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# Strong Vector Equilibrium Problems on Noncompact Sets

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**Abstract.** In this paper, by using the famous Brouwer fixed point theorem, some existence theorems of strong efficient solutions for the strong vector equilibrium problems are obtained on noncompact sets of general real Hausdorff topological vector spaces without assuming that the dual of the ordering cone has a weak* compact base. Moreover, the closeness and the convexity of the strong efficient solution sets are discussed.

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## 1. Introduction

Let $E$ be a real Hausdorff topological vector space, $X$ a nonempty subset of $E$, and $\varphi : X \times X \to R$ a bifunction such that $\varphi(x, x) \geq 0$ for all $x \in X$. Then the scalar equilibrium problem consists in finding $\bar{x} \in X$ such that

$$\varphi(\bar{x}, y) \geq 0, \ \forall y \in X.$$

It provides a unifying framework for many important problems, such as, optimization problems, variational inequality problems, complementary problems, minimax inequality problems and fixed point problems, and has been widely applied to study the problems arising in economics, mechanics and engineering science (see Blum and Oettli [6]).

Let $Z$ be a real Hausdorff topological vector space, $C \subseteq Z$ a closed convex pointed cone. Let $f : X \times X \to Z$ be a mapping. It is well known that the vector equilibrium problem includes three basic types. The first type is the weak vector equilibrium problem (for short, WVEP), which consists in finding $\bar{x} \in X$ such that

$$(1.1) \qquad \text{(WVEP)} \qquad f(\bar{x}, y) \notin -\text{int } C, \quad \forall y \in X,$$

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where int $C$ denotes the topological interior of $C$. The second type is the Stampacchia vector equilibrium problem (for short, VEP), which consists in finding $\bar{x} \in X$ such that

$$(1.2) \qquad \text{(VEP)} \qquad f(\bar{x}, y) \notin -C \backslash \{0\}, \quad \forall\, y \in X.$$

And the third type is the strong vector equilibrium problem (for short, SVEP), which consists in finding $\bar{x} \in X$ such that

$$(1.3) \qquad \text{(SVEP)} \qquad f(\bar{x}, y) \in C, \quad \forall\, y \in X.$$

Each of the above three types of vector equilibrium problems constitutes a valid extension of the scalar equilibrium problem. If int $C \neq \emptyset$ and $\bar{x}$ satisfies (1.1), then we call $\bar{x}$ a weak efficient solution for the vector equilibrium problem, and denote by $V_W(f, X)$ the set of all weak efficient solutions. If $\bar{x}$ satisfies (1.2), then we call $\bar{x}$ an efficient solution for the vector equilibrium problem, and denote by $V(f, X)$ the set of all efficient solutions. If $\bar{x}$ satisfies (1.3), then we call $\bar{x}$ a strong efficient solution for the vector equilibrium problem, and denote by $V_S(f, X)$ the set of all strong efficient solutions.

Up to now, many authors have studied the vector equilibrium problems (see, for example [2–6, 8–10, 12–23, 26–30, 32–34]), focusing mainly on the study of the existence of weak efficient solution. However, if int $C = \emptyset$, then the weak vector equilibrium problem can not be studied. It is well known that, in many cases, the ordering cone has an empty interior. For example, in the classical Banach spaces $\ell^p$ and $L^p(\Omega)$, where $1 < p < \infty$, the standard ordering cone has an empty interior (see [24]). In this case, we can study the existence of the efficient solution and the strong efficient solution, and the properties of these solution sets. Giannessi *et al.* [16] studied the properties of the efficient solution set for the vector equilibrium problem.

When int $C = \emptyset$, in order to study the vector equilibrium problem, Gong [17, 18] introduced the concepts of proper efficient solutions, such as Henig efficient solution and super efficient solution. In addition, Gong [17, 18] gave the scalarization results for the proper efficient solutions and studied the existence and the properties of the sets of proper efficient solutions.

Recently, Fu [13], Fu and Wang [14], Fu *et al.* [15], Lin *et al.* [27] and Wang *et al.* [34] studied the efficient solution for the vector equilibrium problem. Ansari *et al.* [2], Fu [12] and Tan [33] studied the strong efficient solution for the vector equilibrium problem. It is worth mentioning that many existence results of the efficient solution and the strong efficient solution for the vector equilibrium problem are obtained under the assumption that the dual $C^*$ of the ordering cone $C$ has a weak$^*$ compact base. As we know, for a normed space, the dual cone $C^*$ has a weak$^*$ compact base if and only if int $C \neq \emptyset$ (see [25]). However, in many cases, the ordering cone has an empty interior. Thus, there is a need for the study of the existence of solutions and the properties of the solution sets for this case.

On the other hand, it is well known that a strong efficient solution for vector equilibrium problem is an ideal solution, which is better than other solutions such as efficient solution, weak efficient solution, Henig efficient solution and supper efficient solution (see, for example, [19]). Thus, it is very important to study the existence of strong efficient solution and the properties of the strong efficient solution set without assuming that the dual of the ordering cone has a weak$^*$ compact base. In this case, the result, as our best knowledge, is very few.

