

## On Gracefulness of Directed Trees with Short Diameters

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**Abstract.** Graceful labelling is studied on undirected graphs since graceful graphs can be used in some  $H$ -decomposition problems. In this note, we investigate the directed graceful problem for many orientations of undirected trees with short diameters, and provide some directed trees that deny any digraceful labelling.

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### 1. Introduction and concepts

Most of the labellings mentioned in [9] and [11] focus on undirected graphs since undirected graphs have been studied much more extensively than directed graphs [1]. However, Bloom and Hsu [3, 4, 5] investigate graceful labelling (also, numberings) problem of digraphs in early 1980's. In [7], the author studies magic labellings on digraphs.

Graceful digraphs are related in a variety of ways to other areas of mathematics. In [4], Bloom and Hsu characterized the graceful labellings of certain classes of digraphs by the existence of particular algebraic structures, including cyclic difference sets and sequenceable groups, and so on. They provided some digraph models of cyclic groups, and cyclic neofields that are used to generate families of graceful labellings for the models. Furthermore, Bloom and Hsu showed some properties and examples of this new class of graph labellings for several families of graceful digraphs including certain orientations of cycles, paths, and the unions of cycles and paths, as well as certain complete digraphs, wheels, windmills, and umbrellas.

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The shorthand symbol  $[m, n]$  stands for a set  $\{m, m + 1, \dots, n\}$ , where  $m$  and  $n$  are non-negative integers with  $m < n$ . The cardinality of a finite set  $X$  is denoted by  $|X|$ . Only non-negative integers are used here. The *underlying graph*  $UG(D)$  of a digraph  $D$  is obtained from  $D$  by removing the direction of each arc of  $D$ . Unless otherwise mentioned, we consider only that  $UG(D)$  is connected and has no loops, multiple edges, and then we say the digraph  $D$  to be *simple-connected*. A *ditree* is a digraph whose underlying graph is a tree.

The vertex and arc sets of a simple-connected digraph  $D$  are denoted as  $V(D)$  and  $A(D) = \{\vec{uv} : u, v \in V(D)\}$ , respectively. An injection  $f : V(D) \rightarrow [0, |A(D)|]$  of  $D$  is *proper* if no two vertices of  $D$  are assigned the same integer, also,  $f$  is called a *labelling* of  $D$ . For an arc  $\vec{uv}$  (or denoted by  $(u, v)$ ) the first vertex  $u$  is its *tail* and the second vertex  $v$  is its *head*. The out-neighbourhood and in-neighbourhood of a vertex  $u$  are defined as  $N_D^+(u) = \{v \in V(D) \setminus \{u\} : \vec{uv} \in A(D)\}$  and  $N_D^-(u) = \{w \in V(D) \setminus \{u\} : \vec{wu} \in A(D)\}$ , respectively. The numbers  $|N_D^+(u)|$  and  $|N_D^-(u)|$ , denoted by  $d_D^+(u)$  and  $d_D^-(u)$  (or  $d^+(u)$  and  $d^-(u)$  for short), are called the *out-degree* and *in-degree* of the vertex  $u$ , respectively.

In [4] Bloom and Hsu defined: A digraph  $D$  is labelled by a labelling  $\theta : V(D) \rightarrow [0, |A(D)|]$ . The vertex values, in turn, induce a value  $\theta(\vec{uv})$  on each arc  $\vec{uv}$ , where  $\theta(\vec{uv}) = \theta(v) - \theta(u) \pmod{|A(D)| + 1}$ . If the arc values are all distinct and nonzero, then the labelling  $\theta$  is *graceful*. A digraph is *graceful* if it has a *graceful labelling*.

The *dual labelling*  $f^*$  of a labelling  $f$  of a digraph  $D$  is defined as  $f^*(x) = |A(D)| - f(x)$  for all  $x \in V(D)$ .

For the convenience of stating proofs, we will define some terminologies through (1) to (4).

