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On the Semigroup of Semi-Continuous Interval-Valued Multihomomorphisms

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Abstract. A characterization of semi-continuous interval-valued multihomomorphisms on $(\mathbb{R}, +)$ has been given as follows: An interval-valued multifunction f on $\mathbb{R}$ is a semi-continuous multihomomorphism on $(\mathbb{R}, +)$ if and only if f is one of the following forms: $f(x) = \{cx\}$, $f(x) = \mathbb{R}$, $f(x) = (0, \infty)$, $f(x) = (-\infty, 0)$, $f(x) = [cx, \infty)$ and $f(x) = (-\infty, cx]$ where c is a constant in $\mathbb{R}$. Denote by $\mathrm{SIM}(\mathbb{R}, +)$ the set of all such multifunctions on $\mathbb{R}$. We show that $\mathrm{SIM}(\mathbb{R}, +)$ is a semigroup under composition and it is a regular semigroup.

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1. Introduction

A *multifunction* from a nonempty set X into a nonempty set Y is a function $f : X \to P(Y) \smallsetminus \{\emptyset\}$ where $P(Y)$ is the power set of Y. By a *multifunction on* X we mean a multifunction from X into itself.

A multifunction f from a group G into a group G' is a *multihomomorphism* if

$$f(xy) = f(x)f(y) \ (= \{st \mid s \in f(x) \text{ and } t \in f(y)\}) \quad \text{for all } x, y \in G.$$

Multihomomorphisms between cyclic groups were characterized in [7]. These characterizations were used in [2] to determine surjective multihomomorphisms between cyclic groups. In [8], the authors provided some necessary conditions of multihomomorphisms from any group into groups of real numbers under the usual addition and multiplication.

A multifunction f from a topological space X into a topological space Y is *upper semi-continuous* at $a \in X$ if for any open set V in Y containing $f(a)$ as a subset, there exists an open set U in X containing a such that $f(U) \subseteq V$. Such an f

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is called a *lower semi-continuous* at $a \in X$ if for any open set V in Y such that $f(a) \cap V \neq \emptyset$, there exists an open set $U \in X$ containing a such that $f(x) \cap V \neq \emptyset$ for all $x \in U$. See [3, p. 261]. If f is upper semi-continuous and lower semi-continuous at $a \in X$, then we call f *semi-continuous* at a. If f is upper semi-continuous [lower semi-continuous, semi-continuous] at every point in X, then f is called *upper semi-continuous* [*lower semi-continuous, semi-continuous*] *on* X. Evidently, the upper and lower semi-continuity as well as the continuity at $a \in X$ of a single-valued function are identical.

Let $\mathbb{R}$ be the set of real numbers. By an *interval-valued multifunction* on $\mathbb{R}$, we mean a multifunction f on $\mathbb{R}$ such that $f(x)$ is an interval in $\mathbb{R}$ for all $x \in \mathbb{R}$. Notice that interval-valued multihomomorphisms on $(\mathbb{R}, +)$ are a generalization of homomorphisms on $(\mathbb{R}, +)$.

It is well known that if f is a continuous homomorphism on $(\mathbb{R}, +)$, then there is a constant $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. This result was extended in [6] as follows:

Theorem 1.1. [6] *Let f be an interval-valued function on $\mathbb{R}$. Then f is an upper semi-continuous multihomomorphism on* $(\mathbb{R}, +)$ *if and only if f is one of the followings:*

(i) *There is a constant $c \in \mathbb{R}$ such that $f(x) = \{cx\}$ for all $x \in \mathbb{R}$.*
(ii) *$f(x) = \mathbb{R}$ for all $x \in \mathbb{R}$.*
(iii) *$f(x) = (0, \infty)$ for all $x \in \mathbb{R}$.*
(iv) *$f(x) = (-\infty, 0)$ for all $x \in \mathbb{R}$.*
(v) *There is a constant $c \in \mathbb{R}$ such that $f(x) = [cx, \infty)$ for all $x \in \mathbb{R}$.*
(vi) *There is a constant $c \in \mathbb{R}$ such that $f(x) = (-\infty, cx]$ for all $x \in \mathbb{R}$.*

In [5], the authors extended the above known result to lower semi-continuous interval-valued multihomomorphisms on $(\mathbb{R}, +)$. The following result was provided in [5].

