BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(1) (2012), 1–14

Jensen's Operator and Applications to Mean Inequalities for Operators in Hilbert Space

[1]MARIO KRNIĆ, [2]NEDA LOVRIČEVIĆ AND [3]JOSIP PEČARIĆ
[1]Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, Croatia
[2]Faculty of Civil Engineering and Architecture, Matice hrvatske 15, 21000 Split, Croatia
[3]Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia
[1]Mario.Krnic@fer.hr, [2]neda.lovricevic@gradst.hr, [3]pecaric@element.hr

Abstract. In this paper we consider Jensen's operator, which includes bounded self-adjoint operator on Hilbert space, and investigate its properties. Due to derived properties, we find lower and upper bound for Jensen's operator, which are multiples of non-weighted Jensen's operator. On the other hand, we also establish some bounds for spectra of Jensen's operator by means of discrete Jensen's functional. The obtained results are then applied to operator means. In such a way, we get refinements and conversions of numerous mean inequalities for Hilbert space operators.

2010 Mathematics Subject Classification: Primary: 47A63; Secondary: 26D15

Keywords and phrases: Jensen's inequality, Jensen's functional, Jensen's operator, Hilbert space, bounded self-adjoint operator, positive invertible operator, arithmetic operator mean, geometric operator mean, harmonic operator mean, superadditivity, monotonicity, refinement, conversion, Kantorovich constant.

1. Introduction

Let H be a Hilbert space and let $\mathcal{B}_h(H)$ be the semi-space of all bounded self-adjoint operators on H. Besides, let $\mathcal{B}^+(H)$ and $\mathcal{B}^{++}(H)$ respectively denote the sets of all positive and positive invertible operators in $\mathcal{B}_h(H)$. The weighted operator harmonic mean $!_\mu$, geometric mean $\sharp_\mu$, and arithmetic mean ∇_μ, for $\mu \in [0,1]$ and $A, B \in \mathcal{B}^{++}(H)$, are defined as follows:

$$A\,!_\mu\,B = \big((1-\mu)A^{-1} + \mu B^{-1}\big)^{-1}, \tag{1.1}$$

$$A\,\sharp_\mu\,B = A^{\frac{1}{2}}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{\mu}A^{\frac{1}{2}}, \tag{1.2}$$

$$A\,\nabla_\mu\,B = (1-\mu)A + \mu B. \tag{1.3}$$

If $\mu = 1/2$, we write $A\,!\,B$, $A\,\sharp\,B$, $A\,\nabla\,B$ for brevity, respectively. The above definitions and notations will be valid throughout the whole paper.

Communicated by Mohammad Sal Moslehian.
Received: March 3, 2011; *Revised:* April 15, 2011.

It is well known the arithmetic-geometric-harmonic mean inequality

$$A \,!_\mu\, B \le A \,\sharp_\mu\, B \le A \,\nabla_\mu\, B, \quad \mu \in [0,1], \tag{1.4}$$

with respect to operator order. Such mean inequalities for Hilbert space operators lie in the fields of interest of numerous mathematicians and we refer here to some recent results.

Inspired by Furuichi's refinement of arithmetic-geometric mean (see [4])

$$A \,\nabla_\mu\, B - A \,\sharp_\mu\, B \ge 2\min\{\mu, 1-\mu\}\left[A \,\nabla\, B - A \,\sharp\, B\right], \quad \mu \in [0,1], \tag{1.5}$$

Zuo *et al.* [15] also obtained refinement of arithmetic-harmonic mean in difference form. More precisely, they have bounded the difference between weighted arithmetic and harmonic mean with the difference of non-weighted means, that is

$$A \,\nabla_\mu\, B - A \,!_\mu\, B \ge 2\min\{\mu, 1-\mu\}\left[A \,\nabla\, B - A \,!\, B\right], \quad \mu \in [0,1], \tag{1.6}$$

where $A, B \in \mathcal{B}^{++}(H)$.

Recently, Kittaneh *et al.* [9] obtained the following refinement and conversion of arithmetic-geometric operator mean inequality,

$$\begin{aligned} & 2\max\{p_1, p_2\}\left[A \,\nabla\, B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right] \\ & \ge (p_1 + p_2)\left[A \,\nabla_{\frac{p_1}{p_1+p_2}}\, B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C\right] \\ & \ge 2\min\{p_1, p_2\}\left[A \,\nabla\, B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}} C\right], \end{aligned} \tag{1.7}$$

where $A, B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$. Here (and throughout the whole paper), $\mathcal{B}^{-1}(H)$ denotes the set of invertible operators on Hilbert space H. Note that operator $C^*\left(C^{*-1}BC^{-1}\right)^\mu C$ represents generalization of geometric mean, since for $C = A^{\frac{1}{2}}$ it becomes $A \,\sharp_\mu\, B$, $\mu \in [0,1]$.

In a similar manner, Zuo *et al.* [15] also improved arithmetic-geometric mean inequality via Kantorovich constant. More precisely, they proved that

$$A \,\nabla_\mu\, B \ge K\left(\frac{M}{m}, 2\right)^{\min\{\mu, 1-\mu\}} A \,\sharp_\mu\, B, \quad \mu \in [0,1], \tag{1.8}$$

where A, B are positive operators satisfying

$$\begin{aligned} & 0 < m' 1_H \le A \le m 1_H < M 1_H \le B \le M' 1_H \quad \text{or} \\ & 0 < m' 1_H \le B \le m 1_H < M 1_H \le A \le M' 1_H, \end{aligned} \tag{1.9}$$

$K(t, 2)$ is the well known Kantorovich constant, i.e. $K(t, 2) = (t+1)^2/4t$, $t > 0$, and 1_H is an identity operator on Hilbert space. Note also that inequality (1.8) improves Furuichi's result from [5], which includes the well known Specht's ratio instead of Kantorovich constant.

The similar problem area, as presented here, was also considered in papers [7, 8, 13, 11, 14]. In addition, for a comprehensive inspection of the recent results about inequalities for bounded self-adjoint operators on Hilbert space, the reader is referred to [6].