- (1) Let  $f$  be a labelling of a digraph  $D$  from  $V$  to  $[0, |A(D)|]$  such that for each arc  $\vec{uv} \in A(D)$ ,  $f(\vec{uv}) = f(u) - f(v)$  if  $f(u) > f(v)$ , otherwise  $f(\vec{uv}) = |A(D)| + 1 + (f(u) - f(v))$ . We call  $f$  a *digraceful labelling* of  $D$  if the *arc label set*  $\{f(\vec{uv}) : \vec{uv} \in A(D)\} = [1, |A(D)|]$ . Therefore,  $D$  is called a *digraceful digraph*.

It is not hard to understand that a digraceful labelling  $f$  of a digraph  $D$  is equivalent to a certain graceful labelling  $\theta$  of  $D$  defined in [4] since  $f(u) = \theta(u)$  for  $u \in V(D)$  and  $f(\vec{uv}) = |A(D)| + 1 - \theta(\vec{uv})$  for  $\vec{uv} \in A(D)$ , and moreover  $f$  can be regarded as the *arc-dual labelling* of  $\theta$ .

- (2) Let  $f$  be a digraceful labelling of a digraph  $D$ . An arc  $\vec{uv}$  is called a *forward-arc* if its label  $f(\vec{uv}) = f(u) - f(v)$ , otherwise, a *backward-arc*. The number of forward-arcs and the number of backward-arcs are denoted by  $Fa_D(f)$  and  $Ba_D(f)$ , respectively. We call number  $\max_f \{Fa_D(f)\}$  spanning over all digraceful labellings  $f$  of  $D$  the *optimal digraceful number* of  $D$ , denoted by  $Odn(D)$ . Furthermore, if  $Fa_D(f) = Odn(D) = |A(D)|$ , then  $f$  is a graceful labelling of the underlying graph  $UG(D)$ .

Clearly, an arc  $\vec{uv}$ , an out-degree  $d^+(u)$ , an arc label  $f(\vec{uv}) = f(u) - f(v)$  and a forward-arc are in the same “direction”, from the left to the right.

- (3) The *converse*  $H$  of a digraph  $D$  is the digraph obtained from  $D$  by conversing all arcs of  $D$  (by reversing the arc  $\vec{uv}$ , it means that we replace the arc  $\vec{uv}$  by the arc  $\vec{vu}$ ). Notice that  $V(H) = V(D)$  and  $|A(H)| = |A(D)|$ . The

conversely digraceful labelling  $h^*$  of a digraceful labelling  $f$  of  $D$  is defined by  $h^*(x) = |A(D)| - f(x)$  for all  $x \in V(H) = V(D)$ .

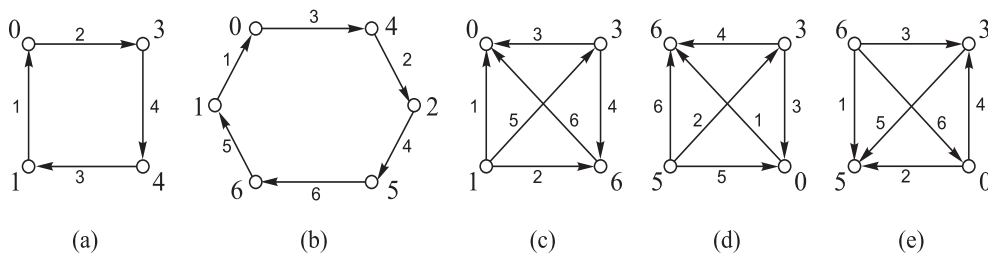


Figure 1. (a) A digraceful orientation of  $C_4$ ; (b) A digraceful orientation of  $C_6$ ; (c) A digraceful orientation of  $K_4$  with a digraceful labelling  $f$ , and  $Fa(f) = Ba(f) = 3$ ; (d) A digraceful orientation of  $K_4$  with the dually digraceful labelling  $f^*$  of the digraceful labelling  $f$  defined in (c); (e) The conversely digraceful labelling  $h^*$  of the converse of an orientation of  $K_4$  shown in (c).

- (4) An undirected graph  $G$  has a *digraceful orientation* if there exists an orientation of  $G$  which admits a digraceful labelling. All orientations of the star  $K_{1,5}$  are digraceful, see Figure 2.