Theorem 1.2. [5] *Let f be an interval-valued multihomomorphism on* $(\mathbb{R}, +)$. *If f is upper semi-continuous on $\mathbb{R}$, then f is semi-continuous on $\mathbb{R}$.*

For a nonempty set X, let $\mathcal{B}(X)$ be the set of all binary relations on X. Then $\mathcal{B}(X)$ is a monoid under the composition defined by

$$\sigma \circ \rho = \{(x, y) \in X \times X \mid (x, z) \in \rho \text{ and } (z, y) \in \sigma \text{ for some } z \in X\},$$

having the identity function on X as its identity [1, p. 13]. If f and $g \in \mathcal{B}(X)$ are multifunctions on X, then gf (the composition of f and g) $\in \mathcal{B}(X)$ is a multifunction on X and

$$\text{for all } x \in X, (gf)(x) = g(f(x)) = \bigcup_{t \in f(x)} g(t).$$

An element a of a semigroup S is called an *idempotent* if $a^2 = a$. An element a of S is called *regular* if $a = axa$ for some $x \in S$. Then every idempotent of S is regular. We call S a *regular semigroup* if every element of S is regular.

Next, let $\mathrm{SIM}(\mathbb{R}, +)$ be the set of all semi-continuous interval-valued multihomomorphisms on $(\mathbb{R}, +)$. By Theorem 1.2, $\mathrm{SIM}(\mathbb{R}, +)$ is the set of all upper semi-continuous interval-valued multihomomorphisms on $(\mathbb{R}, +)$. Also, $\mathrm{SIM}(\mathbb{R}, +) \subseteq$

$\mathcal{B}(\mathbb{R})$. For convenience, for $c \in \mathbb{R}$, let g_c, $h_{\mathbb{R}}$, $h_{(0,\infty)}$, $h_{(-\infty,0)}$, $\overline{k}_c$ and $\underline{k}_c$ be the interval-valued multifunctions on $\mathbb{R}$ defined by

$$\begin{aligned} &g_c(x) = \{cx\}, \\ &h_{\mathbb{R}}(x) = \mathbb{R}, \quad h_{(0,\infty)}(x) = (0,\infty), \quad h_{(-\infty,0)}(x) = (-\infty,0), \\ &\overline{k}_c(x) = [cx,\infty), \underline{k}_c(x) = (-\infty, cx] \end{aligned}$$

for all $x \in \mathbb{R}$. By Theorem 1.1,

$$\mathrm{SIM}(\mathbb{R},+) = \{g_c \mid c \in \mathbb{R}\} \cup \{h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}\} \cup \{\overline{k}_c, \underline{k}_c \mid c \in \mathbb{R}\}.$$

Notice that g_1 is the identity function on $\mathbb{R}$.

In this paper, we show that $\mathrm{SIM}(\mathbb{R},+)$ is a semigroup under composition and it is a regular semigroup. In addition, the idempotents of the semigroup $\mathrm{SIM}(\mathbb{R},+)$ are determined.

2. Main results

First, we show that $\mathrm{SIM}(\mathbb{R},+)$ is a semigroup under composition, that is, $\mathrm{SIM}(\mathbb{R},+)$ is a subsemigroup of $\mathcal{B}(\mathbb{R})$. The following two lemmas are needed.

Lemma 2.1. *For $f \in \mathrm{SIM}(\mathbb{R},+), f(\mathbb{R}) = \mathbb{R}$ if and only if f is one of the followings: $g_c, h_{\mathbb{R}}, \overline{k}_c$ and $\underline{k}_c$ where $c \in \mathbb{R} \smallsetminus \{0\}$.*

Proof. We have that $g_0(\mathbb{R}) = \{0\}$, $h_{(0,\infty)}(\mathbb{R}) = (0,\infty)$, $h_{(-\infty,0)}(\mathbb{R}) = (-\infty,0)$, $\overline{k}_0(\mathbb{R}) = [0,\infty)$ and $\underline{k}_0(\mathbb{R}) = (-\infty,0]$. If $c \in \mathbb{R} \smallsetminus \{0\}$, then

