Extended Upper Sets in BE-Algebras

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Abstract. As a generalization of a BCK-algebra, a BE-algebras was introduced. In this paper, we investigate several properties of upper sets in BE-algebras, and we introduce more extended upper sets of BE-algebras, and obtain some relations with filters of BE-algebras. Also, the notion of Krull dimension of a BE-algebra and the notion of regular sequence in a BE-algebra are introduced.

1. Introduction

The study of BCK/BCI-algebras was initiated by Iséki as a generalization of the concept of set-theoretic difference and propositional calculus [2, 3]. In [6], Neggers and Kim introduced the notion of d-algebras which is a generalization of BCK-algebras. Moreover, Jun, Roh and Kim [4] introduced a new notion, called a BH-algebra, which is a generalization of BCK/BCI-algebras. Recently, as another generalization of BCK-algebras, the notion of a BE-algebra was introduced by Kim and Kim [5]. They provided an equivalent condition of the filters in BE-algebras using the notion of upper sets. In [1], Ahn and So gave several descriptions of ideals in BE-algebras. Also, the fuzzification of ideals in BE-algebras was studied by Jun *et al.* [7]. In this paper, we investigate several properties of upper sets in BE-algebras, and we introduce more extended upper sets of BE-algebras, and obtain some relations with filters of BE-algebras. Also, the notion of Krull dimension of a BE-algebra and the notion of regular sequences in a BE-algebra are introduced.

2. Preliminaries

By a *BE-algebra* we mean an algebra $(X; *, 1)$ of type $(2, 0)$ satisfying the following identities: for any $x, y, z \in X$,

(BE1) $x * x = 1$;
(BE2) $x * 1 = 1$;
(BE3) $1 * x = x$;
(BE4) $x * (y * z) = y * (x * z)$.

The *BE*-algebra determines an relation on X: $x \leq y \Leftrightarrow x * y = 1$ [5].

Definition 2.1. [5] *A non-empty subset F of a BE-algebra X is called a* filter *of X if*

(F1) $1 \in F$;
(F2) $x * y \in F$ and $x \in F$ implies $y \in F$.

Example 2.1.

(1) Let X be a finite (or infinite) set with element 1. Define a binary operation on X as follows: for any $x, y \in X$, $x * y = 1$ if $x = y$ and $x * y = y$ if $x \neq y$. Then $(X; *, 1)$ is a *BE*-algebra, and every non-empty subset containing 1 is a filter of X.

(2) Let $X := \{1, a, b, c\}$ be a set with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Then $(X; *, 1)$ is a *BE*-algebra. Also, $\{1, a\}$ and $\{1, b\}$ are filters of X, but $\{1, c\}$, $\{1, a, b\}$, $\{1, a, c\}$ and $\{1, b, c\}$ are not filters of X.

(3) Let $X := \{1, a, b, c, d\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X; *, 1)$ is a *BE*-algebra [5]. It is easy to see that $\{1, a\}$, $\{1, a, b\}$ and $\{1, a, c\}$ are filters of X, but $\{1, a, b, c\}$ is not a filter of X.

(4) Let $X := \{1, a, b, c, d, 0\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra, and $\{1, a, b\}$ is a filter of X, but $\{1, a\}$ is not a filter of X [5].

Proposition 2.1. *Let X be a BE-algebra. If F_i are filters of X, then $\bigcap_{i \in I} F_i$ is a filter of X.*

Proof. Straightforward. ∎

3. Upper sets

Let X be a BE-algebra. For any $x, y \in X$, we define

$$A(x) := \{z \in X \mid x * z = 1\} \quad \text{and} \quad A(x, y) := \{z \in X \mid x * (y * z) = 1\}.$$

The set $A(x)$ (resp. $A(x, y)$) is called an *upper set* of x (resp. of x and y). Obviously, $1, x \in A(x)$ and $1, x, y \in A(x, y)$. We know that $A(1) = \{1\}$ is always a filter of X. But the sets $A(x)$ and $A(x, y)$ need not be filters of X in general, since $A(a) = A(a, 1) = \{1, a\}$ is not a filter of X in Example 2.1(4).