Each vertex of degree one in an undirected tree is called a *leaf*. A *caterpillar* is an undirected tree  $T$  such that the resulting graph obtained by deleting all leaves of  $T$  is just a path. A *lobster*  $H$  is an undirected tree such that the graph obtained by deleting all leaves from  $H$  is just a caterpillar. An *in-zero-out ditree* is a ditree if, for its every vertex  $u$ , one of out-degree  $d^+(u)$  and in-degree  $d^-(u)$  is always equal to zero. A *rooted ditree*  $T$  at a fixed vertex  $u$  satisfies that in-degree  $d^-(u) = 0$ , out-degree  $d^+(u) \geq 1$ , and in-degrees  $d^-(x) = 1$  for all  $x \in V(T) \setminus \{u\}$ . A vertex  $u$  of a ditree  $T$  is called an *in-leaf* if  $d_T^+(u) = 0$  and  $d_T^-(u) = 1$ , and an *out-leaf* as if  $d_T^-(u) = 0$  and  $d_T^+(u) = 1$ .

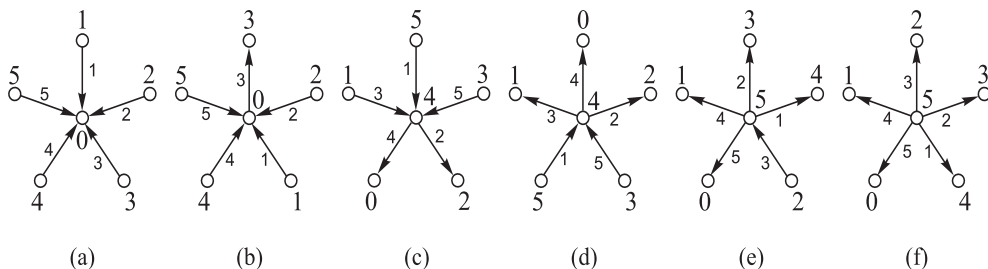


Figure 2. All digraceful orientations of  $K_{1,5}$ .

Undefined terminologies follow [1, 2], and undefined labellings can be found in [9].

The first section is to fix some terminologies and notations, and to present a simple introduction on digraceful digraphs. In Section 2, some properties on digraceful digraphs will be given. We will aim to demonstrate the existence of digraceful labellings for some undirected trees with short diameters. Some ditrees, although their underlying trees are graceful, are verified for denying any digraceful labelling. We will look for digraceful labellings for particular ditrees. In the last section, Section 3, we will propose problems about the digraceful problem on digraphs.

## 2. Main results

### 2.1. Some properties on digraceful labellings

In this subsection we will show several properties of digraceful labellings on simple-connected digraphs.

**Lemma 2.1.** *Let  $H$  be the converse of a simple-connected digraph  $D$  having a labelling  $f$ . Let  $f^*$  and  $h^*$  be the dual labelling and converse labelling of  $f$ , respectively. If  $f$  is digraceful, so are  $f^*$  and  $h^*$ . Furthermore,  $f$  is also a digraceful labelling of  $H$ .*

**Observation 1.** Let  $D$  be a simple-connected digraph with  $m$  arcs and a digraceful labelling  $f$ , and let  $H$  be the converse of  $D$ . Let  $f^*$  and  $h^*$  be the dual labelling and converse labelling of  $f$ , respectively. Then

- (i)  $Fa_D(f) + Ba_D(f) = m$ .
- (ii)  $Fa_D(f) + Fa_D(f^*) = Ba_D(f) + Ba_D(f^*)$ .
- (iii)  $Fa_D(f) + Ba_D(f) = Fa_H(h^*) + Ba_H(h^*)$ .
- (iv)  $(m + 1)(m - 2Ba_D(f)) = 2 \sum_{u \in V(D)} (d_D^+(u) - d_D^-(u))f(u)$ .