On the other hand, it is well known that all inequalities between classical means can be established by means of one of the most important inequality in mathematical analysis, that is Jensen's inequality. Furthermore, Dragomir *et al.* (see [1]) investigated the properties of discrete Jensen's functional

$$J_n(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right), \tag{1.10}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$, $n \geq 2$, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is positive n-tuple of real numbers with $P_n = \sum_{i=1}^{n} p_i$. They obtained that such functional is superadditive on the set of positive real n-tuples, that is

$$J_n(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geq J_n(f,\mathbf{x},\mathbf{p}) + J_n(f,\mathbf{x},\mathbf{q}). \tag{1.11}$$

Further, above functional is also increasing in the same setting, that is,

$$J_n(f,\mathbf{x},\mathbf{p}) \geq J_n(f,\mathbf{x},\mathbf{q}) \geq 0, \tag{1.12}$$

where $\mathbf{p} \geq \mathbf{q}$ (i.e. $p_i \geq q_i$, $i = 1, 2, \ldots, n$). Monotonicity property of discrete Jensen's functional was proved few years before (see [12, p.717]). Above mentioned properties provided improvements of numerous classical inequalities. For more details about such extensions see [1].

Guided by latter idea, in this paper we shall establish Jensen's operator, which includes self-adjoint operator on Hilbert space, and investigate its properties. We are going to provide the properties of superadditivity and monotonicity to hold more generally, by analyzing such operator in above mentioned setting. Moreover, we shall obtain all operator inequalities presented in this Introduction as a simple consequences of more general results.

The paper is organized in the following way: After Introduction, in Section 2 we present some auxiliary results that would be used in the sequel. Further, in Section 3 we establish general Jensen's operator, deduce its important properties and incorporate them with the results presented in Introduction. Finally, in Section 4 we establish lower and upper bounds for spectra of Jensen's operator by means of one interesting property of discrete Jensen's functional, concerning its monotonicity. In such a way we get both refinements and conversions of previously known mean inequalities for operators in Hilbert space.

The techniques that will be used in the proofs are mainly based on classical real and functional analysis, especially on the well known monotonicity property for operator functions.

2. Preliminaries

In this short section we point out two important facts that will help us in establishing our general results.

First of them deals with the well known monotonicity property for bounded self-adjoint operators on Hilbert space: If $X \in \mathcal{B}_h(H)$ with a spectra $\mathrm{Sp}(X)$, then

$$f(t) \geq g(t),\ t \in \mathrm{Sp}(X) \quad \Longrightarrow \quad f(X) \geq g(X), \tag{2.1}$$

provided that f and g are real valued continuous functions. For more details about this property and its consequences the reader is referred to [6].

The other fact, a recent result from [10], is a kind of monotonicity of Jensen's functional, observed as a function in one variable. It is a content of the following lemma.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, and let $\delta \in [a, b]$, $p \in \langle 0, 1\rangle$ be fixed parameters. Then, the function $\varphi : [a, b] \to \mathbb{R}$, defined by*

$$\varphi(t) = (1 - p)f(\delta) + pf(t) - f((1 - p)\delta + pt),$$

satisfies the following properties:

(i) *Function φ is decreasing on $[a, \delta]$;*
(ii) *Function φ is increasing on $[\delta, b]$.*

For the proof of Lemma 2.1, the reader is reffered to [10].

3. Jensen's operator and its properties

The starting point in this section is a general definition of Jensen's operator in Hilbert space.

Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function and let $\mathcal{F}([a, b], \mathbb{R})$ denotes the set of all continuous convex functions on interval $[a, b]$. We define Jensen's operator $\mathcal{J} : \mathcal{F}([a, b], \mathbb{R}) \times \mathcal{B}_h(H) \times [a, b] \times \mathbb{R}^2_+ \to \mathcal{B}^+(H)$ as

$$\mathcal{J}(f, D, \delta, \mathbf{p}) = p_1 f(D) + p_2 f(\delta)1_H - (p_1 + p_2)f\left(\frac{p_1 D + p_2\delta 1_H}{p_1 + p_2}\right), \tag{3.1}$$

where $\mathbf{p} = (p_1, p_2)$, $a1_H \le D \le b1_H$, and 1_H denotes identity operator on Hilbert space H.

Note that operator $\mathcal{J}$ is well-defined. Namely, positivity of underlying operator $\mathcal{J}(f, D, \delta, \mathbf{p})$ follows from Jensen's inequality and monotonicity property (2.1) for operator functions. For the sake of simplicity, positive operator $\mathcal{J}(f, D, \delta, \mathbf{p})$ will also be referred to as Jensen's operator.

Now, we are ready to state and prove our first result that shows that the properties of superadditivity and monotonicity hold in a more general manner.

Theorem 3.1. *Suppose $\mathcal{J}$ is an operator defined by (3.1). Then it satisfies the following properties:*

(i) *$\mathcal{J}(f, D, \delta, \cdot)$ is superadditive on $\mathbb{R}^2_+$, that is*

$$\mathcal{J}(f, D, \delta, \mathbf{p} + \mathbf{q}) \ge \mathcal{J}(f, D, \delta, \mathbf{p}) + \mathcal{J}(f, D, \delta, \mathbf{q}). \tag{3.2}$$

(ii) *If $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2_+$ with $\mathbf{p} \ge \mathbf{q}$ (i.e. $p_1 \ge q_1$, $p_2 \ge q_2$), then*

$$\mathcal{J}(f, D, \delta, \mathbf{p}) \ge \mathcal{J}(f, D, \delta, \mathbf{q}) \ge 0, \tag{3.3}$$

i.e. $\mathcal{J}(f, D, \delta, \cdot)$ is increasing on $\mathbb{R}^2_+$.