Very recently, without assuming that the dual of the ordering cone has a weak$^*$ compact base, Gong [19] established an existence theorem of strong efficient solution for a strong vector equilibrium problem by using the separation theorem for convex set and discussed the closeness of the strong efficient solution set; Gong [20] derived an existence theorem of strong efficient solution for a symmetric strong vector quasiequilibrium problem by using Kakutani-Fan-Glicksberg fixed point theorem; Hou, Gong and Yang [21] derived an existence theorem of strong efficient solution for a generalized strong vector equilibrium problem by using Kakutani-Fan-Glicksberg fixed point theorem and discussed the stability of strong efficient solutions; Long, Huang and Teo [30] extended the main result of [21] from single-valued mapping to set-valued mapping, and showed the closeness of the strong efficient solution set. It should be mentioned that most of the existence results of strong efficient solution are obtained on compact sets in locally convex spaces.

Motivated and inspired by the research works mentioned above, in this paper, we further consider the strong efficient solution for the vector equilibrium problem. Let $F : X \times X \to 2^Z$ be a set-valued mapping. We consider the following set-valued version of strong vector equilibrium problem (for short, MSVEP): find $\bar{x} \in X$ such that

$$\text{(MSVEP)} \qquad F(\bar{x}, y) \subseteq C, \quad \forall\, y \in X. \tag{1.4}$$

We denote by $V_S^M(F, X)$ the set of all strong efficient solutions for (MSVEP). The main purpose of this paper is to discuss the existence of strong efficient solutions and study the properties of the strong efficient solution sets for (SVEP) and (MSVEP) without assuming that the dual cone $C^*$ of the ordering cone $C$ has a weak$^*$ compact base. On noncompact sets of general real Hausdorff topological vector spaces (not necessarily locally convex), we obtain some existence theorems of strong efficient solutions for (SVEP) and (MSVEP) by using the famous Brouwer fixed point theorem. Moreover, we study the closeness and the convexity of the strong efficient solution sets for (SVEP) and (MSVEP).

## 2. Preliminaries

In this section, we shall recall some definitions and lemmas used in the sequel.

**Definition 2.1.** [1] *Let $X$ and $Y$ be two topological spaces. A set-valued mapping $T : X \to 2^Y$ is said to be*

- (i) *upper semicontinuous (for short, u.s.c.) at $x \in X$ if, for each open set $V$ in $Y$ with $T(x) \subseteq V$, there exists an open neighborhood $U(x)$ of $x$ such that $T(x') \subseteq V$ for all $x' \in U(x)$;*
- (ii) *lower semicontinuous (for short, l.s.c.) at $x \in X$ if, for each open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of $x$ such that $T(x') \cap V \neq \emptyset$ for all $x' \in U(x)$;*
- (iii) *u.s.c. (resp. l.s.c.) on $X$ if it is u.s.c. (resp. l.s.c.) at every point $x \in X$;*
- (iv) *continuous on $X$ if it is both u.s.c. and l.s.c. on $X$.*

**Lemma 2.1.** [1] *Let $X$ and $Y$ be two topological spaces, $F : X \to 2^Y$ a set-valued mapping. $F$ is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ for all $\alpha$ and $y_\alpha \to y$.*

**Definition 2.2.** [31] *Let $E$ and $Z$ be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty subset and $C \subseteq Z$ a closed convex pointed cone. Let $F : X \to 2^Z$ be a set-valued mapping. $F$ is said to be*

(i) *upper [resp. lower] $C$-continuous at $x \in X$ if, for any neighborhood $V$ of the origin in $Z$, there exists a neighborhood $U$ of $x$ such that, for all $x' \in U \cap X$, $F(x') \subseteq F(x) + V + C$ [resp. $F(x) \subseteq F(x') + V - C$]; $F$ is said to be upper [resp. lower ] $C$-continuous on $X$ if $F$ is upper [resp. lower] $C$-continuous at every point $x \in X$;*

(ii) *$C$-continuous on $X$ if $F$ is both upper $C$-continuous and lower $C$-continuous on $X$.*

**Lemma 2.2.** *Let $E$ and $Z$ be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty subset and $C \subseteq Z$ a closed convex pointed cone. Let $F : X \to 2^Z$ be a set-valued mapping.*

(i) *If $F$ is u.s.c., then $F$ is upper $C$-continuous;*

(ii) *If $F$ is single-valued, then $F$ is upper $C$-continuous $\Leftrightarrow$ $F$ is lower $C$-continuous $\Leftrightarrow$ $F$ is $C$-continuous.*