**Lemma 2.2.** *Let  $f$  be a digraceful labelling of a simple-connected digraph  $D$  with  $m$  arcs.*

- (i) *For odd  $m$ , a digraph  $H_1$  obtained by conversing a certain arc of  $D$  admits the digraceful labelling  $f$ .*
- (ii) *There are two arcs  $\overrightarrow{uv}$  and  $\overrightarrow{xy}$  with  $f(\overrightarrow{uv}) + f(\overrightarrow{xy}) = m + 1$  such that  $f$  is also a digraceful labelling of the digraph  $H_2$  obtained by conversing both  $\overrightarrow{uv}$  and  $\overrightarrow{xy}$  of  $D$ .*
- (iii) *If both  $\overrightarrow{uv}$  and  $\overrightarrow{xy}$  are forward arcs (or backward arcs) in  $D$ , then  $f(u) + f(y) \neq f(x) + f(v)$ .*
- (iv) *If  $f(u) = 0$  and  $Ba_D(f) = 0$ , then there is a new digraph  $H_3$  obtained by adding new vertices  $w_i$  to  $D$  and joining  $w_i$  with  $u$  by an arc  $\overrightarrow{w_i u}$  for  $i \in [1, k]$  is digraceful.*
- (v) *The digraceful labelling  $f$  is a graceful labelling of the underlying graph  $UG(D)$  if and only if there are no forward-arc  $\overrightarrow{uv}$  and backward-arc  $\overrightarrow{xy}$  such that  $f(u) + f(x) = f(v) + f(y)$ .*

*Proof.* (i) Since  $m + 1$  is even, there exists an arc  $\overrightarrow{uv}$  that holds  $f(\overrightarrow{uv}) = f(u) - f(v) = m + 1 + (f(v) - f(u))$ . Hence, it is easy to obtain the digraph  $H_1$  generated from  $D$  by conversing arc  $\overrightarrow{uv}$  which admits the digraceful labelling  $f$ .

(ii) We take an arc  $\overrightarrow{uv}$  from the simple-connected digraph  $D$  of size  $m$  with a digraceful labelling  $f$  such that  $f(u) - f(v) \neq m + 1 + (f(v) - f(u))$ .

**Case 1.**  $f(\overrightarrow{uv}) = f(u) - f(v)$ , that is,  $f(u) > f(v)$ . We converse the arc  $\overrightarrow{uv}$  of  $D$ , and then obtain a digraph  $D'$ . Therefore, there is an arc  $\overrightarrow{xy}$  of  $D'$  such that  $f(\overrightarrow{vu}) = m + 1 + (f(v) - f(u)) = f(\overrightarrow{xy}) = f(x) - f(y)$ , since  $f(\overrightarrow{uv}) \neq f(\overrightarrow{xy})$  in  $D$ . Next, we have a new digraph  $H$  obtained by conversing the arc  $\overrightarrow{xy}$  of  $D'$ . Notice that  $f(\overrightarrow{yx}) = m + 1 + (f(y) - f(x)) = f(u) - f(v) = f(\overrightarrow{uv})$ , we know that any two arcs of  $H$  are assigned distinct labels under the labelling  $f$ . As a result,  $H$  is digraceful.

**Case 2.**  $f(\overrightarrow{uv}) = m + 1 + (f(u) - f(v))$ , that is,  $f(u) < f(v)$ . Let  $D''$  be the digraph generated from  $D$  by conversing the arc  $\overrightarrow{uv}$  of  $D$ . Therefore, there exists an arc  $\overrightarrow{wz}$  in  $D''$  such that  $f(\overrightarrow{vu}) = f(v) - f(u) = f(w) - f(z) = f(\overrightarrow{wz})$  since  $f(\overrightarrow{uv}) \neq f(\overrightarrow{wz})$  in  $D$ . We have a digraph  $H_2$  obtained by conversing the arc  $\overrightarrow{wz}$  of  $D''$ . It is easy to see that

$$\begin{aligned} f(\overrightarrow{zw}) &= m + 1 + (f(z) - f(w)) = m + 1 - f(\overrightarrow{wz}) \\ &= m + 1 - f(\overrightarrow{vu}) = m + 1 - (f(v) - f(u)) = f(\overrightarrow{uv}). \end{aligned}$$

Thereby,  $H_2$  is digraceful.

(iii) This assertion is obvious.