$$g_c(\mathbb{R}) = c\mathbb{R} = \mathbb{R},$$

$$\overline{k}_c(\mathbb{R}) = \bigcup_{x\in\mathbb{R}} [cx,\infty) = \begin{cases} c\left(\bigcup_{x\in\mathbb{R}}[x,\infty)\right) = c\mathbb{R} = \mathbb{R} & \text{if } c > 0, \\ c\left(\bigcup_{x\in\mathbb{R}}(-\infty,x]\right) = c\mathbb{R} = \mathbb{R} & \text{if } c < 0, \end{cases}$$

$$\underline{k}_c(\mathbb{R}) = \bigcup_{x\in\mathbb{R}} (-\infty,cx] = \begin{cases} c\left(\bigcup_{x\in\mathbb{R}}(-\infty,x]\right) = c\mathbb{R} = \mathbb{R} & \text{if } c > 0, \\ c\left(\bigcup_{x\in\mathbb{R}}[x,\infty)\right) = c\mathbb{R} = \mathbb{R} & \text{if } c < 0. \end{cases}$$

Since $\mathrm{SIM}(\mathbb{R},+) = \{g_c \mid c \in \mathbb{R}\} \cup \{h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}\} \cup \{\overline{k}_c, \underline{k}_c \mid c \in \mathbb{R}\}$, the result follows. ∎

Lemma 2.2. *The following statements hold for $c, d \in \mathbb{R}$.*

(i) *For a constant multifunction f on $\mathbb{R}$ and a multifunction l on $\mathbb{R}, fl = f$. In particular, if $f \in \{g_0, h_{\mathbb{R}}, h_{(0,\infty)}, h_{(-\infty,0)}, \overline{k}_0, \underline{k}_0\}$, then $fl = f$ for every multifunction l on $\mathbb{R}$.*

(ii) $g_c g_d = g_{cd}$,

(iii) $g_c h_{\mathbb{R}} = \overline{k}_c h_{\mathbb{R}} = \underline{k}_c h_{\mathbb{R}} = h_{\mathbb{R}}$ *if* $c \neq 0$,

(iv) $g_c h_{(0,\infty)} = \begin{cases} h_{(0,\infty)} & \text{if } c > 0, \\ h_{(-\infty,0)} & \text{if } c < 0, \end{cases} \quad g_c h_{(-\infty,0)} = \begin{cases} h_{(-\infty,0)} & \text{if } c > 0, \\ h_{(0,\infty)} & \text{if } c < 0, \end{cases}$

(v) $g_c \overline{k}_d = \begin{cases} \overline{k}_{cd} & \text{if } c > 0, \\ \underline{k}_{cd} & \text{if } c < 0, \end{cases} \quad g_c \underline{k}_d = \begin{cases} \underline{k}_{cd} & \text{if } c > 0, \\ \overline{k}_{cd} & \text{if } c < 0, \end{cases}$

$\overline{k}_c g_d = \overline{k}_{cd}$, $\underline{k}_c g_d = \underline{k}_{cd}$,

(vi) $\overline{k}_c h_{(0,\infty)} = \begin{cases} h_{(0,\infty)} & \textit{if } c > 0, \\ h_{\mathbb{R}} & \textit{if } c < 0, \end{cases}$ $\underline{k}_c h_{(0,\infty)} = \begin{cases} h_{\mathbb{R}} & \textit{if } c > 0, \\ h_{(-\infty,0)} & \textit{if } c < 0, \end{cases}$

$\overline{k}_c h_{(-\infty,0)} = \begin{cases} h_{\mathbb{R}} & \textit{if } c > 0, \\ h_{(0,\infty)} & \textit{if } c < 0, \end{cases}$ $\underline{k}_c h_{(-\infty,0)} = \begin{cases} h_{(-\infty,0)} & \textit{if } c > 0, \\ h_{\mathbb{R}} & \textit{if } c < 0, \end{cases}$

(vii) $\overline{k}_c \overline{k}_d = \begin{cases} \overline{k}_{cd} & \textit{if } c > 0, \\ h_{\mathbb{R}} & \textit{if } c < 0, \end{cases}$ $\overline{k}_c \underline{k}_d = \begin{cases} h_{\mathbb{R}} & \textit{if } c > 0, \\ \overline{k}_{cd} & \textit{if } c < 0, \end{cases}$

$\underline{k}_c \overline{k}_d = \begin{cases} h_{\mathbb{R}} & \textit{if } c > 0, \\ \underline{k}_{cd} & \textit{if } c < 0, \end{cases}$ $\underline{k}_c \underline{k}_d = \begin{cases} \underline{k}_{cd} & \textit{if } c > 0, \\ h_{\mathbb{R}} & \textit{if } c < 0. \end{cases}$