Example 3.1.
  (1) Consider a BE-algebra X in Example 2.1(1). For any $x, y \in X$, we have that $A(x) = \{1, x\}$ and $A(x, y) = \{1, x, y\}$. Also, every upper set in X is a filter of X.
  (2) Let X be a BE-algebra in Example 2.1(2). Then $A(1) = \{1\}$, $A(a) = A(a, 1) = A(a, a) = \{1, a\}$, $A(b) = A(b, 1) = A(b, b) = \{1, b\}$, and $A(c) = A(c, 1) = A(c, c) = A(a, b) = A(a, c) = A(b, c) = X$.
  (3) In Example 2.1(3), we obtain that $A(a) = \{1, a\}$, $A(c) = \{1, a, c\}$, $A(a, b) = \{1, a, b\}$, $A(b, c) = X$, etc.

Proposition 3.1. *If X is a BE-algebra, then $A(x) \subseteq A(x, y)$ for any $x, y \in X$.*

Proof. Let $z \in A(x)$. Then $x * z = 1$. By (BE2) and (BE4), we have that $x * (y * z) = y * (x * z) = y * 1 = 1$, and hence $z \in A(x, y)$. ∎

Proposition 3.2. *Let X be a BE-algebra and $x \in X$. Then*

$$A(x) = \bigcap_{y \in X} A(x, y).$$

Proof. By Proposition 3.1, we have $A(x) \subseteq \bigcap_{y \in X} A(x, y)$. If $z \in \bigcap_{y \in X} A(x, y)$, then $z \in A(x, y)$ for any $y \in X$, and so $z \in A(x, 1)$. Hence $1 = x * (1 * z) = x * z$, which proves $z \in A(x)$. This means that $\bigcap_{y \in X} A(x, y) \subseteq A(x)$. ∎

Using Proposition 3.1 and Proposition 3.2 we obtain the following:

Corollary 3.1. *Let X be a BE-algebra. Then for any $x \in X$, we have that $A(x) = A(x, 1) = \bigcap_{y \in X} A(x, y)$.*

Proposition 3.3. *If X is a BE-algebra, then $A(x, y) = A(y, x)$ for any $x, y \in X$.*

Proof. It follows immediately from (BE4). ∎

Proposition 3.4. *Let X be a BE-algebra and $\alpha \in X$. Then the followings are equivalent:*

    (i) $\alpha \leq x$ *for any* $x \in X$,
    (ii) $X = A(\alpha)$,
    (iii) $X = A(\alpha, x) = A(x, \alpha)$ *for any* $x \in X$.

Proof. (i) $\Longleftrightarrow$ (ii) Straightforward.
    (ii) $\Longrightarrow$ (iii) $X = A(\alpha) \subseteq A(\alpha, x) \subseteq X$, by Proposition 3.1.
    (iii) $\Longrightarrow$ (ii) $X = A(\alpha, 1) = A(\alpha)$, by Corollary 3.1. ∎

Using the notion of upper set $A(x, y)$, H. S. Kim and Y. H. Kim obtained an equivalent condition of the filter in BE-algebras.

Theorem 3.1. [5] *Let F be a non-empty subset of a BE-algebra X. Then F is a filter of X if and only if $A(x, y) \subseteq F$ for any $x, y \in F$.*

From this theorem and Proposition 3.1 we immediately obtain the following result.

Corollary 3.2. *Let X be a BE-algebra. If F is a filter of X, then $A(x) \subseteq F$ for any $x \in F$.*

However, the converse of Corollary 3.2 need not be true in general. In Example 2.1(4), $F := \{1, a\}$ contains $A(1)$ and $A(a)$, but F is not a filter of X.

Theorem 3.2. [5] *If F is a filter of a BE-algebra X, then $F = \bigcup\limits_{x, y \in F} A(x, y)$.*

Corollary 3.3. [5] *If F is a filter of a BE-algebra X, then*

$$F = \bigcup_{x \in F} A(x, 1).$$

Corollary 3.4. *If F is a filter of a BE-algebra X, then $F = \bigcup\limits_{x \in F} A(x)$.*

Proof. By Corollary 3.1 and Corollary 3.3, we have that

$$F = \bigcup_{x \in F} A(x, 1) = \bigcup_{x \in F} A(x).$$ ∎

Definition 3.1. [5] *A BE-algebra $(X; *, 1)$ is said to be* self distributive *if $x*(y*z) = (x * y) * (x * z)$ for any $x, y, z \in X$.*

The BE-algebras X in Example 2.1(1), (2) and (3) are self distributive, but the BE-algebra X in Example 2.1(4) is not self distributive, since $d * (a * 0) \neq (d * a) * (d * 0)$.