(iv) Let  $H_3$  be a digraph obtained by adding new vertices  $w_i$  to  $D$  and joining  $w_i$  with  $u$  by arcs  $\overrightarrow{w_i u}$  ( $i \in [1, k]$ ), where  $f(u) = 0$ . It is straightforward to defined a digraceful labelling  $g$  for  $H_3$ . Let  $g(x) = f(x)$  for  $x \in V(D) \subset V(H_3)$ , and  $g(w_i) = m + i$  for  $i \in [1, k]$ . Obviously,  $g$  is digraceful since  $Bad_D(f) = 0$ .

(v) The necessary condition of the assertion (v) is evident, so we only present the proof of the sufficient condition. Notice that there are no forward-arc  $\overrightarrow{uv}$  and backward-arc  $\overrightarrow{xy}$  such that  $f(u) + f(x) = f(v) + f(y)$  by the assertion (iii). Suppose that there are two edges  $x_1y_1$  and  $x_2y_2$  of the underlying graph  $UG(D)$  such that  $|f(x_1) - f(y_1)| = |f(x_2) - f(y_2)|$  for the digraceful labelling  $f$  of  $D$ . Clearly, there is no  $f(\overrightarrow{x_1y_1}) = f(\overrightarrow{x_2y_2})$  as if both  $\overrightarrow{x_1y_1}$  and  $\overrightarrow{x_2y_2}$  are forward-arcs (or backward-arcs). We assume that  $\overrightarrow{x_1y_1}$  is a backward-arc and  $\overrightarrow{x_2y_2}$  is a forward-arc. Therefore, from  $f(y_1) - f(x_1) = f(x_2) - f(y_2)$  and  $f(\overrightarrow{x_1y_1}) = m + 1 + (f(x_1) - f(y_1))$ , we have

$$m + 1 + (f(x_1) - f(y_1)) = m + 1 - (f(y_1) - f(x_1)) = m + 1 - (f(x_2) - f(y_2)),$$

immediately,  $f(x_1) + f(x_2) = f(y_1) + f(y_2)$ , a contradiction. ■

## 2.2. Results on particular orientations of trees

Verifying the digracefulness of each orientation of a tree  $T$  seems to be difficult, since such a verification is harder than finding only one graceful labelling for  $T$ . Usually, one provide a digraceful labelling for an orientation of  $T$ . In fact, providing a graceful labelling for a given tree was widely used in attacking the Graceful Tree Conjecture [9, 8]: Every tree is graceful.

If the underlying tree  $T$  of an in-zero-out ditree is a caterpillar, we denote this ditree as  $D(T; ca)$ .

**Theorem 2.1.** *Let  $D(T; ca)$  be an in-zero-out ditree whose underlying tree  $T$  is a caterpillar. Then  $D(T; ca)$  is digraceful.*

*Proof.* Let  $D(T; ca)$  be an in-zero-out ditree whose underlying tree  $T$  is a caterpillar. Deleting all leaves of the caterpillar  $T$ , the remainder is a path  $P = u_1 u_2 \cdots u_n$ . We, without loss of generalization, assume that  $d^+(u_1) = 0$ , so  $d^-(u_2) = 0$ , and then  $d^+(u_3) = 0$  by the definition of an in-zero-out ditree. Thereby, we have  $d^+(u_{2i-1}) = 0$  and  $d^-(u_{2i}) = 0$  in  $D(T; ca)$ ,  $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$ . A digraceful  $D(T; ca)$  can be obtained by the way described in the following.

First, we consider a directed star  $H_1$  with vertex set  $V(H_1) = \{u_1, u_2, u_{1,j} : j \in [1, m_1]\}$  and arc set  $A(H_1) = \{\overrightarrow{u_2 u_1}, \overrightarrow{u_{1,j} u_1} : j \in [1, m_1]\}$ . Clearly,  $H_1$  is an orientation of the star  $K_{1, m_1}$ . Consequently, we define the labelling  $f_1$  of  $H_1$  in the way:  $f_1(u_1) = m_1 + 1$ ,  $f_1(u_{1,j}) = j$  for  $j \in [1, m_1]$ , and  $f_1(u_2) = 0$ . Hence,  $H_1$  is digraceful by the definition of  $f_1$ .