Proof. Consider Jensen's functional (1.10) for $n = 2$, and $\mathbf{x} = (x, \delta)$, that is

$$j(f, x, \delta, \mathbf{p}) = p_1 f(x) + p_2 f(\delta) - (p_1 + p_2)f\left(\frac{p_1 x + p_2\delta}{p_1 + p_2}\right). \tag{3.4}$$

Clearly, superadditivity and monotonicity properties (1.11) and (1.12) provide inequalities

$$j(f, x, \delta, \mathbf{p} + \mathbf{q}) \ge j(f, x, \delta, \mathbf{p}) + j(f, x, \delta, \mathbf{q}), \tag{3.5}$$

and

$$j(f, x, \delta, \mathbf{p}) \geq j(f, x, \delta, \mathbf{q}), \qquad \mathbf{p} \geq \mathbf{q}. \tag{3.6}$$

On the other hand, according to monotonicity property (2.1) for operator functions, inequalities (3.5) and (3.6) are also valid in Hilbert space if we replace x with an operator $D \in \mathcal{B}_h(H)$, assuming $a1_H \leq D \leq b1_H$. The proof is now completed since $j(f, D, \delta, \mathbf{p}) = \mathcal{J}(f, D, \delta, \mathbf{p})$. ∎

Superadditivity and monotonicity properties of Jensen's operator are very important properties, considering the numerous applications that will follow from them. First, regarding monotonicity property (3.3), we give the consequence of Theorem 3.1, which includes the lower and upper bound for operator $\mathcal{J}(f, D, \delta, \mathbf{p})$, by means of non-weighted operator.

Corollary 3.1. *Suppose $\mathcal{J}$ is an operator defined by* (3.1). *Then,*

$$2\max\{p_1, p_2\}\mathcal{J}_{\mathcal{N}}(f, D, \delta) \geq \mathcal{J}(f, D, \delta, \mathbf{p}) \geq 2\min\{p_1, p_2\}\mathcal{J}_{\mathcal{N}}(f, D, \delta), \tag{3.7}$$

where

$$\mathcal{J}_{\mathcal{N}}(f, D, \delta) = \frac{f(D) + f(\delta)1_H}{2} - f\left(\frac{D + \delta 1_H}{2}\right).$$

Proof. It is natural to compare ordered pair $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$ with the constant ordered pairs

$$\mathbf{p}_{\max} = \big(\max\{p_1, p_2\}, \max\{p_1, p_2\}\big) \text{ and } \mathbf{p}_{\min} = \big(\min\{p_1, p_2\}, \min\{p_1, p_2\}\big).$$

Clearly, $\mathbf{p}_{\max} \geq \mathbf{p} \geq \mathbf{p}_{\min}$, hence yet another use of property (3.3) yields interpolating series of inequalities:

$$\mathcal{J}(f, D, \delta, \mathbf{p}_{\max}) \geq \mathcal{J}(f, D, \delta, \mathbf{p}) \geq \mathcal{J}(f, D, \delta, \mathbf{p}_{\min}).$$

Finally, since $\mathcal{J}(f, D, \delta, \mathbf{p}_{\max}) = 2\max\{p_1, p_2\}\mathcal{J}_{\mathcal{N}}(f, D, \delta)$ and $\mathcal{J}(f, D, \delta, \mathbf{p}_{\min}) = 2\min\{p_1, p_2\}\mathcal{J}_{\mathcal{N}}(f, D, \delta)$, we get relation (3.7). ∎

Our next consequence of Theorem 3.1 provides Jensen's operator deduced from (3.1), which includes several self-adjoint operators on Hilbert space. More precisely, if $\mathcal{J}$ is Jensen's operator defined by (3.1), we consider the operator

$$\begin{aligned} &C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right)C \\ &= p_1 C^* f(C^{*-1}BC^{-1})C + p_2 f(\delta)A \\ &\quad - (p_1 + p_2)C^* f\bigg(\frac{p_1 C^{*-1}BC^{-1} + p_2\delta 1_H}{p_1 + p_2}\bigg)C, \end{aligned} \tag{3.8}$$

where $A \in \mathcal{B}^{++}(H)$, $B \in \mathcal{B}_h(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, and $aA \leq B \leq bA$. For the sake of simplicity we are going to use abbreviation $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right)C$ for the operator defined by (3.8).

Remark 3.1. We easily conclude that the operator $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right)C$ is well-defined under above assumptions. Namely, condition $aA \leq B \leq bA$ implies $a1_H \leq C^{*-1}BC^{-1} \leq b1_H$, that is, spectra of operator $C^{*-1}BC^{-1}$ belongs to domain of function f.

Our next two results show that deduced operator $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right)C$ possess the same properties as original Jensen's operator defined by (3.1).

Theorem 3.2. *Let $A \in \mathcal{B}^{++}(H)$, $B \in \mathcal{B}_h(H)$, $C \in \mathcal{B}^{-1}(H)$ be the operators satisfying $A = C^*C$ and $aA \le B \le bA$. If $\mathcal{J}$ is an operator defined by (3.1), then the operator $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right) C$ has the following properties:*

(i) *$C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \cdot\right) C$ is superadditive on $\mathbb{R}_+^2$, that is*

$$\begin{aligned} &C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}+\mathbf{q}\right) C \\ &\ge C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right) C + C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{q}\right) C. \end{aligned} \tag{3.9}$$

(ii) *If $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^2$ with $\mathbf{p} \ge \mathbf{q}$ (i.e. $p_1 \ge q_1$, $p_2 \ge q_2$), then*

$$C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right) C \ge C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{q}\right) C \ge 0, \tag{3.10}$$

i.e. $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \cdot\right) C$ is increasing on $\mathbb{R}_+^2$.

Proof. If operator $\mathcal{J}$ is defined by (3.1), then, according to Remark 3.1, the operator $\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right)$ is well-defined. Due to superadditivity property (3.2) of operator $\mathcal{J}$ we conclude that operator

$$\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}+\mathbf{q}\right) - \mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right) - \mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{q}\right) \tag{3.11}$$

is positive, i.e. belongs to $\mathcal{B}^+(H)$. Now if we multiply operator (3.11) by C^* on the left and by C on the right, we again get positive operator. Obviously, as a consequence, we get superadditivity property (3.9). Monotonicity property (3.10) is deduced in the same way. ∎

Remark 3.2. If C is the square root of operator A, that is $C = A^{\frac{1}{2}}$, then operator (3.8) takes form $A^{\frac{1}{2}}\mathcal{J}\left(f, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, \delta, \mathbf{p}\right) A^{\frac{1}{2}}$. Of course, that operator is also superadditive and increasing on $\mathbb{R}_+^2$.