Theorem 3.3. [5] *Let X be a self distributive BE-algebra. Then the upper set $A(x, y)$ is a filter of X for any $x, y \in X$.*

Combining Proposition 2.1 and Proposition 3.2 with Theorem 3.3, we have the following result.

Corollary 3.5. *If X is a self distributive BE-algebra, then the upper set $A(x)$ is a filter of X for any $x \in X$.*

We discuss some relations between $A(x)$ and $A(x, y)$ in a self distributive BE-algebra.

Proposition 3.5. *Let X be a self distributive BE-algebra and let $x, y \in X$. Then $y \in A(x)$ if and only if $A(x) = A(x, y)$.*

Proof. Assume that $y \in A(x)$. Then $x * y = 1$. By Proposition 3.1, $A(x) \subseteq A(x, y)$. For any $z \in A(x, y)$, we have $1 = x * (y * z) = (x * y) * (x * z) = 1 * (x * z) = x * z$, and so $z \in A(x)$. Hence $A(x) = A(x, y)$. Conversely, if $A(x) = A(x, y)$, then $y \in A(x, y) = A(x)$. ∎

From this proposition we obtain the fact that $y \notin A(x)$ if and only if $A(x) \subsetneqq A(x, y)$. In Example 3.1(3), we observe that $A(c) = \{1, a, c\} = A(c, 1) = A(c, a) = A(c, c)$, $A(c) \subsetneqq A(c, b) = X$, and $A(a) = \{1, a\} \subsetneqq A(a, b) = \{1, a, b\}$.

Theorem 3.4. *Let X be a self distributive BE-algebra and let $x, y \in X$. Then $x \leq y$ if and only if $A(y) \subseteq A(x)$.*

Proof. Let $x \leq y$. Then $x * y = 1$. For any $z \in A(y)$, we have $y * z = 1$. It follows from the self distributive law that $x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1$, and so $z \in A(x)$. Hence $A(y) \subseteq A(x)$. Conversely, if $A(y) \subseteq A(x)$, then $y \in A(x)$, and hence $x \leq y$. ∎

In Example 2.1(2), we see that there exists only $c \leq a$ and $c \leq b$ except trivial cases. Also, we observe that there exists only $A(a) \subseteq A(c)$ and $A(b) \subseteq A(c)$ except trivial cases. See Example 3.1(2).

Corollary 3.6. *Let X be a self distributive BE-algebra and let $x, y \in X$. Then $x \leq y$ and $y \leq x$ if and only if $A(y) = A(x)$.*

Example 3.2. Let $X := \{1, a, b, c\}$ be a set with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	c	1	c
c	1	1	b	1

Then we see that $(X; *, 1)$ is a self distributive BE-algebra. Here, it is easy to obtain that $a \leq c$, $c \leq a$ and $A(a) = A(c) = \{1, a, c\}$.

4. Extended upper sets

In this section, let X and $\mathbb{N}$ denote a BE-algebra and the set of all positive integers, respectively, unless otherwise specified.

For any elements $x_1, x_2, \cdots, x_n \in X$ and $n \in \mathbb{N}$, we define

$$A(x_1, x_2, \cdots, x_n) := \{z \in X \mid \prod_{i=1}^{n} x_i * z = 1\},$$

where $\prod_{i=1}^{n} x_i * z := x_n * (x_{n-1} * (\cdots * (x_1 * z) \cdots))$. We call it an *extended upper set* of $x_1, x_2, \cdots, x_n$. It follows from (BE4) that

$$x_n * (x_{n-1} * (\cdots * (x_1 * z) \cdots)) = x_1 * (x_2 * (\cdots * (x_n * z) \cdots))$$

for any $x_1, x_2, \cdots, x_n \in X$. Obviously, $1, x_i \in A(x_1, x_2, \cdots, x_n)$ for any $i = 1, 2, \cdots, n$. For example, we observe that

$$A(x_1, x_2, \cdots, x_n) = \{1, x_1, x_2, \cdots, x_n\}$$

in Example 2.1(1), and $A(a, b, c) = A(a, b, d) = A(a, c, d) = X$ in Example 2.1(3).

