious. ∎

Proposition 2.2. *The subsets $U_5^2, U_5^3, U_5^4, U_5^5, U_5^6, U_5^7, U_5^8$ and U_5^9 are single orbits with representatives* $L(1,0,1,0), L(0,0,1,0), L(0,1,0,1), L(0,1,0,0), L(1,0,0,1), L(1,0,0,0), L(0,0,0,1)$ *and* $L(0,0,0,0)$, *respectively.*

Proof. To prove it, we give the appropriate values of A_0, A_1 and B_1 in the base change (as for other $A_i,\ B_j,\ i,j = 2,3,4$ they are any, except where specified otherwise).

For U_5^2 :

$$\begin{aligned}
e'_0 &= A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4,\\
e'_1 &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4,\\
e'_2 &= A_0B_1e_2 + A_0B_2e_3 + (A_1B_1b_{1,1} + A_0B_3)\, e_4,\\
e'_3 &= A_0^2B_1e_3 + A_0^2B_2e_4,\\
e'_4 &= {A_0}^3B_1e_4,
\end{aligned}$$

where $A_0 = \frac{\Delta^{\frac{1}{4}}}{\sqrt{2}}$, $A_1 = -\frac{b_{0,1}\,\Delta^{\frac{1}{4}}}{2\sqrt{2}b_{1,1}}$ and $B_1 = \frac{\Delta^{\frac{3}{4}}}{2\sqrt{2}b_{1,1}}$.

For U_5^3 :

$$\begin{aligned}
e'_0 &= A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4,\\
e'_1 &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4,\\
e'_2 &= A_0B_1e_2 + A_0B_2e_3 + (A_1B_1b_{1,1} + A_0B_3)\, e_4,\\
e'_3 &= A_0^2B_1e_3 + {A_0}^2B_2e_4,\\
e'_4 &= {A_0}^3B_1e_4,
\end{aligned}$$

where $A_0 \in \mathbb{C}^*$, $A_1 = -\frac{A_0b_{0,1}}{2\,b_{1,1}}$ and $B_1 = \frac{A_0^3}{b_{1,1}}$.

For U_5^4 :

$$e'_0 = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4,$$

$$
\begin{aligned}
e_1' &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4, \\
e_2' &= A_0B_1e_2 + A_0B_2e_3 + (A_0B_3 + (A_2B_1 - A_1B_2)b_{1,2})e_4, \\
e_3' &= A_0^2B_1e_3 + (A_0^2B_2 - A_1A_0B_1b_{1,2})e_4, \\
e_4' &= {A_0}^3B_1e_4,
\end{aligned}
$$

where $A_0^2 = b_{0,1},\ A_1 = -\frac{A_0b_{0,0}}{b_{0,1}}$ and $B_1 = \frac{A_0^2}{b_{1,2}}$.

For U_5^5 :

$$
\begin{aligned}
e_0' &= A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4, \\
e_1' &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4, \\
e_2' &= A_0B_1e_2 + A_0B_2e_3 + A_0B_3e_4, \\
e_3' &= A_0^2B_1e_3 + A_0^2B_2e_4, \\
e_4' &= A_0^3B_1e_4,
\end{aligned}
$$

where $A_0^2 = b_{0,1},\ A_1 = -\frac{b_{0,0}}{\sqrt{b_{0,1}}}$ and $B_1 \in \mathbb{C}^*$.

For U_5^6 :

$$
\begin{aligned}
e_0' &= A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4, \\
e_1' &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4, \\
e_2' &= A_0B_1e_2 + A_0B_2e_3 + (A_0B_3 + (A_2B_1 - A_1B_2)b_{1,2})e_4, \\
e_3' &= A_0^2B_1e_3 + (A_0^2B_2 - A_1A_0B_1b_{1,2})e_4, \\
e_4' &= {A_0}^3B_1e_4,
\end{aligned}
$$

where $A_0^3 = b_{0,0}b_{1,2},\ A_1 \in \mathbb{C}$ and $B_1 = \frac{b_{0,0}^{\frac{2}{3}}}{b_{1,2}^{\frac{1}{3}}}$.

For U_5^7 :

$$
\begin{aligned}
e_0' &= A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4, \\
e_1' &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4, \\
e_2' &= A_0B_1e_2 + A_0B_2e_3 + A_0B_3e_4, \\
e_3' &= A_0^2B_1e_3 + A_0^2B_2e_4, \\
e_4' &= A_0^3B_1e_4,
\end{aligned}
$$

where $A_0 \in \mathbb{C}^*,\ A_1 \in \mathbb{C}$ and $B_1 = \frac{b_{0,0}}{A_0}$.