Our next result yields bounds for operator $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right) C$ expressed in terms of non-weighted operator. Let's also mention that such result will enable us to deduce some interpolating inequalities which include operator means presented in Introduction.

Corollary 3.2. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous convex function and $\delta \in [a, b]$. Suppose $A \in \mathcal{B}^{++}(H)$, $B \in \mathcal{B}_h(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$. If $aA \le B \le bA$ then*

$$\begin{aligned} &2\max\{p_1, p_2\} C^*\mathcal{J}_\mathcal{N}(f, C^{*-1}BC^{-1}, \delta) C \ge C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right) C \\ &\ge 2\min\{p_1, p_2\} C^*\mathcal{J}_\mathcal{N}(f, C^{*-1}BC^{-1}, \delta) C, \end{aligned} \tag{3.12}$$

where

$$\mathcal{J}_\mathcal{N}(f, C^{*-1}BC^{-1}, \delta) = \frac{f(C^{*-1}BC^{-1}) + f(\delta)1_H}{2} - f\left(\frac{C^{*-1}BC^{-1} + \delta 1_H}{2}\right),$$

and $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right) C$ is defined by (3.8).
In particular,

$$\begin{aligned} &2\max\{p_1, p_2\} A^{\frac{1}{2}}\mathcal{J}_\mathcal{N}\left(f, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, \delta\right) A^{\frac{1}{2}} \ge A^{\frac{1}{2}}\mathcal{J}\left(f, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, \delta, \mathbf{p}\right) A^{\frac{1}{2}} \\ &\ge 2\min\{p_1, p_2\} A^{\frac{1}{2}}\mathcal{J}_\mathcal{N}\left(f, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, \delta\right) A^{\frac{1}{2}}. \end{aligned} \tag{3.13}$$

Proof. The proof follows the same lines as the proof of Corollary 3.1, except we use monotonicity property (3.10) from Theorem 3.2 instead of monotonicity property (3.3) from Theorem 3.1. ∎

As we have already mentioned, Corollary 3.2 enables us to provide numerous refinements and conversions of means inequalities for operators in Hilbert space. Recall, inequality (1.6) from Introduction provides refinement of arithmetic-harmonic mean inequality. We also get conversion of above mentioned inequality as a special case of relation (3.13).

Remark 3.3. Consider series of inequalities (3.13) equipped with continuous convex function $f : \langle 0, \infty\rangle \to \mathbb{R}$, $f(x) = 1/x$, and parameter $\delta = 1$. In that setting, $A, B \in \mathcal{B}^{++}(H)$. Furthermore, if we replace operators A and B, respectively with A^{-1} and B^{-1}, then the right inequality in (3.13) becomes inequality (1.6), that is refinement of arithmetic-harmonic mean inequality. On the other hand, the left inequality in (3.13) yields conversion of arithmetic-harmonic mean inequality, that is

$$2\max\{p_1, p_2\}\left[A \nabla B - A \,!\, B\right] \geq (p_1 + p_2)\left[A \nabla_{\frac{p_1}{p_1+p_2}} B - A \,!_{\frac{p_1}{p_1+p_2}} B\right]. \tag{3.14}$$

Remark 3.4. Now, we consider series of inequalities (3.12) equipped with convex function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \exp x$, and parameter $\delta = 0$. In addition, we consider operator $\log\left(C^{*-1}BC^{-1}\right)$, assuming $B \in \mathcal{B}^{++}(H)$, instead of operator $C^{*-1}BC^{-1}$. In described setting, relation (3.12) becomes series of inequalities (1.7), that is refinement and conversion of arithmetic-geometric mean inequality. Clearly, the operator $\log\left(C^{*-1}BC^{-1}\right)$ is well defined since $C^{*-1}BC^{-1} \in \mathcal{B}^{++}(H)$.

In the next section we develop yet another method for bounding of Jensen's operator $\mathcal{J}(f, D, \delta, \mathbf{p})$. Let's mention here that some different Jensen's type inequalities including self-adjoint operators on Hilbert space were recently obtained in paper [2].

4. Bounds for spectra of Jensen's operator and applications to mean inequalities

In the previous section we were concerned with the bounding of Jensen's operator $\mathcal{J}(f, D, \delta, \mathbf{p})$ with non-weighted operator $\mathcal{J}_{\mathcal{N}}(f, D, \delta)$. As distinguished from Section 3, in this section we use a different method for bounding of Jensen's operator. Namely, regarding monotonicity properties of Jensen's functional, considered as a function of one variable (see Lemma 2.1), we get, under certain conditions, bounds for operator $\mathcal{J}(f, D, \delta, \mathbf{p})$ that are multiples of identity operator 1_H. Obviously, such estimates can be interpreted as the bounds for the spectra of operator $\mathcal{J}(f, D, \delta, \mathbf{p})$. Now follows our main result concerning above discussion.