*Proof.* (a) the assertion (i) holds obviously since $0 \in C$;
(b) Suppose that $F$ is single-valued and upper $C$-continuous at $x_0 \in X$, then for each neighborhood $V$ of the origin in $Z$, there exists a neighborhood $U$ of $x_0$ such that

$$F(x) \in F(x_0) + V + C, \quad \forall\, x \in U \cap X.$$

Thus, for each $x \in U \cap X$, there exists some $c \in C$ such that

$$F(x) - F(x_0) - c \in V.$$

Since $Z$ is a real Hausdorff topological vector space, there exists a balanced neighborhood $V_0 \subseteq V$ of the origin in $Z$ such that $F(x) - F(x_0) - c \in V_0$. Notice that $V_0$ is balanced, i.e., $V_0 = -V_0$. It follows that

$$\begin{aligned} F(x_0) &\in F(x) - c - V_0 = F(x) - c + V_0 \\ &\subseteq F(x) + V_0 - C \\ &\subseteq F(x) + V - C. \end{aligned}$$

By the arbitrary of $x$, we know that $F$ is lower $C$-continuous at $x_0$, and so $F$ is $C$-continuous at $x_0$.

Similarly, we can show that if $F$ is single-valued and lower $C$-continuous at $x_0 \in X$, then $F$ is upper $C$-continuous at $x_0 \in X$, and thus $F$ is $C$-continuous at $x_0$. ∎

**Definition 2.3.** *Let $E$ and $Z$ be two real topological vector spaces, $X \subseteq E$ a nonempty convex subset and $C \subseteq Z$ a closed convex pointed cone. Let $F : X \to 2^Z$ be a set-valued mapping. $F$ is said be*

(i) *$C$-convex if, for any $x, y \in X$ and $t \in [0, 1]$, one has*

$$F(tx + (1-t)y) \subseteq tF(x) + (1-t)F(y) - C;$$

*$F$ is said to be $C$-concave if $-F$ is $C$-convex;*

(ii) *affine if, for any $x, y \in X$ ($X$ is a vector subspace of $E$) and $t \in R$, one has*

$$F(tx + (1-t)y) = tF(x) + (1-t)F(y).$$

**Definition 2.4.** Let $E$ and $Z$ be two real topological vector spaces, $X \subseteq E$ a nonempty convex subset and $C \subseteq Z$ a closed convex pointed cone. Let $F : X \times X \to 2^Z$ be a set-valued mapping. $F$ is said be

(i) $C$-diagonally convex if, for any finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq X$ and any $t_i \geq 0, i = 1, 2, \cdots, n$ with $\sum_{i=1}^n t_i = 1$, $x = \sum_{i=1}^n t_i x_i$, one has

$$F(x, x) \subseteq \sum_{i=1}^n t_i F(x, x_i) - C;$$

(ii) properly $C$-diagonally quasiconvex if, for any finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq X$ and any $t_i \geq 0, i = 1, 2, \cdots, n$ with $\sum_{i=1}^n t_i = 1$, $x = \sum_{i=1}^n t_i x_i$, there exists some $i_0 \in \{1, 2, \cdots, n\}$ such that

$$F(x, x_{i_0}) \subseteq F(x, x) + C.$$

**Remark 2.1.** The above concepts of $C$-diagonally convexity and properly $C$-diagonally quasiconvex generalize the concepts of convexity and properly quasiconvexity of [11, 12], respectively.

The following example shows that there is no implication between $C$-diagonally convexity and properly $C$-diagonally quasiconvexity.

**Example 2.1.** Let $E = Z = R$, $X = [0, 1]$ and $C = R_+ = [0, +\infty)$. Let

$$f(x, y) = [\min\{x, y\} - \max\{x, y\}, 1], \quad g(x, y) = [0, \min\{x, y\}], \ \forall\, x, y \in X.$$

Then $f, g : X \times X \to 2^R$. For any finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq X$ and any $t_i \geq 0, i = 1, 2, \cdots, n$ with $\sum_{i=1}^n t_i = 1$, $x = \sum_{i=1}^n t_i x_i$, there exists some $i_0 \in \{1, 2, \cdots, n\}$ such that $x_{i_0} \leq x$. Notice that $0 \in C$. Thus, we have

$$\begin{aligned} g(x, x_{i_0}) &= [0, x_{i_0}] \subseteq [0, x] = g(x, x) \subseteq g(x, x) + C; \\ f(x, x_i) &\supseteq [0, 1], \quad \forall\, i = 1, 2, \cdots, n. \end{aligned}$$

It follows that

$$\begin{aligned} f(x, x) &= [0, 1] \\ &= [0, t_1 + t_2 + \cdots + t_n] \\ &= [0, t_1] + [0, t_2] + \cdots + [0, t_n] \\ &= t_1[0, 1] + t_2[0, 1] + \cdots + t_n[0, 1] \\ &= \sum_{i=1}^n t_i[0, 1] \subseteq \sum_{i=1}^n t_i f(x, x_i) \\ &\subseteq \sum_{i=1}^n t_i f(x, x_i) - C. \end{aligned}$$

Hence, $f$ is $C$-diagonally convex and $g$ is properly $C$-diagonally quasiconvex.