For U_5^8 :

$$
\begin{aligned}
e_0' &= A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4, \\
e_1' &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4, \\
e_2' &= A_0B_1e_2 + A_0B_2e_3 + (A_0B_3 + (A_2B_1 - A_1B_2)b_{1,2})e_4, \\
e_3' &= A_0^2B_1e_3 + (A_0^2B_2 - A_1A_0B_1b_{1,2})e_4, \\
e_4' &= {A_0}^3B_1e_4,
\end{aligned}
$$

where $A_0 \in \mathbb{C}^*$, $A_1 \in \mathbb{C}$ and $B_1 = \frac{A_0^2}{b_{1,2}}$.

For U_5^9 :

$$\begin{aligned}
e_0' &= A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4,\\
e_1' &= B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4,\\
e_2' &= A_0B_1e_2 + A_0B_2e_3 + A_0B_3e_4,\\
e_3' &= A_0^2B_1e_3 + A_0^2B_2e_4,\\
e_4' &= A_0^3B_1e_4,
\end{aligned}$$

where $A_0, B_1 \in \mathbb{C}^*$ and $A_1 \in \mathbb{C}$. ∎

2.2. 6-dimensional case

This section is devoted to the description of $TLeib_6$. This class can be represented by the following table of multiplication:

$$TLeib_6 = \begin{cases}
[e_i, e_0] = e_{i+1}, & 1 \le i \le 4,\\
[e_0, e_i] = -e_{i+1}, & 2 \le i \le 4,\\
[e_0, e_0] = b_{0,0}e_5, \quad [e_0, e_1] = -e_2 + b_{0,1}e_5, \quad [e_1, e_1] = b_{1,1}e_5, & \\
[e_1, e_2] = -[e_2, e_1] = b_{1,2}e_4 + b_{1,3}e_5, & \\
[e_1, e_3] = -[e_3, e_1] = b_{1,2}e_5, & \\
[e_1, e_4] = -[e_4, e_1] = -[e_2, e_3] = [e_3, e_2] = -b_{2,3}e_5. &
\end{cases}$$

The elements of $TLeib_6$ will be denoted by $L(\alpha)$, where $\alpha = (b_{0,1}, b_{1,1}, b_{1,2}, b_{1,3}, b_{2,3})$.

Theorem 2.2 (Isomorphism criterion for $TLeib_6$)**.** *Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from $TLeib_6$ are isomorphic, if and only if there exist $A_0, A_1, B_1, B_2, B_3 \in \mathbb{C}$: such that $A_0B_1(A_0 + A_1\, b_{2,3}) \neq 0$ and the following equalities hold:*

$$\begin{aligned}
b_{0,0}' &= \frac{A_0^2 b_{0,0} + A_0A_1b_{0,1} + A_1^2 b_{1,1}}{A_0^3B_1(A_0 + A_1\, b_{2,3})},\\
b_{0,1}' &= \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^3(A_0 + A_1\, b_{2,3})},\\
b_{1,1}' &= \frac{B_1b_{1,1}}{A_0^3(A_0 + A_1\, b_{2,3})},\\
b_{1,2}' &= \frac{B_1\, b_{1,2}}{A_0^2},\\
b_{1,3}' &= \frac{2\,A_0A_1B_1^2b_{1,2}^2 + A_0^2B_1^2b_{1,3} + \left(A_0^2\left(-2\,B_1B_3 + B_2^2\right) + A_1^2B_1^2b_{1,2}^2\right)b_{2,3}}{A_0^4B_1\left(A_0 + A_1b_{2,3}\right)},\\
b_{2,3}' &= \frac{B_1\, b_{2,3}}{A_0 + A_1\, b_{2,3}}.
\end{aligned} \tag{2.6}$$

Proof. The proof is the similar to that of Theorem 2.1. ∎

Represent $TLeib_6$ as a union of the following subsets:

$TLeib_6^1$

$U_6^1 = \{L(\alpha) \in TLeib_6 : b_{2,3} \neq 0,\ b_{1,1} \neq 0\};$

$$
\begin{aligned}
U_6^2 &= \{L(\alpha) \in TLeib_6 : b_{2,3} \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\};\\
U_6^3 &= \{L(\alpha) \in TLeib_6 : b_{2,3} \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{1,2} \neq 0,\ b_{0,0} \neq 0\};\\
U_6^4 &= \{L(\alpha) \in TLeib_6 : b_{2,3} \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{1,2} \neq 0,\ b_{0,0} = 0\};\\
U_6^5 &= \{L(\alpha) \in TLeib_6 : b_{2,3} \neq 0,\ b_{1,1} = b_{0,1} = b_{1,2} = 0,\ b_{0,0} \neq 0\};\\
U_6^6 &= \{L(\alpha) \in TLeib_6 : b_{2,3} \neq 0,\ b_{1,1} = b_{0,1} = b_{1,2} = b_{0,0} = 0\};\\
U_6^7 &= \{L(\alpha) \in TLeib_6 : b_{2,3} = 0,\ b_{1,2} \neq 0,\ b_{1,1} \neq 0\};\\
U_6^8 &= \{L(\alpha) \in TLeib_6 : b_{2,3} = 0,\ b_{1,2} \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\};\\
U_6^9 &= \{L(\alpha) \in TLeib_6 : b_{2,3} = 0,\ b_{1,2} \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\};\\
U_6^{10} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = 0,\ b_{1,2} \neq 0,\ b_{1,1} = b_{0,1} = b_{0,0} = 0\};\\
U_6^{11} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = 0,\ b_{1,1} \neq 0,\ b_{1,3} \neq 0\};\\
U_6^{12} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = 0,\ b_{1,1} \neq 0,\ b_{1,3} = 0,\ \Delta \neq 0\};\\
U_6^{13} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = 0,\ b_{1,1} \neq 0,\ b_{1,3} = \Delta = 0\};\\
U_6^{14} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,3} \neq 0\};\\
U_6^{15} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,3} = 0\};\\
U_6^{16} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,3} \neq 0\};\\
U_6^{17} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,3} = 0\};\\
U_6^{18} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = 0,\ b_{1,3} \neq 0\};\\
U_6^{19} &= \{L(\alpha) \in TLeib_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = b_{1,3} = 0\}.
\end{aligned}
$$

Proposition 2.3.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_6^1 are isomorphic, if and only if*

$$\left(\frac{b'_{2,3}}{2\,b'_{1,1} - b'_{0,1}b'_{2,3}}\right)^2 \Delta' = \left(\frac{b_{2,3}}{2\,b_{1,1} - b_{0,1}b_{2,3}}\right)^2 \Delta$$

and

$$\frac{\left(2\,b'_{1,1} - b'_{2,3}b'_{0,1}\right)^3 b'^3_{1,2}}{b'^2_{2,3}b'^4_{1,1}} = \frac{\left(2\,b_{1,1} - b_{2,3}b_{0,1}\right)^3 b^3_{1,2}}{b^2_{2,3}b^4_{1,1}}.$$

(2) *For any $\lambda_1, \lambda_2 \in \mathbb{C}$, there exists $L(\alpha) \in U_6^1$: $\left(\frac{b_{2,3}}{2\,b_{1,1}-b_{0,1}b_{2,3}}\right)^2 \Delta = \lambda_1$, $\frac{(2\,b_{1,1}-b_{2,3}b_{0,1})^3 b^3_{1,2}}{b^2_{2,3}b^4_{1,1}} = \lambda_2$.*

Then orbits from the set U_6^1 can be parameterized as $L(\lambda_1, 0, 1, \lambda_2, 0, 1)$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proposition 2.4.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_6^2 are isomorphic, if and only if*

$$\frac{\left(b'_{0,1} - b'_{2,3}b'_{0,0}\right)^4 b'^3_{1,2}}{b'^3_{2,3}b'^5_{0,1}} = \frac{\left(b_{0,1} - b_{2,3}b_{0,0}\right)^4 b^3_{1,2}}{b^3_{2,3}b^5_{0,1}}.$$

(2) *For any* $\lambda \in \mathbb{C}$*, there exists* $L(\alpha) \in U_6^2 : \frac{(b_{0,1}-b_{2,3}b_{0,0})^4 b_{1,2}^3}{b_{2,3}^3 b_{0,1}^5} = \lambda.$

Therefore orbits from U_6^2 *can be parameterized as* $L\,(0,1,0,\lambda,0,1)\,,\ \lambda \in \mathbb{C}.$

Proposition 2.5.