Theorem 4.1. *Suppose $\mathcal{J}$ is an operator defined by* (3.1)*, and let $\gamma, \delta \in [a, b]$. If*

$$a1_H \leq D \leq \gamma 1_H \leq \delta 1_H \quad \text{or} \quad \delta 1_H \leq \gamma 1_H \leq D \leq b1_H \tag{4.1}$$

then

$$\mathcal{J}(f, D, \delta, \mathbf{p}) \geq \jmath(f, \gamma, \delta, \mathbf{p})1_H \geq 2\min\{p_1, p_2\}\jmath_{\mathcal{N}}(f, \gamma, \delta)1_H, \tag{4.2}$$

where

$$\jmath(f,\gamma,\delta,\mathbf{p}) = p_1 f(\gamma) + p_2 f(\delta) - (p_1+p_2) f\left(\frac{p_1\gamma + p_2\delta}{p_1+p_2}\right),$$

and

$$\jmath_{\mathcal{N}}(f,\gamma,\delta) = \frac{f(\gamma)+f(\delta)}{2} - f\left(\frac{\gamma+\delta}{2}\right).$$

In addition, if

$$a1_H \le \gamma 1_H \le D \le \delta 1_H \quad \text{or} \quad \delta 1_H \le D \le \gamma 1_H \le b1_H, \tag{4.3}$$

then

$$\mathcal{J}(f,D,\delta,\mathbf{p}) \le \jmath(f,\gamma,\delta,\mathbf{p}) 1_H \le 2\max\{p_1,p_2\}\jmath_{\mathcal{N}}(f,\gamma,\delta)1_H. \tag{4.4}$$

Proof. We consider Jensen's functional

$$\jmath(f,x,\delta,\mathbf{p}) = p_1 f(x) + p_2 f(\delta) - (p_1+p_2) f\left(\frac{p_1 x + p_2\delta}{p_1+p_2}\right)$$

as a function in variable x, and use properties (i) and (ii) from Lemma 2.1, concerning monotonicity of $\jmath(f,x,\delta,\mathbf{p})$ on intervals $[a,\delta]$ and $[\delta,b]$.

More precisely, if $a \le x \le \gamma \le \delta$ or $\delta \le \gamma \le x \le b$, then properties (i) and (ii) from Lemma 2.1, together with monotonicity property (1.12) of Jensen's functional, yield the following series of inequalities:

$$\jmath(f,x,\delta,\mathbf{p}) \ge \jmath(f,\gamma,\delta,\mathbf{p}) \ge 2\min\{p_1,p_2\}\jmath_{\mathcal{N}}(f,\gamma,\delta). \tag{4.5}$$

Now, if $D \in \mathcal{B}_h(H)$ satisfies (4.1), then, according to monotonicity property (2.1) for operator functions, we can insert D in series of inequalities (4.5). As a result we get relation (4.2), as required.

It remains to prove relation (4.4). We act in the same way as in the first part of the proof. If $a \le \gamma \le x \le \delta$ or $\delta \le x \le \gamma \le b$, then (1.12) and properties (i), (ii) from Lemma 2.1 provide relation

$$\jmath(f,x,\delta,\mathbf{p}) \le \jmath(f,\gamma,\delta,\mathbf{p}) \le 2\max\{p_1,p_2\}\jmath_{\mathcal{N}}(f,\gamma,\delta). \tag{4.6}$$

Finally, if $D \in \mathcal{B}_h(H)$ satisfies one of conditions in (4.3), then relation (4.6) implies (4.4), due to monotonicity property (2.1) for operator functions. ∎

As a first application, we give an analogue of Corollary 3.2, regarding the method developed in Theorem 4.1.

Corollary 4.1. *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous convex function. Suppose $A \in \mathcal{B}^{++}(H)$, $B \in \mathcal{B}_h(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, $\mathbf{p} = (p_1,p_2) \in \mathbb{R}_+^2$, and $\gamma,\delta \in [a,b]$. If*

$$aA \le B \le \gamma A \le \delta A \quad \text{or} \quad \delta A \le \gamma A \le B \le bA, \tag{4.7}$$

then

$$C^*\mathcal{J}\left(f, C^{*-1}BC^{-1},\delta,\mathbf{p}\right)C \ge \jmath(f,\gamma,\delta,\mathbf{p})A \ge 2\min\{p_1,p_2\}\jmath_{\mathcal{N}}(f,\gamma,\delta)A, \tag{4.8}$$

where $C^*\mathcal{J}\left(f, C^{*-1}BC^{-1},\delta,\mathbf{p}\right)C$ is defined by (3.8) *and the functionals $\jmath(f,\gamma,\delta,\mathbf{p})$, $\jmath_{\mathcal{N}}(f,\gamma,\delta)$ are defined in Theorem* 4.1.
In addition, if

$$aA \le \gamma A \le B \le \delta A \quad \text{or} \quad \delta A \le B \le \gamma A \le bA, \tag{4.9}$$

then

$$C^*\mathcal{J}\left(f, C^{*-1}BC^{-1}, \delta, \mathbf{p}\right)C \leq \jmath(f,\gamma,\delta,\mathbf{p})A \leq 2\max\{p_1,p_2\}\jmath_{\mathcal{N}}(f,\gamma,\delta)A. \tag{4.10}$$

Proof. Follows simply from Theorem 4.1. Namely, we easily see that the conditions (4.1) and (4.3), rewritten for operator $D = C^{*-1}BC^{-1}$, are equivalent respectively to conditions (4.7) and (4.9). Hence, if we replace D in (4.2) and (4.4) with $C^{*-1}BC^{-1}$, and then, multiply obtained series of inequalities by C^* on the left, and by C on the right, we get (4.8) and (4.10). ∎

Remark 4.1. If C is the square root of operator A, i.e. $C = A^{\frac{1}{2}}$, then, under the same assumptions as in Corollary 4.1, the series of inequalities (4.8) and (4.10) respectively read

$$A^{\frac{1}{2}}\mathcal{J}\left(f, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, \delta, \mathbf{p}\right)A^{\frac{1}{2}} \geq \jmath(f,\gamma,\delta,\mathbf{p})A \geq 2\min\{p_1,p_2\}\jmath_{\mathcal{N}}(f,\gamma,\delta)A, \tag{4.11}$$

and

$$A^{\frac{1}{2}}\mathcal{J}\left(f, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, \delta, \mathbf{p}\right)A^{\frac{1}{2}} \leq \jmath(f,\gamma,\delta,\mathbf{p})A \leq 2\max\{p_1,p_2\}\jmath_{\mathcal{N}}(f,\gamma,\delta)A. \tag{4.12}$$

In the sequel we focus to applications of Theorem 4.1 and Corollary 4.1 which provide refinements and conversions of above mentioned operator means in so called difference form. Our next result refers to the difference between arithmetic and harmonic operator mean.