On the other hand, we choose

$$x_1 = 0, \quad x_2 = 1, \quad t_1 = t_2 = \frac{1}{2}, \quad x_0 = t_1 x_1 + t_2 x_2 = \frac{1}{2}.$$

Then

$$f(x_0, x_0) = f\left(\frac{1}{2}, \frac{1}{2}\right) = [0, 1],$$

$$f(x_0, x_1) = f\left(\frac{1}{2}, 0\right) = \left[-\frac{1}{2}, 1\right],$$

$$f(x_0, x_2) = f\left(\frac{1}{2}, 1\right) = \left[-\frac{1}{2}, 1\right]$$

and

$$g(x_0, x_0) = g\left(\frac{1}{2}, \frac{1}{2}\right) = \left[0, \frac{1}{2}\right],$$

$$g(x_0, x_1) = g\left(\frac{1}{2}, 0\right) = \{0\},$$

$$g(x_0, x_2) = g\left(\frac{1}{2}, 1\right) = \left[0, \frac{1}{2}\right].$$

It follows that

$$f(x_0, x_0) + C = [0, 1] + [0, +\infty) = [0, +\infty)$$

and

$$\sum_{i=1}^{2} t_i g(x_0, x_i) - C = \frac{1}{2}\{0\} + \frac{1}{2}\left[0, \frac{1}{2}\right] - [0, +\infty) = \left[0, \frac{1}{4}\right] + (-\infty, 0] = \left(-\infty, \frac{1}{4}\right].$$

Hence,

$$f(x_0, x_i) \not\subseteq f(x_0, x_0) + C, \quad \forall\, i = 1, 2$$

and

$$g(x_0, x_0) = \left[0, \frac{1}{2}\right] \not\subseteq \left(-\infty, \frac{1}{4}\right] = \sum_{i=1}^{2} t_i g(x_0, x_i) - C.$$

Therefore, $f$ is not properly $C$-diagonally quasiconvex and $g$ is not $C$-diagonally convex.

**Lemma 2.3.** [7, Brouwer Fixed Point Theorem] *Let $X$ be a nonempty, compact and convex subset of a finite dimensional space $E$ and $f : X \to X$ be a mapping. If $f$ is continuous, then there exists $\bar{x} \in X$ such that $f(\bar{x}) = \bar{x}$.*

## 3. Main results

In this section, we shall apply the famous Brouwer fixed point theorem to establish some existence results of strong efficient solutions and discuss the closeness and the convexity of the strong efficient solution sets for strong vector equilibrium problems.

**Theorem 3.1.** *Let $E$ and $Z$ be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty closed convex subset and $C \subseteq Z$ a closed convex pointed cone. Let $F : X \times X \to 2^Z$ be a set-valued mapping. Suppose that*

(i) *for any* $x \in X$, $F(x,x) \subseteq C$;
(ii) *for any* $x \in X$, *the set* $\{y \in X : F(x,y) \not\subseteq C\}$ *is empty or convex;*
(iii) *for any* $y \in X$, *the set* $\{x \in X : F(x,y) \subseteq C\}$ *is closed;*
(iv) *there exists a nonempty compact convex subset* $D \subseteq X$ *such that, for each* $x \in X \backslash D$, *there exists some* $y_0 \in D$ *such that* $F(x,y_0) \not\subseteq C$.

*Then* $V_S^M(F,X) \neq \emptyset$. *Moreover,* $V_S^M(F,X)$ *is closed. Further, if the following condition also holds:*

(v) *for any* $y \in X$, *the set* $\{x \in X : F(x,y) \subseteq C\}$ *is empty or convex,*

*then* $V_S^M(F,X)$ *is convex.*

*Proof.* Define a set-valued mapping $G : X \to 2^D$ by

$$G(y) = \{x \in D : F(x,y) \subseteq C\}, \ \forall\, y \in X.$$

Then, to prove that $V_S^M(F,X) \neq \emptyset$ is equivalent to prove that

$$\bigcap_{y \in X} G(y) \neq \emptyset. \tag{3.1}$$

Notice that

$$G(y) = D \cap \{x \in X : F(x,y) \subseteq C\}.$$

By assumption (iii), it is easy to see that for every $y \in X$, $G(y)$ is closed in $D$. Noting that $D$ is compact, thus, in order to show (3.1), we need only to show that the family of sets $\{G(y) : y \in X\}$ has the finite intersection property.