(1) *Two algebras* $L(\alpha)$ *and* $L(\alpha')$ *from* U_6^7 *are isomorphic, if and only if*

$$\frac{4\,b'_{0,0}b'^4_{1,2} - 2\,b'_{1,3}b'_{0,1}b'^2_{1,2} + b'^2_{1,3}b'_{1,1}}{b'_{1,2}b'^2_{1,1}} = \frac{4\,b_{0,0}b^4_{1,2} - 2\,b_{1,3}b_{0,1}b^2_{1,2} + b^2_{1,3}b_{1,1}}{b_{1,2}b^2_{1,1}}$$

and

$$\frac{\left(b'_{0,1}b'^2_{1,2} - b'_{1,3}b'_{1,1}\right)^2}{b'_{1,2}b'^3_{1,1}} = \frac{\left(b_{0,1}b^2_{1,2} - b_{1,3}b_{1,1}\right)^2}{b_{1,2}b^3_{1,1}}.$$

(2) *For any* $\lambda_1, \lambda_2 \in \mathbb{C}$*, there exists* $L(\alpha) \in U_6^7$:

$$\frac{4\,b_{0,0}b^4_{1,2} - 2\,b_{1,3}b_{0,1}b^2_{1,2} + b^2_{1,3}b_{1,1}}{b_{1,2}b^2_{1,1}} = \lambda_1\ ,\quad \frac{\left(b_{0,1}b^2_{1,2} - b_{1,3}b_{1,1}\right)^2}{b_{1,2}b^3_{1,1}} = \lambda_2.$$

The orbits from U_6^7 *can be parameterized as* $L\,(\lambda_1,\lambda_2,1,1,0,0)\,,\ \lambda_1,\lambda_2 \in \mathbb{C}.$

Proposition 2.6.

(1) *Two algebras* $L(\alpha)$ *and* $L(\alpha')$ *from* U_6^8 *are isomorphic, if and only if*

$$\frac{\left(2\,b'_{0,0}b'^2_{1,2} - b'_{1,3}b'_{0,1}\right)^3}{b'^3_{1,2}b'^4_{0,1}} = \frac{\left(2\,b_{0,0}b^2_{1,2} - b_{1,3}b_{0,1}\right)^3}{b^3_{1,2}b^4_{0,1}}.$$

(2) *For any* $\lambda \in \mathbb{C}$*, there exists* $L(\alpha) \in U_6^8 : \frac{\left(2\,b_{0,0}b^2_{1,2} - b_{1,3}b_{0,1}\right)^3}{b^3_{1,2}b^4_{0,1}} = \lambda.$

The orbits from the set U_6^8 *can be parameterized as* $L\,(\lambda,1,0,1,0,0)\,,\ \lambda \in \mathbb{C}.$

Proposition 2.7.

(1) *Two algebras* $L(\alpha)$ *and* $L(\alpha')$ *from* U_6^{11} *are isomorphic, if and only if*

$$\left(\frac{b'_{1,3}}{b'_{1,1}}\right)^6 \Delta' = \left(\frac{b_{1,3}}{b_{1,1}}\right)^6 \Delta.$$

(2) *For any* $\lambda \in \mathbb{C}$*, there exists* $L(\alpha) \in U_6^{11} : \left(\frac{b_{1,3}}{b_{1,1}}\right)^6 \Delta = \lambda.$

The orbits from U_6^{11} *can be parameterized as* $L\,(\lambda,0,1,0,1,0)\,,\ \lambda \in \mathbb{C}.$

Proposition 2.8. *The subsets* $U_6^3,\ U_6^4,\ U_6^5,\ U_6^6,\ U_6^9,\ U_6^{10},\ U_6^{12},\ U_6^{13},\ U_6^{14},\ U_6^{15},$ $U_6^{16},\ U_6^{17},\ U_6^{18}$ *and* U_6^{19} *are single orbits with representatives* $L(1,0,0,1,0,1),$ $L(0,0,0,1,0,1), L(1,0,0,0,0,1),\ L(0,0,0,0,0,1),\ L(1,0,0,1,0,0), L(0,0,0,1,0,0),$ $L(1,0,1,0,0,0), L(0,0,1,0,0,0), L(0,1,0,0,1,0), L(0,1,0,0,0,0),\ L(1,0,0,0,1,0),$ $L(1,0,0,0,0,0), L(0,0,0,0,1,0)$ *and* $L(0,0,0,0,0,0)$*, respectively.*

3. Conclusion

(1) In $TLeib_5$, we distinguished nine isomorphism classes (one parametric family and eight concrete) of three dimensional Leibniz algebras and shown that they exhaust all possible cases.

(2) In the case of $TLeib_6$, there are 19 isomorphism classes (five parametric families and 14 concrete) and they exhaust all possible cases.

Remark 3.1. It should be pointed out that the filiform Lie algebras case is covered by U_5^8, U_5^9 in 5- and U_6^4, U_6^6, U_6^{10}, U_6^{18}, U_6^{19} in 6-dimensional cases, respectively. Therefore, the list of filiform Lie algebras in the paper agrees with the list given in [2].

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