Corollary 4.2. *Suppose H is a Hilbert space, $A, B \in \mathcal{B}^{++}(H)$, $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$, and $\gamma > 0$. If*

$$\gamma A \leq A \leq \gamma B \quad \text{or} \quad \gamma B \leq A \leq \gamma A, \tag{4.13}$$

then,

$$\begin{aligned}(p_1+p_2)\left[A\,\nabla_{\frac{p_1}{p_1+p_2}}\,B - A\,!_{\frac{p_1}{p_1+p_2}}\,B\right] &\geq \frac{(\gamma-1)^2p_1p_2}{\gamma(p_1\gamma+p_2)}A \\ &\geq \min\{p_1,p_2\}\frac{(\gamma-1)^2}{\gamma(\gamma+1)}A.\end{aligned} \tag{4.14}$$

Furthermore, if

$$\gamma A \leq \gamma B \leq A \quad \text{or} \quad A \leq \gamma B \leq \gamma A, \tag{4.15}$$

then

$$\begin{aligned}(p_1+p_2)\left[A\,\nabla_{\frac{p_1}{p_1+p_2}}\,B - A\,!_{\frac{p_1}{p_1+p_2}}\,B\right] &\leq \frac{(\gamma-1)^2p_1p_2}{\gamma(p_1\gamma+p_2)}A \\ &\leq \max\{p_1,p_2\}\frac{(\gamma-1)^2}{\gamma(\gamma+1)}A.\end{aligned} \tag{4.16}$$

Proof. Relations (4.14) and (4.16) are immediate consequences of relations (4.11) and (4.12). Namely, we consider (4.11) and (4.12) endowed with convex function $f : \langle 0,\infty\rangle \to \mathbb{R}$, $f(x) = 1/x$, parameter $\delta = 1$ and operators A and B respectively replaced with A^{-1} and B^{-1}. In that setting, the left expression in (4.11) and (4.12) becomes

$$(p_1+p_2)\left[A\,\nabla_{\frac{p_1}{p_1+p_2}}\,B - A\,!_{\frac{p_1}{p_1+p_2}}\,B\right],$$

Jensen's functionals $\jmath(f,\gamma,\delta,\mathbf{p})$ and $\jmath_{\mathcal{N}}(f,\gamma,\delta)$ respectively read

$$\jmath(f,\gamma,\delta,\mathbf{p}) = \frac{(\gamma-1)^2 p_1 p_2}{\gamma(p_1\gamma+p_2)} \quad \text{and} \quad \jmath_{\mathcal{N}}(f,\gamma,\delta) = \frac{(\gamma-1)^2}{2\gamma(\gamma+1)},$$

which yields required relations (4.14) and (4.16).

It remains to justify the conditions under which the inequalities in (4.14) and (4.16) are valid. More precisely, inequalities in (4.14) are obtained from relation (4.11) which is valid under conditions (4.7). Taking into account that we have replaced operators A and B respectively with A^{-1} and B^{-1}, conditions (4.7) in described setting respectively read $B^{-1} \le \gamma A^{-1} \le A^{-1}$ or $A^{-1} \le \gamma A^{-1} \le B^{-1}$. Clearly, these conditions are equivalent to those in (4.13), due to operator monotonicity of function $g(t) = -1/t$ on $\langle 0,\infty\rangle$. The similar discussion holds for relation (4.16) and associated conditions (4.15). ∎

Note that in relations (4.14) and (4.16), the difference between arithmetic and harmonic mean was bounded by multiple of one of the operator included in definitions of mentioned means. The same form can be established for the difference between arithmetic and geometric operator mean. To get such a result, it is more convenient to use Theorem 4.1 directly, than Corollary 4.1.

Corollary 4.3. *Suppose H is a Hilbert space, $A,B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, $\mathbf{p} = (p_1,p_2) \in \mathbb{R}^2_+$, and $\gamma > 0$. If*

$$B \le \gamma A \le A \quad \text{or} \quad A \le \gamma A \le B, \tag{4.17}$$

then,

$$\begin{aligned} &(p_1+p_2)\left[A \,\nabla_{\frac{p_1}{p_1+p_2}}\, B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C\right] \\ &\ge \left[p_1\gamma + p_2 - (p_1+p_2)\gamma^{\frac{p_1}{p_1+p_2}}\right] A \ge \min\{p_1,p_2\}(\sqrt{\gamma}-1)^2 A. \end{aligned} \tag{4.18}$$

Further, if

$$\gamma A \le B \le A \quad \text{or} \quad A \le B \le \gamma A, \tag{4.19}$$

then

$$\begin{aligned} &(p_1+p_2)\left[A \,\nabla_{\frac{p_1}{p_1+p_2}}\, B - C^*\left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C\right] \\ &\le \left[p_1\gamma + p_2 - (p_1+p_2)\gamma^{\frac{p_1}{p_1+p_2}}\right] A \le \max\{p_1,p_2\}(\sqrt{\gamma}-1)^2 A. \end{aligned} \tag{4.20}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \exp x$, let $\delta = 0$, and let $D = \log\left(C^{*-1}BC^{-1}\right)$. Regarding notations from Theorem 4.1, we have

$$\jmath(f,\gamma,\delta,\mathbf{p}) = p_1 \exp\gamma + p_2 - (p_1+p_2)\exp\left(\frac{p_1\gamma}{p_1+p_2}\right),$$

$$\jmath_{\mathcal{N}}(f,\gamma,\delta) = \frac{\left(\exp\left(\frac{\gamma}{2}\right)-1\right)^2}{2},$$

while Jensen's operator (3.1) becomes

$$p_1 C^{*-1}BC^{-1} + p_2 1_H - (p_1+p_2)\left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}}.$$

Further, to apply relation (4.2), operator D must satisfy one of conditions in (4.1), i.e. $\log\left(C^{*-1}BC^{-1}\right) \le \gamma 1_H \le 0$ or $0 \le \gamma 1_H \le \log\left(C^{*-1}BC^{-1}\right)$, which reduces to $B \le \exp\gamma A \le A$ or $A \le \exp\gamma A \le B$. Since exponential function is injective, we can replace $\exp\gamma$ with γ in all expressions, assuming $\gamma > 0$. At last, if we substitute obtained expressions in (4.2), and then, multiply associated inequalities by C^* on the left and by C on the right, we get (4.18), as required.