For any finite subset $\{y_1, y_2, \cdots, y_n\} \subseteq X$, let $B = co(D \cup \{y_1, y_2, \cdots, y_n\})$. Then $B$ is a compact and convex subset of $X$. Now, we consider the following set-valued mapping $H : B \to 2^B$ defined by

$$H(y) = \{x \in B : F(x,y) \subseteq C\}, \ \forall\, y \in B.$$

Firstly, we want to show that $\cap_{y \in B} H(y) \neq \emptyset$. Suppose that it is not the case, then, for each $x \in B$, there exists some $y \in B$ such that $x \notin H(y)$, i.e.,

$$F(x,y) \not\subseteq C. \tag{3.2}$$

For every $y \in B$, define the set $N_y$ as follows:

$$N_y = \{x \in B : F(x,y) \not\subseteq C\}. \tag{3.3}$$

By assumption (iii), the set $N_y$ is open in $B$ and hence from (3.2), it follows that the family of sets $\{N_y : y \in B\}$ is an open cover of $B$. Since $B$ is compact, there exists a finite subset $\{u_1, u_2, \cdots, u_m\} \subseteq B$ such that $B = \cup_{i=1}^m N_{u_i}$. It follows that there exists a continuous partition of unity $\{\beta_1, \beta_2, \cdots, \beta_m\}$ subordinate to the open cover $\{N_{u_1}, N_{u_2}, \cdots, N_{u_m}\}$ such that, for all $x \in B$, $\beta_i(x) \geq 0, i = 1, 2, \cdots, m$, $\sum_{i=1}^m \beta_i(x) = 1$, and $\beta_i(x) > 0$ whenever $x \in N_{u_i}$, $\beta_i(x) = 0$ whenever $x \notin N_{u_i}$.

Define a mapping $h : B \to Z$ as follows:

$$h(x) = \sum_{i=1}^m \beta_i(x) u_i, \quad \forall\, x \in B. \tag{3.4}$$

Since $\beta_i$ is continuous for each $i$, it follows from (3.4) that $h$ is continuous. Let $S = co\{u_1, u_2, \cdots, u_m\}$. Then $S \subseteq B$ is a simplex of a finite dimensional space and

$h$ maps $S$ into $S$. By Lemma 2.3, there exists some $x^* \in S$ such that $h(x^*) = x^*$. Let $I_0 = \{i : \beta_i(x^*) > 0\}$. Clearly, $I_0 \neq \emptyset$. Moreover,

$$x^* = h(x^*) = \sum_{i \in I_0} \beta_i(x^*) u_i \in co\{u_i : i \in I_0\}. \tag{3.5}$$

On the other hand, for every $i \in I_0$, $\beta_i(x^*) > 0$. So $x^* \in N_{u_i}$, i.e.,

$$F(x^*, u_i) \nsubseteq C, \quad \forall\, i \in I_0. \tag{3.6}$$

It follows that

$$u_i \in \{y \in X : F(x^*, y) \nsubseteq C\}, \quad \forall\, i \in I_0. \tag{3.7}$$

By (3.5),(3.7) and the assumption (ii), we have $x^* \in \{y \in X : F(x^*, y) \nsubseteq C\}$, i.e.,

$$F(x^*, x^*) \nsubseteq C. \tag{3.8}$$

which contradicts the assumption (i). Hence $\cap_{y \in B} H(y) \neq \emptyset$.

Let $x_0 \in \cap_{y \in B} H(y)$, then we have

$$F(x_0, y) \subseteq C, \quad \forall\, y \in B. \tag{3.9}$$

We assert that $x_0 \in D$. Suppose to the contrary that $x_0 \notin D$, then we have $x_0 \in B \backslash D \subseteq X \backslash D$. It follows from the assumption (iv) that there exists some $y_0 \in D$ such that

$$F(x_0, y_0) \nsubseteq C. \tag{3.10}$$

Since $D \subseteq B$, we can see that (3.10) contradicts (3.9). So $x_0 \in D$. Notice that $\{y_1, y_2, \cdots, y_n\} \subseteq B$. It follows from (3.9) that $x_0 \in \cap_{i=1}^n G(y_i)$, which implies that the family of sets $\{G(y) : y \in X\}$ has the finite intersection property. Hence, $\cap_{y \in X} G(y) \neq \emptyset$.

Now we shall show that $V_S^M(F, X)$ is closed. Notice that

$$V_S^M(F, X) = \bigcap_{y \in X} \{x \in X : F(x, y) \subseteq C\}. \tag{3.11}$$

Then, by the assumption (iii), it is easy to see that $V_S^M(F, X)$ is closed.

Further, if condition (v) is also satisfied, i.e., for any $y \in X$, the set $\{x \in X : F(x, y) \subseteq C\}$ is empty or convex, it is sufficient to show that $V_S^M(F, X)$ is convex. Indeed, since $V_S^M(F, X) = \cap_{y \in X}\{x \in X : F(x, y) \subseteq C\} \neq \emptyset$, it follows that for any $y \in X$, the set $\{x \in X : F(x, y) \subseteq C\} \neq \emptyset$ and so is convex. Thus, by (3.11), it is easy to see that $V_S^M(F, X)$ is convex. This completes the proof. ∎

**Remark 3.1.** Theorem 3.1 is quite different from Theorem 3 of Ansari *et al.* [2] in the following aspects:

(a) In Theorem 3.1, the existence of strong efficient solution for (MSVEP) is obtained on a nonempty closed convex subset of a real Hausdorff topological vector space, while in Theorem 3 of Ansari *et al.* [2], it was obtained on a nonempty compact convex subset of a locally convex Hausdorff topological vector space.