Proposition 4.1. *For any $x_1, x_2, \cdots, x_n \in X$ and $n \in \mathbb{N}$, we have*

$$A(x_1) \subseteq A(x_1, x_2) \subseteq \cdots \subseteq A(x_1, x_2, \cdots, x_n).$$

Proof. For any $k = 1, 2, \cdots, n - 1$, let $z \in A(x_1, x_2, \cdots, x_k)$. Then $\prod_{i=1}^{k} x_i * z = 1$, and hence $\prod_{i=1}^{k+1} x_i * z = x_{k+1} * (\prod_{i=1}^{k} x_i * z) = x_{k+1} * 1 = 1$, proving that $z \in A(x_1, x_2, \cdots, x_{k+1})$. This completes the proof. ∎

Proposition 4.2. *For any $x_1, x_2, \cdots, x_n, y \in X$ and $n \in \mathbb{N}$, we have*

$$A(x_1, x_2, \cdots, x_n) = \bigcap_{y \in X} A(x_1, x_2, \cdots, x_n, y).$$

Proof. Let $z \in \bigcap_{y \in X} A(x_1, x_2, \cdots, x_n, y)$. Then $z \in A(x_1, x_2, \cdots, x_n, y)$ for any $y \in X$, and so $z \in A(x_1, x_2, \cdots, x_n, 1)$. Thus we have $1 = \prod_{i=1}^{n} x_i * (1 * z) = \prod_{i=1}^{n} x_i * z$. This means that $z \in A(x_1, x_2, \cdots, x_n)$. The converse of the proof follows from Proposition 4.1. ∎

Similarly, the following result holds by using Proposition 4.1 and Proposition 4.2.

Corollary 4.1. *For any $x_1, x_2, \cdots, x_n, y \in X$ and $n \in \mathbb{N}$, we obtain that*

$$A(x_1, x_2, \cdots, x_n) = A(x_1, x_2, \cdots, x_n, 1) = \bigcap_{y \in X} A(x_1, x_2, \cdots, x_n, y).$$

Proposition 4.3. *Let $x_1, x_2, \cdots, x_n \in X$ and $n \in \mathbb{N}$. Then we have*

$$A(x_1, x_2, \cdots, x_n) = A(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}),$$

where σ is a permutation on $\{1, 2, \cdots, n\}$.

Proposition 4.4. *For $\alpha \in X$, the followings are equivalent:*
  (i) *$\alpha \leq x$ for any $x \in X$,*
  (ii) *$X = A(\alpha)$,*
  (iii) *$X = A(\alpha, x_1, x_2, \cdots, x_n) = A(x_1, \alpha, x_2, \cdots, x_n) = \cdots$*
      $= A(x_1, x_2, \cdots, x_n, \alpha)$ for any $x_1, x_2, \cdots, x_n \in X$.

The proof of Proposition 4.3 is straightforward and the proof of Proposition 4.4 is similar to Proposition 3.4.

The next theorems are similar to Theorem 3.1, Theorem 3.2 and Theorem 3.3.

Theorem 4.1. *Let F be a non-empty subset of X and $n \in \mathbb{N}$. Then F is a filter of X if and only if $A(x_1, x_2, \cdots, x_n) \subseteq F$ for any $x_1, x_2, \cdots, x_n \in F$, where $n \geq 2$.*

Proof. Assume that F is a filter of X. If $z \in A(x_1, x_2, \cdots, x_n)$, then $\prod_{i=1}^{n} x_i * z = 1 \in F$. Since each $x_i \in F$, by (F2), $z \in F$. Conversely, let $A(x_1, x_2, \cdots, x_n) \subseteq F$ for any $x_1, x_2, \cdots, x_n \in F$, where $n \geq 2$. Then $A(x_1, x_2) = A(x_1, x_2, 1, \cdots, 1) \subseteq F$ for any $x_1, x_2 \in F$. By Theorem 3.1, F is a filter of X. ∎

Remark 4.1. The necessity of Theorem 4.1 always holds for any $n \in \mathbb{N}$. But the sufficiency of Theorem 4.1 does not hold when $n = 1$. See Corollary 3.2 below.