Proof. The proofs of (i) and (ii) are evident, (iii) follows directly from Lemma 2.1 while (iv) and (v) are obviously seen.
(vi) If $x \in \mathbb{R}$, then

$$\begin{aligned}
\overline{k}_c h_{(0,\infty)}(x) &= \bigcup_{t\in(0,\infty)} [ct, \infty) \\
&= \begin{cases} c\left(\bigcup_{t\in(0,\infty)}[t,\infty)\right) = c(0,\infty) = (0,\infty) = h_{(0,\infty)}(x) & \text{if } c > 0, \\ c\left(\bigcup_{t\in(0,\infty)}(-\infty,t]\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\underline{k}_c h_{(0,\infty)}(x) &= \bigcup_{t\in(0,\infty)} (-\infty, ct] \\
&= \begin{cases} c\left(\bigcup_{t\in(0,\infty)}(-\infty,t]\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c > 0, \\ c\left(\bigcup_{t\in(0,\infty)}[t,\infty)\right) = c(0,\infty) = (-\infty,0) = h_{(-\infty,0)}(x) & \text{if } c < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\overline{k}_c h_{(-\infty,0)}(x) &= \bigcup_{t\in(-\infty,0)} [ct, \infty) \\
&= \begin{cases} c\left(\bigcup_{t\in(-\infty,0)}[t,\infty)\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c > 0, \\ c\left(\bigcup_{t\in(-\infty,0)}(-\infty,t]\right) = c(-\infty,0) = (0,\infty) = h_{(0,\infty)}(x) \\ \quad \text{if } c < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
\underline{k}_c h_{(-\infty,0)}(x) &= \bigcup_{t\in(-\infty,0)} (-\infty, ct] \\
&= \begin{cases} c\left(\bigcup_{t\in(-\infty,0)}(-\infty,t]\right) = c(-\infty,0) = (-\infty,0) = h_{(-\infty,0)}(x) \\ \quad \text{if } c > 0, \\ c\left(\bigcup_{t\in(-\infty,0)}[t,\infty)\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c < 0, \end{cases}
\end{aligned}$$

so (vi) is proved.

(vii) Let $x \in \mathbb{R}$. Then

$$\overline{k}_c\overline{k}_d(x) = \bigcup_{t\in[dx,\infty)} [ct,\infty)$$

$$= \begin{cases} c\left(\bigcup_{t\in[dx,\infty)}[t,\infty)\right) = c[dx,\infty) = [cdx,\infty) = \overline{k}_{cd}(x) & \text{if } c > 0, \\ c\left(\bigcup_{t\in[dx,\infty)}(-\infty,t]\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c < 0, \end{cases}$$

$$\overline{k}_c\underline{k}_d(x) = \bigcup_{t\in(-\infty,dx]} [ct,\infty)$$

$$= \begin{cases} c\left(\bigcup_{t\in(-\infty,dx]}[t,\infty)\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) \quad \text{if } c > 0, \\ c\left(\bigcup_{t\in(-\infty,dx]}(-\infty,t]\right) = c(-\infty,dx] = [cdx,\infty) = \overline{k}_{cd}(x) \\ \quad \text{if } c < 0, \end{cases}$$

$$\underline{k}_c\overline{k}_d(x) = \bigcup_{t\in[dx,\infty)} (-\infty,ct]$$

$$= \begin{cases} c\left(\bigcup_{t\in[dx,\infty)}(-\infty,t]\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) & \text{if } c > 0, \\ c\left(\bigcup_{t\in[dx,\infty)}[t,\infty)\right) = c[dx,\infty) = (-\infty,cdx] = \underline{k}_{cd}(x) & \text{if } c < 0, \end{cases}$$

$$\underline{k}_c\underline{k}_d(x) = \bigcup_{t\in(-\infty,dx]} (-\infty,ct]$$

$$= \begin{cases} c\left(\bigcup_{t\in(-\infty,dx]}(-\infty,t]\right) = c(-\infty,dx] = (-\infty,cdx] = \underline{k}_{cd}(x) \\ \quad \text{if } c > 0, \\ c\left(\bigcup_{t\in(-\infty,dx]}[t,\infty)\right) = c\mathbb{R} = \mathbb{R} = h_{\mathbb{R}}(x) \quad \text{if } c < 0. \end{cases}$$

Hence the proof is complete. ∎

The following theorem is directly obtained from Lemma 2.2.