(b) Theorem 3.1 shows both the existence of strong efficient solution and the closeness and the convexity of the strong efficient solution sets, while Theorem 3 of Ansari *et al.* [2] only showed the existence of strong efficient solution.

(c) The proof method is different. In fact, Theorem 3.1 is proved by using the famous Brouwer fixed point theorem, while Theorem 3 of Ansari *et al.* [2] was proved by using the Kakutani-Fan-Glicksberg fixed point theorem.

**Example 3.1.** Let $E = Z = R, X = C = R_+ = [0, +\infty)$, and $F : X \times X \to 2^Z$ be defined as follows:

$$F(x, y) = [y - x, +\infty), \quad \forall\, x, y \in X.$$

If we take $D = [0, 1]$ and $y_0 = 1$, then it is easy to check that all the conditions (i)–(v) of Theorem 3.1 are satisfied and so Theorem 3.1 implies that $V_S^M(F, X)$ is nonempty, closed and convex. Indeed, we can see that $V_S^M(F, X) = \{0\}$.

**Corollary 3.1.** *Let $E, Z$ and $C$ be as in Theorem 3.1. Let $X \subseteq E$ be a nonempty compact convex subset and $F : X \times X \to 2^Z$ a set-valued mapping. Suppose that*

(i) *for any $x \in X$, $F(x, x) \subseteq C$;*
(ii) *for any $x \in X$, the set $\{y \in X : F(x, y) \not\subseteq C\}$ is empty or convex;*
(iii) *for any $y \in X$, the set $\{x \in X : F(x, y) \subseteq C\}$ is closed;*

*Then, $V_S^M(F, X) \neq \emptyset$. Moreover, $V_S^M(F, X)$ is closed. Further, if the following condition also holds:*

(iv) *for any $y \in X$, the set $\{x \in X : F(x, y) \subseteq C\}$ is empty or convex,*

*then $V_S^M(F, X)$ is convex.*

*Proof.* Take $D = X$. Then, by the assumptions, it is easy to see that all the conditions of Theorem 3.1 are satisfied and so Theorem 3.1 yields the conclusion. This completes the proof. ∎

**Corollary 3.2.** *Let $E, Z, X$ and $C$ be as in Theorem 3.1. Let $f : X \times X \to Z$ be a given mapping. Suppose that*

(i) *for any $x \in X$, $f(x, x) \in C$;*
(ii) *for any $x \in X$, the set $\{y \in X : f(x, y) \notin C\}$ is empty or convex;*
(iii) *for any $y \in X$, the set $\{x \in X : f(x, y) \in C\}$ is closed;*
(iv) *there exists a nonempty compact convex subset $D \subseteq X$ such that, for each $x \in X \backslash D$, there exists some $y_0 \in D$ such that $f(x, y_0) \notin C$.*

*Then, $V_S(f, X) \neq \emptyset$. Moreover, $V_S(f, X)$ is closed. Further, if the following condition is satisfied:*

(v) *for any $y \in X$, the set $\{x \in X : f(x, y) \in C\}$ is empty or convex,*

*then $V_S(f, X)$ is convex.*

From Theorem 3.1, we can also obtain the following result.

**Theorem 3.2.** *Let $E, Z, X, C$ and $F$ be as in Theorem 3.1. Assume that the conditions* (i), (ii), (iv) *of Theorem 3.1 and one of the following conditions hold:*

(iii)′ *for any $y \in X$, $F(x, y)$ is l.s.c. in $x$;*
(iii)″ *for any $y \in X$, $F(x, y)$ is lower $(-C)$-continuous in $x$.*

*Then the conclusion of Theorem 3.1 holds.*

*Proof.* We need only to show that for each $y \in X$, the set

$$Q(y) = \{x \in X : F(x,y) \subseteq C\}$$

is closed in $X$.

Indeed, let $\{x_\alpha\} \subseteq Q(y)$ be an arbitrary net such that $x_\alpha \to x_0$. We need to show that $x_0 \in Q(y)$. Since $\{x_\alpha\} \subseteq X$ and $X$ is closed, we have $x_0 \in X$. In addition, for each $\alpha$,

$$F(x_\alpha, y) \subseteq C. \tag{3.12}$$

(I) If the assumption (iii)$'$ holds, then it follows from lemma 2.1 that for each $z_0 \in F(x_0, y)$, there exists a net $\{z_\alpha\}$ such that $z_\alpha \in F(x_\alpha, y)$ for all $\alpha$ and $z_\alpha \to z_0$. Further, by (3.12), we have $z_\alpha \in C$ for all $\alpha$. By the closeness of $C$, it follows that $z_0 \in C$. Thus, by the arbitrary of $z_0$, we have $F(x_0, y) \subseteq C$. Hence $x_0 \in Q(y)$, and so $Q(y)$ is closed.