Second, we construct a directed caterpillar  $H_2$  such that  $V(H_2) = V(H_1) \cup \{u_3, u_{2,j} : j \in [1, m_2]\}$  and arc set  $A(H_2) = A(H_1) \cup \{\overrightarrow{u_2 u_3}, \overrightarrow{u_{2,j} u_2} : j \in [1, m_2]\}$ . Since  $d^+(u_1) = 0$  and  $d^-(u_2) = 0$  in  $H_2$ , so  $H_2$  is an in-zero-out ditree. We have the labelling  $f_2$  of  $H_2$  defined by  $f_2(x) = f_1(x)$  if  $x \in V(H_1) \subset V(H_2)$ , and  $f_2(u_{2,j}) = m_1 + 1 + j$  for  $j \in [1, m_2]$ , and  $f_2(u_3) = m_1 + m_2 + 2$ . Notice that each member of  $A(H_2)$  is a backward arc under the labelling  $f_2$ , and  $|A(H_2)| = m_1 + m_2 + 2$ . The following arc labels

$$f_2(\overrightarrow{u_{1,j} u_1}) = |A(H_2)| + 1 + (f_2(u_{1,j}) - f_2(u_1)) = m_2 + 1 - j, j \in [1, m_1],$$

provide numbers  $m_2 + 3, m_2 + 4, \dots, |A(H_2)| - 1, |A(H_2)|$ . Next,  $f_2(\overrightarrow{u_2 u_3}) = |A(H_2)| + 1 + (0 - (m_1 + 1)) = m_2 + 2$ . Furthermore, the following arc labels

$$f_2(\overrightarrow{u_2 u_{2,j}}) = |A(H_2)| + 1 + (f_2(u_2) - f_2(u_{2,j})) = m_2 + 2 - j, j \in [1, m_2]$$

yield numbers  $2, 3, \dots, m_2, m_2 + 1$ . Notice that  $f_2(\overrightarrow{u_2 u_3}) = |A(H_2)| + 1 + (f_2(u_2) - f_2(u_3)) = 1$ . Thereby,  $H_2$  is digraceful.

Third, we construct a directed caterpillar  $H_3$  with vertex set  $V(H_3) = V(H_2) \cup \{u_4, u_{3,j} : j \in [1, m_3]\}$  and arc set  $A(H_3) = A(H_2) \cup \{\overrightarrow{u_3 u_4}, \overrightarrow{u_{3,j} u_3} : j \in [1, m_3]\}$ . Clearly,  $d^+(u_1) = 0$ ,  $d^-(u_2) = 0$  and  $d^+(u_3) = 0$  in  $H_3$ , it means that  $H_3$  is an in-zero-out ditree. Next, we define a labelling  $f_3$  to  $H_3$  in the way:  $f_3(x) = |A(H_2)| - f_2(x)$  if  $x \in V(H_2) \subset V(H_3)$ , and  $f_3(u_{3,j}) = |A(H_2)| + j$  for  $j \in [1, m_3]$ , and  $f_3(u_4) = |A(H_3)| = \sum_{i=1}^3 (1 + m_i)$ . Notice that every member of  $A(H_3)$  is a forward arc under  $f_3$ . We have  $\{f_3(\overrightarrow{xy}) : xy \in A(H_2) \subset A(H_3)\} = [1, |A(H_2)|]$  and  $\{f_3(\overrightarrow{u_4 u_3}), f_3(\overrightarrow{u_{3,j} u_3}) : j \in [1, m_3]\} = [|A(H_2)| + 1, |A(H_3)|]$ . It follows that  $H_3$  is digraceful.

In general, we construct a directed caterpillar  $H_{2i}$  ( $i \geq 2$ ) with vertex set  $V(H_{2i}) = V(H_{2i-1}) \cup \{u_{2i+1}, u_{2i,j} : j \in [1, m_{2i}]\}$  and arc set  $A(H_{2i}) = A(H_{2i-1}) \cup \{\overrightarrow{u_{2i} u_{2i+1}}, \overrightarrow{u_{2i} u_{2i,j}} : j \in [1, m_{2i}]\}$ , and define a digraceful labelling  $f_{2i}$  in the way:  $f_{2i}(x) = |A(H_{2i-1})| - f_{2i-1}(x)$  if  $x \in V(H_{2i-1}) \subset V(H_{2i})$ , and  $f_{2i}(u_{2i,j}) = |A(H_{2i-1})| + j$  for  $j \in [1, m_{2i}]$ , and  $f_{2i}(u_{2i+1}) = |A(H_{2i})| = \sum_{i=1}^{2i} (1 + m_i)$ . It is very similar to Second step above to verify that  $f_{2i}$  is a digraceful labelling of  $H_{2i}$ . Notice that each  $H_{2i}$  is an in-zero-out ditree.