Theorem 2.1. $\mathrm{SIM}(\mathbb{R},+)$ *is a semigroup under composition.*

Theorem 2.2. *All the idempotents of the semigroup* $\mathrm{SIM}(\mathbb{R},+)$ *are* g_0, g_1, $h_{\mathbb{R}}$, $h_{(0,\infty)}$, $h_{(-\infty,0)}$, $\overline{k}_0$, $\overline{k}_1$, $\underline{k}_0$ *and* $\underline{k}_1$.

Proof. It follows from Lemma 2.2(i), (ii) and (vii) that g_0, g_1, $h_{\mathbb{R}}$, $h_{(0,\infty)}$, $h_{(-\infty,0)}$, $\overline{k}_0, \overline{k}_1, \underline{k}_0, \underline{k}_1$ are idempotents of the semigroup $\mathrm{SIM}(\mathbb{R},+)$. If $c \in \mathbb{R}\setminus\{0\}$, by Lemma 2.2(ii), $g_c^2 = g_c$ and then $g_{c^2} = g_c$ which shows $c^2 = g_{c^2}(1) = g_c(1) = c$ and $c = 1$, respectively. Similarly, if $\overline{k}_c^2 = \overline{k}_c$ then $\overline{k}_{c^2} = \overline{k}_c$ and $c > 0$ from Lemma 2.2(vii) which implies $[c^2,\infty) = \overline{k}_{c^2}(1) = \overline{k}_c(1) = [c,\infty)$ and $c = 1$, respectively. Similarly, if $\underline{k}_c^2 = \underline{k}_c$ then $\underline{k}_{c^2} = \underline{k}_c$ and $c > 0$ from Lemma 2.2(vii) which implies $(-\infty,c^2] = \underline{k}_{c^2}(1) = \underline{k}_c(1) = (-\infty,c]$ so $c = 1$.

Therefore the result follows, as desired. ∎

Theorem 2.3. *The semigroup* $\mathrm{SIM}(\mathbb{R},+)$ *is a regular semigroup.*

Proof. Since every idempotent is a regular element, it follows from Theorem 2.2 that g_0, g_1, $h_{\mathbb{R}}$, $h_{(0,\infty)}$, $h_{(-\infty,0)}$, $\overline{k}_0$, $\overline{k}_1$, $\underline{k}_0$ and $\underline{k}_1$ are regular elements of the semigroup $\mathrm{SIM}(\mathbb{R},+)$. Let $c \in \mathbb{R} \setminus \{0\}$. Then by Lemma 2.2(ii), $g_c g_{c^{-1}} g_c = g_{cc^{-1}c} = g_c$. Also,

$$\begin{aligned}\overline{k}_c g_{c^{-1}} \overline{k}_c &= \left(\overline{k}_c g_{c^{-1}}\right) \overline{k}_c \\ &= \overline{k}_{cc^{-1}} \overline{k}_c \quad \text{from Lemma 2.2(v)} \\ &= \overline{k}_1 \overline{k}_c \\ &= \overline{k}_{1c} = \overline{k}_c \quad \text{from Lemma 2.2(vii)},\end{aligned}$$

and

$$\begin{aligned}\underline{k}_c g_{c^{-1}} \underline{k}_c &= \left(\underline{k}_c g_{c^{-1}}\right) \underline{k}_c \\ &= \underline{k}_{cc^{-1}} \underline{k}_c \quad \text{from Lemma 2.2(v)} \\ &= \underline{k}_1 \underline{k}_c \\ &= \underline{k}_{1c} = \underline{k}_c \quad \text{from Lemma 2.2(vii)}.\end{aligned}$$