Theorem 4.2. *If F is a filter of X and $n \in \mathbb{N}$, then*

$$F = \bigcup_{x_i \in F} A(x_1, x_2, \cdots, x_n).$$

Proof. Let F be a filter of X. By Theorem 4.1, $A(x_1, x_2, \cdots, x_n) \subseteq F$ for any $x_1, x_2, \cdots, x_n \in F$, and hence $\bigcup_{x_i \in F} A(x_1, x_2, \cdots, x_n) \subseteq F$. Also, it follows from Corollary 3.4 and Corollary 4.1 that

$$F = \bigcup_{x \in F} A(x) = \bigcup_{x \in F} A(x, 1, \cdots, 1) \subseteq \bigcup_{x_i \in F} A(x_1, x_2, \cdots, x_n).$$ ∎

Theorem 4.3. *If X is self distributive, then $A(x_1, x_2, \cdots, x_n)$ is a filter of X for any $x_1, x_2, \cdots, x_n \in X$ and $n \in \mathbb{N}$.*

Proof. Clearly $1 \in A(x_1, x_2, \cdots, x_n)$. Let $x * y \in A(x_1, x_2, \cdots, x_n)$ and $x \in A(x_1, x_2, \cdots, x_n)$. Then $\prod_{i=1}^{n} x_i * (x * y) = 1$ and $\prod_{i=1}^{n} x_i * x = 1$. It follows from the self distributive law that

$$1 = \prod_{i=1}^{n} x_i * (x * y) = \left(\prod_{i=1}^{n} x_i * x\right) * \left(\prod_{i=1}^{n} x_i * y\right)$$

$$= 1 * \left(\prod_{i=1}^{n} x_i * y\right) = \prod_{i=1}^{n} x_i * y,$$

and hence $y \in A(x_1, x_2, \cdots, x_n)$. This proves that $A(x_1, x_2, \cdots, x_n)$ is a filter of X for any $x_1, x_2, \cdots, x_n \in X$ and $n \in \mathbb{N}$. ∎

Proposition 4.5. *Let X be self distributive and let $x_1, x_2, \cdots, x_n, y \in X$ and $n \in \mathbb{N}$. Then $y \in A(x_1, x_2, \cdots, x_n)$ if and only if $A(x_1, x_2, \cdots, x_n) = A(x_1, x_2, \cdots, x_n, y)$.*

Proof. The proof is similar to Proposition 3.5. ∎

Proposition 4.6. *Let X be self distributive and let $x_1, x_2, \cdots, x_n \in X$ and $n \in \mathbb{N}$. If $x_1 \leq x_2 \leq \cdots \leq x_n$, then we have*

$$A(x_1) = A(x_1, x_2) = \cdots = A(x_1, x_2, \cdots, x_n).$$

Proof. If $x_1 \leq x_2$, then $x_1 * x_2 = 1$, and so $x_2 \in A(x_1)$. By Proposition 4.5, $A(x_1) = A(x_1, x_2)$. If $x_2 \leq x_3$, then $x_2 * x_3 = 1$, and so $x_3 \in A(x_2) \subseteq A(x_1, x_2)$. Also we have $A(x_1, x_2) = A(x_1, x_2, x_3)$ by Proposition 4.5. Continuing this process, we obtain our result. ∎

The converse of Proposition 4.6 may not be true, since $A(a) = A(d, a) = A(d, a, b) = X$, but $a \nleq b$ in Example 2.1(3).

5. Krull dimension and regular sequences

In this section, we introduce the notion of Krull dimension of a BE-algebra X and the notion of regular sequences in a BE-algebra X, and provide the relation between these ones. Let us denote X and $\mathbb{N}$ by as before in Section 4, unless otherwise specified.

Now, we consider a chain

$$\mathbb{F}: \quad F_0 = \{1\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = X$$

of distinct filters in X. If n is finite, then we say that $\mathbb{F}$ is a *finite chain* in X and n is the *length* of $\mathbb{F}$. Otherwise, $\mathbb{F}$ is said to be an *infinite chain* in X. In Example 2.1(3), if we let

$$\mathbb{F}: \quad F_0 = \{1\} \subset F_1 = \{1, a, c\} \subset F_2 = X,$$
$$\mathbb{G}: \quad G_0 = \{1\} \subset G_1 = \{1, a\} \subset G_2 = \{1, a, b\} \subset G_3 = X,$$
$$\mathbb{H}: \quad H_0 = \{1\} \subset H_1 = \{1, a\} \subset H_2 = \{1, a, c\} \subset H_3 = X,$$

then $\mathbb{F}$ is a finite chain of length 2, and both $\mathbb{G}$ and $\mathbb{H}$ are finite chains of length 3 in X.