(II) If the assumption (iii)$''$ holds, then it is sufficient to show that

$$F(x_0, y) \subseteq C. \tag{3.13}$$

Indeed, since $F(x,y)$ is lower $(-C)$-continuous in $x$, it follows that for each neighborhood $V$ of the origin in $Z$, there exists some $\alpha_0$ such that

$$F(x_0, y) \subseteq F(x_\alpha, y) + V + C, \quad \forall\, \alpha \geq \alpha_0. \tag{3.14}$$

Noting that $C$ is a convex cone, for any $\alpha \geq \alpha_0$, by (3.14) and (3.12), we have

$$F(x_0, y) \subseteq F(x_\alpha, y) + V + C \subseteq C + V + C \subseteq C + V. \tag{3.15}$$

By the arbitrary of $V$, we can show that $F(x_0, y) \subseteq C$. In fact, suppose that it is not the case, then there exists some $a_0 \in F(x_0, y)$ such that $a_0 \notin C$. Since $C$ is closed, there exists a neighborhood $V_0$ of the origin in $Z$ such that $(a_0 + V_0) \cap C = \emptyset$. Since $Z$ is a real Hausdorff topological vector space, there exists a balanced neighborhood $V_1$ of the origin in $Z$ such that $V_1 \subseteq V_0$. Then, we have $(a_0 + V_1) \cap C = \emptyset$. Notice that $V_1$ is balanced, i.e., $V_1 = -V_1$. So $(a_0 - V_1) \cap C = \emptyset$. It follows that

$$0 \notin C - (a_0 - V_1) = -a_0 + V_1 + C,$$

i.e.,

$$a_0 \notin V_1 + C,$$

which contradicts (3.15). Consequently $F(x_0, y) \subseteq C$. Thus $x_0 \in Q(y)$, and so $Q(y)$ is closed. This completes the proof. ∎

**Theorem 3.3.** *Let $E, Z, X, C$ and $F$ be as in Theorem 3.1. Assume that the conditions* (i), (iii), (iv) *of Theorem 3.1 and the following condition hold:*

(ii)$'$ *$F$ is properly $C$-diagonally quasiconvex.*

*Then, $V_S^M(F,X) \neq \emptyset$. Moreover, $V_S^M(F,X)$ is closed. Further, if one of the following conditions also holds:*

(v)$'$ *for any $y \in X$, $F(x,y)$ is $C$-concave in $x$;*

(v)$''$ *for any $y \in X$, $F(x,y)$ is affine in $x$,*

*then $V_S^M(F,X)$ is convex.*

*Proof.* For the first part of the conclusion, we can proceed the proof exactly as that of Theorem 3.1 except for using the assumptions (ii)′ and (i) to get a contradiction with (3.6) and so is omitted.

For the second part of the conclusion, we need only to show that for each $y \in X$, the set

$$Q(y) = \{x \in X : F(x, y) \subseteq C\}$$

is empty or convex in $X$.

Indeed, suppose that $Q(y) \neq \emptyset$ and $x_1, x_2 \in Q(y)$, we need to show that for each $t \in [0, 1]$, $x_t = tx_1 + (1 - t)x_2 \in Q(y)$. Noting that $x_1, x_2 \in X$ and $X$ is convex, we have $x_t \in X$. In addition,

$$F(x_i, y) \subseteq C, \quad i = 1, 2. \tag{3.16}$$

(I) If the assumption (v)′ holds, then we have

$$F(x_t, y) \subseteq tF(x_1, y) + (1 - t)F(x_2, y) + C \subseteq C + C + C \subseteq C. \tag{3.17}$$

(II) If the assumption (v)″ holds, then we have

$$F(x_t, y) = tF(x_1, y) + (1 - t)F(x_2, y) \subseteq C + C \subseteq C. \tag{3.18}$$

From (3.17) and (3.18), we know that, for each $t \in [0, 1]$, $x_t \in Q(y)$, and so $Q(y)$ is convex. This completes the proof. ∎

By Lemma 2.2, Theorems 3.2 and 3.3, we can obtain the following result.