To obtain series of inequalities in (4.20), we act in the same way as above, considering operators satisfying (4.19). ∎

Note that in previous two corollaries the difference between two operator means was bounded by a multiple of one of the operator included in definitions of considered means. Regarding the method developed in Theorem 4.1, for various choices of convex functions, we can get some other forms of inequalities for operator means, which will be clarified in the sequel.

Emphasize once more that inequality (1.8) from Introduction provided refinement of arithmetic-geometric inequality via Kantorovich constant $K(t,2) = (t+1)^2/4t$, $t > 0$. More precisely, arithmetic mean was bounded by a multiple of geometric mean. That constant factor was certain power of Kantorovich constant. Our next consequence of Theorem 4.1 yields even better constant factor then actual power of Kantorovich constant, as well as conversion of above mentioned operator inequality.

Corollary 4.4. *Suppose H is a Hilbert space, $A, B \in \mathcal{B}^{++}(H)$, $C \in \mathcal{B}^{-1}(H)$, $A = C^*C$, $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$, and $\gamma > 0$. If*

$$B \le \gamma A \le A \quad \text{or} \quad A \le \gamma A \le B, \tag{4.21}$$

then,

$$\begin{aligned} A \nabla_{\frac{p_1}{p_1+p_2}} B &\ge \frac{p_1\gamma + p_2}{p_1+p_2} \cdot \gamma^{-\frac{p_1}{p_1+p_2}} C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C \\ &\ge K(\gamma,2)^{\frac{\min\{p_1,p_2\}}{p_1+p_2}} C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C. \end{aligned} \tag{4.22}$$

In addition, if

$$\gamma A \le B \le A \quad \text{or} \quad A \le B \le \gamma A, \tag{4.23}$$

then

$$\begin{aligned} A \nabla_{\frac{p_1}{p_1+p_2}} B &\le \frac{p_1\gamma + p_2}{p_1+p_2} \cdot \gamma^{-\frac{p_1}{p_1+p_2}} C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C \\ &\le K(\gamma,2)^{\frac{\max\{p_1,p_2\}}{p_1+p_2}} C^* \left(C^{*-1}BC^{-1}\right)^{\frac{p_1}{p_1+p_2}} C. \end{aligned} \tag{4.24}$$

Proof. The proof is direct use of Theorem 4.1. We consider Jensen's operator (3.1) equipped with convex function $f : \langle 0, \infty\rangle \to \mathbb{R}$, $f(x) = -\log x$, parameter $\delta = 1$ and operator $D \in \mathcal{B}^{++}(H)$ satisfying

$$D \le \gamma 1_H \le 1_H \quad \text{or} \quad 1_H \le \gamma 1_H \le D. \tag{4.25}$$

Now, taking into account notations from Theorem 4.1, we have

$$\mathcal{J}(f, D, \delta, \mathbf{p}) = (p_1+p_2)\log\left(\frac{p_1 D + p_2 1_H}{p_1+p_2} \cdot D^{-\frac{p_1}{p_1+p_2}}\right),$$

$$\jmath(f,\gamma,\delta,\mathbf{p}) = (p_1+p_2)\log\left(\frac{p_1\gamma+p_2}{p_1+p_2}\cdot\gamma^{-\frac{p_1}{p_1+p_2}}\right),$$
$$\jmath_{\mathcal{N}}(f,\gamma,\delta) = \log\left(\frac{\gamma+1}{2\sqrt{\gamma}}\right),$$

so relation (4.2) takes form

$$\log\left(\frac{p_1D+p_2 1_H}{p_1+p_2}\cdot D^{-\frac{p_1}{p_1+p_2}}\right) \geq \log\left(\frac{p_1\gamma+p_2}{p_1+p_2}\cdot\gamma^{-\frac{p_1}{p_1+p_2}}\right)1_H \geq \log K(\gamma,2)^{\frac{\min\{p_1,p_2\}}{p_1+p_2}}1_H,$$

that is,

$$\frac{p_1D+p_2 1_H}{p_1+p_2} \geq \frac{p_1\gamma+p_2}{p_1+p_2}\cdot\gamma^{-\frac{p_1}{p_1+p_2}}D^{\frac{p_1}{p_1+p_2}} \geq K(\gamma,2)^{\frac{\min\{p_1,p_2\}}{p_1+p_2}}D^{\frac{p_1}{p_1+p_2}}. \tag{4.26}$$

On the other hand, if operators $A = C^*C, B \in \mathcal{B}^{++}(H)$ satisfy (4.21), it follows that operator $D = C^{*-1}BC^{-1}$ satisfy (4.25). Finally, if we replace D in (4.26) with $C^{*-1}BC^{-1}$ and multiply expressions in (4.26) by C^* on the left, and by C on the right, we get (4.22).

The series of inequalities in (4.24) is obtained in the same way as (4.22), considering relation (4.4) and operators $A = C^*C, B \in \mathcal{B}^{++}(H)$ satisfying one of the conditions in (4.23). ∎

The following two remarks describe connection between our Corollary 4.4 and inequality (1.8) in detail.