Again, we make a directed caterpillar  $H_{2i+1}$  ( $i \geq 2$ ) with vertex set  $V(H_{2i+1}) = V(H_{2i}) \cup \{u_{2i+2}, u_{2i+1,j} : j \in [1, m_{2i+1}]\}$  and arc set  $A(H_{2i+1}) = A(H_{2i}) \cup \{\overrightarrow{u_{2i+1} u_{2i+2}}, \overrightarrow{u_{2i+1} u_{2i+1,j}} : j \in [1, m_{2i+1}]\}$ .

$\overrightarrow{u_{2i+1,j}u_{2i+1}} : j \in [1, m_{2i+1}]$ , and then define a digraceful labelling  $f_{2i+1}$  by setting  $f_{2i+1}(x) = |A(H_{2i})| - f_{2i}(x)$  if  $x \in V(H_{2i}) \subset V(H_{2i+1})$ , and  $f_{2i+1}(u_{2i+1,j}) = |A(H_{2i})| + j$  for  $j \in [1, m_{2i+1}]$ , and  $f_{2i+1}(u_{2i+2}) = |A(H_{2i+1})| = \sum_{i=1}^{2i+1} (1 + m_i)$ . To verify  $f_{2i+1}$  is a digraceful labelling of  $H_{2i+1}$ , it is as the same as that in Third step above. Notice that each  $H_{2i+1}$  is an in-zero-out ditree.

Summarizing all of the above arguments together, we claim that the in-zero-out ditree  $D(T; ca)$  is digraceful. ■

Combining Lemma 2.2 with Theorem 2.1 together, we can obtain a number of digraceful ditrees whose underlying trees are caterpillars. Furthermore, by the proof of Theorem 2.1, we have

**Corollary 2.1.** *Let  $D$  be a ditree with a digraceful labelling  $f$  such that  $Ba_D(f) = 0$ . There is a digraceful ditree  $H$  obtained by identifying a vertex of  $D$  with a vertex of an in-zero-out ditree  $D(T; ca)$ .*

### 2.3. Results on many orientations of a tree with short diameter

**Theorem 2.2.** *Let  $\overrightarrow{K}_{1,m}$  be an orientation of a star  $K_{1,m}$  on  $(m+1)$  vertices. Then  $\overrightarrow{K}_{1,m}$  is digraceful if  $m$  is odd, and one of out-degree and in-degree of the center  $w$  of  $\overrightarrow{K}_{1,m}$  must be even when  $m$  is even.*

*Proof.* Let  $\overrightarrow{K}_{1,m}$  be an orientation of a star  $\overrightarrow{K}_{1,m}$  with the center  $w$ . If  $d^+(w) = 0$ , that is, any arc of  $\overrightarrow{K}_{1,m}$  is as the form  $\overrightarrow{yw}$ , clearly,  $\overrightarrow{K}_{1,m}$  admits a digraceful labelling. Analogously,  $\overrightarrow{K}_{1,m}$  admits a digraceful labelling as if  $d^-(w) = 0$ . Next, we consider that  $\overrightarrow{K}_{1,m}$  contains  $s$  ( $\geq 1$ ) arcs as form  $\overrightarrow{wx}$  and  $t$  ( $\geq 1$ ) arcs as form  $\overrightarrow{yw}$ . Let the vertex set and arc set of  $\overrightarrow{K}_{1,m}$  be  $V(\overrightarrow{K}_{1,m}) = \{w, u_i, v_j : i \in [1, s], j \in [1, t]\}$  and  $A(\overrightarrow{K}_{1,m}) = V_s \cup V_t$  where  $V_s = \{\overrightarrow{wu_i} : i \in [1, s]\}$  and  $V_t = \{\overrightarrow{v_jw} : j \in [1, t]\}$ , respectively.