**Corollary 3.3.** *Let $E, Z, X, C$ and $f$ be as in Corollary 3.2. Assume that the conditions* (i), (iv) *of Corollary 3.2 and one of the following conditions hold:*

(ii) *for any $x \in X$, the set $\{y \in X : f(x, y) \notin C\}$ is empty or convex;*
(ii)′ *$f$ is properly $C$-diagonally quasiconvex;*

*and one of the following conditions holds:*

(iii)′ *for any $y \in X$, $f(x, y)$ is continuous in $x$;*
(iii)″ *for any $y \in X$, $f(x, y)$ is upper $(-C)$-continuous in $x$;*
(iii)‴ *for any $y \in X$, $f(x, y)$ is lower $(-C)$-continuous in $x$;*
(iii)⁗ *for any $y \in X$, $f(x, y)$ is $(-C)$-continuous in $x$.*

*Then, $V_S(f, X) \neq \emptyset$. Moreover, $V_S(f, X)$ is closed. Further, if one of the following conditions also holds:*

(v)′ *for any $y \in X$, $f(x, y)$ is $C$-concave in $x$;*
(v)″ *for any $y \in X$, $f(x, y)$ is affine in $x$,*

*then $V_S(f, X)$ is convex.*

**Theorem 3.4.** *Let $Z$ and $C$ be as in Theorem 3.1. Let $E$ be a real reflexive Banach space, and $X \subseteq E$ be a nonempty closed convex subset. Let $F : X \times X \to 2^Z$ be a set-valued mapping. Suppose that*

(i) *for any $x \in X$, $F(x, x) \subseteq C$;*
(ii) *for any $x \in X$, the set $\{y \in X : F(x, y) \not\subseteq C\}$ is empty or convex;*
(iii) *for any $y \in X$, the set $\{x \in X : F(x, y) \subseteq C\}$ is weakly closed;*
(iv) *there exists a nonempty bounded closed and convex subset $D \subseteq X$ such that, for each $x \in X \backslash D$, there exists some $y_0 \in D$ such that $F(x, y_0) \not\subseteq C$.*

*Then, $V_S^M(F,X) \neq \emptyset$. Moreover, $V_S^M(F,X)$ is weakly closed, and so is closed. Further, if the following condition also holds:*

(v) *for any $y \in X$, the set $\{x \in X : F(x,y) \subseteq C\}$ is empty or convex,*

*then $V_S^M(F,X)$ is convex.*

*Proof.* Since $E$ is a real reflexive Banach space, $X \subseteq E$ is a nonempty closed convex subset and $D \subseteq X$ is a nonempty bounded closed convex subset, so $X$ is closed and $D$ is compact with respect to the weak topology of $E$. Thus, by endowed with $E$ with weak topology, it is easy to see that all the conditions of Theorem 3.1 are satisfied with respect to the weak topology of $E$ and so Theorem 3.1 yields the conclusion. This completes the proof. ∎

**Remark 3.2.** By the same argument of Theorem 3.3, we can show that (a) the condition (ii) of Theorem 3.4 can be replaced by (ii)$'$ of Theorem 3.3; (b) the condition (v) of Theorem 3.4 can be replaced by (v)$'$ or (v)$''$ of Theorem 3.3.

**Example 3.2.** Let $E = Z = R, X = C = R_+ = [0, +\infty)$, and $F : X \times X \to 2^Z$ be defined as follows:

$$F(x,y) = [y - x + 1, +\infty), \quad \forall\, x, y \in X.$$

If we take $D = [0,1]$ and $y_0 = 0$, then it is easy to check that all the conditions (i)–(v) of Theorem 3.4 are satisfied and so Theorem 3.4 implies that $V_S^M(F,X)$ is nonempty, weakly closed and convex. Indeed, we can see that $V_S^M(F,X) = [0,1]$.

**Corollary 3.4.** *Let $E, Z, X, C$ and $F$ be as in Theorem 3.4. Suppose that*

(i) *for any $x \in X$, $F(x,x) \subseteq C$;*

(ii) *$F$ is properly $C$-diagonally quasiconvex or for any $x \in X$, the set $\{y \in X : F(x,y) \nsubseteq C\}$ is empty or convex;*

iii) *for any $y \in X$, $F(x,y)$ is l.s.c. or lower $(-C)$-continuous in $x$;*

(iv) *for any $y \in X$, $F(x,y)$ is $C$-concave or affine in $x$;*

(v) *there exists a nonempty bounded closed and convex subset $D \subseteq X$ such that, for each $x \in X \backslash D$, there exists some $y_0 \in D$ such that $F(x,y_0) \nsubseteq C$.*

*Then, $V_S^M(F,X) \neq \emptyset$. Moreover, $V_S^M(F,X)$ is weakly closed and convex, and so is closed.*

*Proof.* We need only to show that the condition (iii) of Theorem 3.4 holds, i.e., for each $y \in X$, the set

$$Q(y) = \{x \in X : F(x,y) \subseteq C\}$$

is weakly closed. Indeed, by the assumption (iii), it follows from the proof of Theorem 3.2 that for each $y \in X$, $Q(y)$ is closed in $X$. In addition, by the assumption (iv), it follows from the proof of Theorem 3.3 that for each $y \in X$, $Q(y)$ is empty or convex. Thus, for each $y \in X$, $Q(y)$ is weakly closed. This completes the proof. ∎

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