Remark 4.2. Relations (4.22) and (4.24) provide respectively refinement and conversion of arithmetic-geometric mean for Hilbert space operators. Namely, if $C = A^{\frac{1}{2}}$ relations (4.22) and (4.24) respectively read

$$A\nabla_{\frac{p_1}{p_1+p_2}}B \geq \frac{p_1\gamma+p_2}{p_1+p_2}\cdot\gamma^{-\frac{p_1}{p_1+p_2}}A\sharp_{\frac{p_1}{p_1+p_2}}B \geq K(\gamma,2)^{\frac{\min\{p_1,p_2\}}{p_1+p_2}}A\sharp_{\frac{p_1}{p_1+p_2}}B, \tag{4.27}$$

and

$$A\nabla_{\frac{p_1}{p_1+p_2}}B \leq \frac{p_1\gamma+p_2}{p_1+p_2}\cdot\gamma^{-\frac{p_1}{p_1+p_2}}A\sharp_{\frac{p_1}{p_1+p_2}}B \leq K(\gamma,2)^{\frac{\max\{p_1,p_2\}}{p_1+p_2}}A\sharp_{\frac{p_1}{p_1+p_2}}B. \tag{4.28}$$

On the other hand, Fujii *et al.* obtained in [3] the inequality

$$A\nabla B \leq \frac{m\nabla M}{m\,\sigma\,M}A\,\sigma\,B, \tag{4.29}$$

where $0 < m1_H \leq A, B \leq M1_H$ and σ is a symmetric operator mean. Now, by letting σ to be the geometric mean, we see that (4.28) represents an extension of (4.29) to non-symmetric case.

Remark 4.3. Let's compare our relation (4.22), i.e. (4.27) with inequality (1.8) from Introduction. We show that inequality (1.8) can be obtained from (4.27). More precisely, if we denote $h = M/m > 1$, then the first condition in (1.9) yields

$$A < hA \leq m\cdot\frac{M}{m}1_H = M1_H \leq B, \quad \text{i.e.} \quad A < hA \leq B.$$

On the other hand, since $1/h < 1$, the second condition in (1.9) yields

$$B \le m1_H = M \cdot \frac{m}{M} 1_H \le \frac{1}{h} A < A \quad \text{i.e.} \quad B \le \frac{1}{h} A < A.$$

Hence, our conditions in (4.21) are equivalent to those in (1.9). Now, inequality (1.8) follows from (4.27) since $K(h,2) = K(1/h,2)$, $h > 0$. Note also that our series of inequalities (4.27) also refines inequality (1.8). Namely, if we take into consideration (4.27), we have interpolated inequality (1.8) with the multiple of geometric mean operator which includes constant factor not less then above mentioned power of Kantorovich constant.

Yet another specific example, regarding mean inequalities for operators, arises directly from Remark 4.2. That is a content of the following remark, with which we conclude this paper.

Remark 4.4. Series of inequalities (4.27) and (4.28) enable us to deduce refinement and conversion of geometric-harmonic operator mean inequality. More precisely, if we replace operators A and B in (4.27) and (4.28) respectively with A^{-1} and B^{-1}, and then, use the fact that $g(t) = -1/t$ is operator monotone on $\langle 0, \infty \rangle$, relations (4.27) and (4.28) become respectively

$$A \,!_{\frac{p_1}{p_1+p_2}}\, B \le \frac{p_1+p_2}{p_1\gamma+p_2} \cdot \gamma^{\frac{p_1}{p_1+p_2}} A \,\sharp_{\frac{p_1}{p_1+p_2}}\, B \le K(\gamma,2)^{-\frac{\min\{p_1,p_2\}}{p_1+p_2}} A \,\sharp_{\frac{p_1}{p_1+p_2}}\, B, \tag{4.30}$$

and

$$A \,!_{\frac{p_1}{p_1+p_2}}\, B \ge \frac{p_1+p_2}{p_1\gamma+p_2} \cdot \gamma^{\frac{p_1}{p_1+p_2}} A \,\sharp_{\frac{p_1}{p_1+p_2}}\, B \ge K(\gamma,2)^{-\frac{\max\{p_1,p_2\}}{p_1+p_2}} A \,\sharp_{\frac{p_1}{p_1+p_2}}\, B. \tag{4.31}$$

Furthermore, if we replace operators A and B respectively with A^{-1} and B^{-1} in conditions (4.21) and (4.23), we see that relation (4.30) holds if $\gamma A \le A \le \gamma B$ or $\gamma B \le A \le \gamma A$, while (4.31) holds if $\gamma A \le \gamma B \le A$ or $A \le \gamma B \le \gamma A$.

Acknowledgement. This research was supported by the Croatian Ministry of Science, Education and Sports, under Research Grants 036–1170889–1054 (first author), 083–0000000–3227 (second author) and 117–1170889–0888 (third author).

References

[1] S. S. Dragomir, J. Pečarić and L. E. Persson, Properties of some functionals related to Jensen's inequality, *Acta Math. Hungar.* **70** (1996), no. 1–2, 129–143.

[2] S. S. Dragomir, Some Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 3, 445–454.

[3] J. I. Fujii, M. Nakamura, J. Pečarić and Y. Seo, Bounds for the ratio and difference between parallel sum and series via Mond-Pečarić method, *Math. Inequal. Appl.* **9** (2006), no. 4, 749–759.

[4] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), no. 1, 21–31.

[5] S. Furuichi, *Refined Young inequalities with Specht's ratio*, ArXiv:1004.0581v2.

[6] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, Mond-Pečarić method in operator inequalities, *Element*, Zagreb, 2005.

[7] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.* **361** (2010), no. 1, 262–269.

[8] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear A.* **59** (2011), no. 9, 1031–1037.

[9] F. Kittaneh, M. Krnić, N. Lovričević and J. Pečarić, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators, *Publ. Math. Debrecen* (to appear).

[10] M. Klaričić Bakula, M. Matić and J. Pečarić, On inequalities complementary to Jensen's inequality, *Mat. Bilten No. 32* (2008), 17–27.

[11] J. Mićić, J. Pečarić and V. Šimić, Inequalities involving the arithmetic and geometric operator means, *Math. Inequal. Appl.* **11** (2008), no. 3, 415–430.

[12] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and its Applications (East European Series), 61, Kluwer Acad. Publ., Dordrecht, 1993.

[13] J. Pečarić, J. Mićić and Y. Seo, Inequalities between operator means based on the Mond-Pečarić method, *Houston J. Math.* **30** (2004), no. 1, 191–207 (electronic).

[14] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Jpn.* **55** (2002), no. 3, 583–588.

[15] H. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.* (to appear).