Definition 5.1. *The maximal length of any chain of distinct filters in X is called the* Krull dimension *of X, denoted by* $\mathrm{Kdim}(X)$*, and this chain with the maximal length is said to be a* maximal chain *of distinct filters in X.*

Example 5.1.
   (1) Let X be as before in Example 2.1(1). If X is an infinite set, then $\mathrm{Kdim}(X) = \infty$, and if $|X| = n$, then $\mathrm{Kdim}(X) = n - 1$, since every non-empty subset containing 1 is a filter of X.
   (2) In Example 2.1(3), we obtain that $\mathrm{Kdim}(X) = 3$, since

$$\mathbb{F}: \quad F_0 = \{1\} \subset F_1 = \{1, a\} \subset F_2 = \{1, a, b\} \subset F_3 = X$$

   is a maximal chain of distinct filters in X.

Proposition 5.1. *Let X be self distributive. If*

$$\mathbb{F}: \quad F_0 = \{1\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = X$$

is a maximal chain of distinct filters in X, then each $F_k = A(x_1, x_2, \cdots, x_k)$ for some $x_i \in F$.

Proof. First we know that $F_0 = \{1\} = A(1)$. Since F_1 is a filter of X, if we take $x_1 \neq 1$ in F_1, then $F_0 = \{1\} \subset A(x_1) \subseteq F_1$ by Corollary 3.2. Since $A(x_1)$ is a filter of X, if $F_1 \neq A(x_1)$, this contradicts to the maximality, and so $F_1 = A(x_1)$. Now, we take $x_2 \in F_2 \setminus F_1$. Then $F_1 = A(x_1) \subset A(x_1, x_2) \subseteq F_2$ by Proposition 4.1 and Theorem 4.1. Since $A(x_1, x_2)$ is also a filter of X, if $F_2 \neq A(x_1, x_2)$, this contradicts to the maximality, and so $F_2 = A(x_1, x_2)$. Continuing this process, we have that each $F_k = A(x_1, x_2, \cdots, x_k)$ for some $x_i \in F$. ∎

Corollary 5.1. *Let X be self distributive. If $\mathrm{Kdim}(X) = n$, then there exists a chain of distinct extended upper sets in X as follows:*

$$\mathbb{F}: \quad A(1) = \{1\} \subset A(x_1) \subset A(x_1, x_2) \subset \cdots \subset A(x_1, x_2, \cdots, x_n) = X.$$

Proof. Assume that $\mathrm{Kdim}(X) = n$. Then there exists a maximal chain

$$\mathbb{F}: \quad F_0 = \{1\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = X$$

of distinct filters in X. It follows from Proposition 5.1 that each F_k has of the form $A(x_1, x_2, \cdots, x_k)$ for some $x_i \in F$. Hence we have our result. ∎

Theorem 5.1. *Let X be self distributive. Then $\mathrm{Kdim}(X) = n$ if and only if there exists a maximal chain of distinct upper sets in X as follows:*

$$\mathbb{F}: \quad A(1) = \{1\} \subset A(x_1) \subset A(x_1, x_2) \subset \cdots \subset A(x_1, x_2, \cdots, x_n) = X.$$

Proof. Assume that $\mathrm{Kdim}(X) = n$. By Corollary 5.1, there exists a chain

$$\mathbb{F}: \quad A(1) = \{1\} \subset A(x_1) \subset A(x_1, x_2) \subset \cdots \subset A(x_1, x_2, \cdots, x_n) = X$$

of distinct upper sets in X. If $\mathbb{G}$ is another chain of distinct upper sets in X with length m, then $n \geq m$, since all upper sets are filters of X by Theorem 4.3. Hence $\mathbb{F}$ is a maximal chain of distinct upper sets in X. Conversely, assume that

$$\mathbb{F}: \quad A(1) = \{1\} \subset A(x_1) \subset A(x_1, x_2) \subset \cdots \subset A(x_1, x_2, \cdots, x_n) = X$$

is a maximal chain of distinct upper sets in X. Since all $A(x_1, x_2, \cdots, x_k)$ are filters of X, $\mathrm{Kdim}(X) \geq n$. Let $\mathrm{Kdim}(X) = m$. By Corollary 5.1, there exists a chain

$$A(1) = \{1\} \subset A(y_1) \subset A(y_1, y_2) \subset \cdots \subset A(y_1, y_2, \cdots, y_m) = X$$

of distinct upper sets in X. Since $\mathbb{F}$ is a maximal chain of distinct upper sets in X, $n \geq m$. Hence $n = m$. ∎

Definition 5.2. *A sequence $a_1, a_2, \cdots, a_n$ in X is called* regular *if it is a maximal sequence in X such that $1 = a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n$ and $A(a_{k-1}) \neq A(a_k)$ for any $k = 1, 2, \cdots, n$.*

Remark 5.1. Let $a_1, a_2, \cdots, a_n$ be a regular sequence in X. Then the condition $A(a_{k-1}) \neq A(a_k)$ is equivalent to $a_{k-1} \nleq a_k$, for any $k = 1, 2, \cdots, n$, since we have that $a_{k-1} \geq a_k$. See Corollary 3.6.