Therefore $\mathrm{SIM}(\mathbb{R},+)$ is a regular semigroup, as desired. ∎

Remark 2.1. It follows from Lemma 2.2(i) and (vii) that for $c \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}\overline{k}_c \overline{k}_{c^{-1}} \overline{k}_c &= \left(\overline{k}_c \overline{k}_{c^{-1}}\right) \overline{k}_c \\ &= \begin{cases} \overline{k}_1 \overline{k}_c = \overline{k}_c & \text{if } c > 0, \\ h_{\mathbb{R}} \overline{k}_c = h_{\mathbb{R}} & \text{if } c < 0, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\underline{k}_c \underline{k}_{c^{-1}} \underline{k}_c &= \left(\underline{k}_c \underline{k}_{c^{-1}}\right) \underline{k}_c \\ &= \begin{cases} \underline{k}_1 \underline{k}_c = \underline{k}_c & \text{if } c > 0, \\ h_{\mathbb{R}} \underline{k}_c = h_{\mathbb{R}} & \text{if } c < 0. \end{cases}\end{aligned}$$

Therefore the equalities $\overline{k}_c \overline{k}_{c^{-1}} \overline{k}_c = \overline{k}_c$ and $\underline{k}_c \underline{k}_{c^{-1}} \underline{k}_c = \underline{k}_c$ hold only the case that $c > 0$.

Remark 2.2. If e is an idempotent of a semigroup S, then the greatest subgroup of S having e as its identity is

$$G_e = \{x \in S \mid xe = ex = x \text{ and } xy = yx = e \text{ for some } y \in S\}.$$

[4, p.10]. Thus if S has an identity 1, then G_1 is the unit group or the group of units of S and

$$G_1 = \{x \in S \mid xy = yx = 1 \text{ for some } y \in S\}.$$

Let us consider G_f of the semigroup $\mathrm{SIM}(\mathbb{R},+)$ where f is an idempotent of $\mathrm{SIM}(\mathbb{R},+)$. By Theorem 2.2, all the idempotents of $\mathrm{SIM}(\mathbb{R},+)$ are

$$g_0,\ g_1,\ h_{\mathbb{R}},\ h_{(0,\infty)},\ h_{(-\infty,0)},\ \overline{k}_0,\ \overline{k}_1,\ \underline{k}_0,\ \underline{k}_1.$$

It follows directly from Lemma 2.2(i) that

$$G_{g_0} = \{g_0\}, \quad G_{h_{\mathbb{R}}} = \{h_{\mathbb{R}}\}, \quad G_{h_{(0,\infty)}} = \{h_{(0,\infty)}\},$$

$$G_{h_{(-\infty,0)}} = \{h_{(-\infty,0)}\}, \quad G_{\overline{k}_0} = \{\overline{k}_0\}, \quad G_{\underline{k}_0} = \{\underline{k}_0\}.$$

Also, it can be seen from Lemma 2.2(i)–(vii) that the unit group of the semigroup $\mathrm{SIM}(\mathbb{R},+)$ is

$$G_{g_1} = \{g_c \mid c \in \mathbb{R} \setminus \{0\}\}$$

which is clearly isomorphic to the group $(\mathbb{R} \setminus \{0\}, \cdot)$. It follows from Lemma 2.2(i)–(vii) that

$$G_{\overline{k}_1} = \{\overline{k}_c \mid c > 0\} \text{ and } G_{\underline{k}_1} = \{\underline{k}_c \mid c > 0\}.$$

Evidently, both $G_{\overline{k}_1}$ and $G_{\underline{k}_1}$ are isomorphic to the group $((0,\infty), \cdot)$.

Notice

$$\bigcup \{G_f \mid f \text{ is an idempotent of } \mathrm{SIM}(\mathbb{R},+)\} \subsetneq \mathrm{SIM}(\mathbb{R},+).$$

This implies that $\mathrm{SIM}(\mathbb{R},+)$ is not a union of groups. A semigroup S is called an *inverse semigroup* if for every $a \in S$, there is a unique element $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. Then every inverse semigroup is a regular semigroup. By Theorem 2.3, the semigroup $\mathrm{SIM}(\mathbb{R},+)$ is a regular semigroup. It is interesting to know whether $\mathrm{SIM}(\mathbb{R},+)$ is an inverse semigroup. It is well known that a semigroup S is an inverse semigroup if and only if S is a regular semigroup and any two idempotents commute with each other [1, p.28]. From Lemma 2.2(i) and Theorem 2.2, we have that g_0 and $h_{\mathbb{R}}$ are idempotents and $g_0 h_{\mathbb{R}} = g_0 \neq h_{\mathbb{R}} = h_{\mathbb{R}} g_0$. Therefore we deduce that $\mathrm{SIM}(\mathbb{R},+)$ is a regular semigroup which is neither an inverse semigroup nor a union of groups.

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