Example 5.2.
  (1) In Example 2.1(1), every regular sequence in X is of the form $1 \geq x$ for any $x \in X$.
  (2) In Example 2.1(2), there are only two regular sequences in X, i.e., $1 \geq a \geq c$ and $1 \geq b \geq c$. And $1 \geq a \geq c \geq d$ is only a regular sequence in X, which was defined in Example 2.1(3).

Lemma 5.1. *Let X be self distributive. If $a_1, a_2, \cdots, a_n$ is a regular sequence in X, then $A(1) = \{1\} \subset A(a_1) \subset A(a_2) \subset \cdots \subset A(a_n)$ is a chain of distinct upper sets in X.*

Proof. It follows from Theorem 3.4 and Definition 5.2. ∎

Proposition 5.2. *Let X be self distributive and let $a_1, a_2, \cdots, a_n$ be a regular sequence in X. Then*

$$\mathbb{F}: \quad A(1) = \{1\} \subset A(a_1) \subset A(a_1, a_2) \subset \cdots \subset A(a_1, a_2, \cdots, a_n) = X$$

is a chain of distinct upper sets in X.

Proof. By Proposition 4.1, we can see that

$$A(1) = \{1\} \subseteq A(a_1) \subseteq A(a_1, a_2) \subseteq \cdots \subseteq A(a_1, a_2, \cdots, a_n)$$

is a chain of upper sets in X. By applying Proposition 4.5 and Lemma 5.1, we have $A(a_k) = A(a_1, a_2, \cdots, a_k)$ for any $k = 1, 2, \cdots, n$. It follows from Lemma 5.1 that $\mathbb{F}$ is a chain of distinct upper sets in X. ∎

Combining this proposition with Theorem 5.1, we have directly the following result.

Corollary 5.2. *Let X be self distributive. If $a_1, a_2, \cdots, a_n$ is a regular sequence in X, then $\mathrm{Kdim}(X) \geq n$.*

Theorem 5.2. *Let X be self distributive and let $a_1, a_2, \cdots, a_n$ be a regular sequence in X. If $A(a_n) = X$, then $\mathrm{Kdim}(X) = n$. Moreover,*

$$\mathbb{F}: \quad A(1) = \{1\} \subset A(a_1) \subset A(a_2) \subset \cdots \subset A(a_n) = X$$

is a maximal chain of distinct upper sets in X.

Proof. By Corollary 5.2, $\mathrm{Kdim}(X) \geq n$. Let $\mathrm{Kdim}(X) = m$. Then we have a maximal chain of distinct upper sets in X as follows:

$$A(1) = \{1\} \subset A(x_1) \subset A(x_1, x_2) \subset \cdots \subset A(x_1, x_2, \cdots, x_m) = X,$$

by Theorem 5.1. Since $A(a_n) = X = A(x_1, x_2, \cdots, x_m)$, $a_n \leq x_k$ for any $k = 1, 2, \cdots, m$ by Proposition 4.4. Now, we show that $A(x_{k-1}) \neq A(x_k)$ for any $k = 1, 2, \cdots, m$, where $x_0 = 1$. For any $k = 1, 2, \cdots, m$, if $A(x_{k-1}) = A(x_k)$, then $x_{k-1} * x_k = 1$, and hence $x_k \in A(x_1, x_2, \cdots, x_{k-1})$. This means that $A(x_1, x_2, \cdots, x_{k-1}) = A(x_1, x_2, \cdots, x_{k-1}, x_k)$ by Proposition 4.5, which is a contradiction. Thus every sequence satisfying the regularity in $\{x_1, x_2, \cdots, x_m\}$ has a length $\leq m + 1$. Since $a_1, a_2, \cdots, a_n$ is a regular sequence in X, $m + 1 \leq n$, and hence $m \leq n$, i.e., $m = n$. Finally, if follows from Lemma 5.1 and Proposition 5.2 that $\mathbb{F}$ is a maximal chain of distinct upper sets in X. ∎

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