Let  $2k = m + 1$ . We define a labelling  $h$  of  $\overrightarrow{K}_{1,m}$  as:  $h(w) = k$ , and each pair of  $(i, 2k - i)$  ( $i \in [1, k - 1]$ ) is assigned to a pair of vertices of  $V_s$  or  $V_t$  such that there no  $x \in V_s$  and  $y \in V_t$  hold  $h(x) = i$  and  $h(y) = 2k - i$ . If  $|V_s|$  is odd, the number zero is assigned to a vertex of  $|V_s|$ . Clearly, there are no  $h(\overrightarrow{wx}) = h(\overrightarrow{wy})$ ,  $h(\overrightarrow{uw}) = h(\overrightarrow{vw})$  and  $h(\overrightarrow{wx}) = h(\overrightarrow{wy}) \pmod{m + 1}$ . Hence,  $\overrightarrow{K}_{1,m}$  is digraceful.

For even  $m$ , we need to prove the following claim first.

**Claim 1.** Then  $\overrightarrow{K}_{1,m}$  is digraceful if and only if the following property holds: There is an integer  $c \in [0, m]$  such that  $[0, m] \setminus \{c\} = V_s \cup V_t$  with  $|V_s| = s$ ,  $|V_t| = t$ ,  $m = s + t \geq 1$ , and furthermore no  $a \in V_s$  and  $b \in V_t$  hold  $a + b = 2c + \delta(m + 1)$ , where  $\delta = -1$  or  $0$  or  $1$ .

*The proof of Claim 1.* We only present the proof of the necessity since the sufficient property is just a digraceful labelling of  $\overrightarrow{K}_{1,m}$ . Let  $f$  be a digraceful labelling of  $\overrightarrow{K}_{1,m}$ . We have

- (1)  $f(w) - f(u_i) \neq m + 1 + (f(v_j) - f(w))$ , so  $2f(w) - (m + 1) \neq f(u_i) + f(v_j)$ .
- (2)  $f(w) - f(u_i) \neq f(v_j) - f(w)$ , that is  $2f(w) \neq f(u_i) + f(v_j)$ .

- (3)  $m + 1 + (f(w) - f(u_i)) \neq f(v_j) - f(w)$ , immediately,  $2f(w) + (m + 1) \neq f(u_i) + f(v_j)$ .

Let  $c = f(w)$ ,  $V_s = \{f(u_i) : i \in [1, s]\}$  and  $V_t = \{f(v_j) : j \in [1, t]\}$ . Therefore,  $[0, m] \setminus \{c\} = V_s \cup V_t$ , and there are no  $a \in V_s$  and  $b \in V_t$  satisfy  $a + b = 2c + \delta(m + 1)$ , where  $\delta = -1$  or  $0$  or  $1$ . The proof of the Claim 1 is completed.  $\blacksquare$

Next, we verify the following claim.

**Claim 2.** Let  $2k = s + t$ , where both integers  $s, t$  are odd. Then for any integer  $c \in [0, 2k]$  we have that  $[0, 2k] \setminus \{c\} = V_s \cup V_t$  with  $|V_s| = s$  and  $|V_t| = t$  such that there is at least a pair of  $a \in V_s$  and  $b \in V_t$  that hold  $a + b = 2c + \delta(2k + 1)$ , where  $\delta = -1$  or  $0$  or  $1$ .

*The proof of Claim 2.* For  $c = 0$ , then we have  $k$  pairs of  $(i, 2k + 1 - i)$  ( $i \in [1, k]$ ) that hold  $a + b = \delta(2k + 1)$ , where  $\delta = 1$ . For  $[0, 2k] \setminus \{c\} = V_s \cup V_t$ , there is at least one pair of  $(i, 2k + 1 - i)$  such that  $i \in V_s$  and  $2k + 1 - i \in V_t$ , since both  $|V_s|, |V_t|$  are odd.

For  $c \geq 1$ , we have  $c$  pairs of  $(i - 1, 2c - i + 1)$  from  $0, 1, 2, \dots, c - 1, c + 1, \dots, 2c - 2, 2c - 1, 2c$  such that  $(i - 1) + (2c - i + 1) = 2c + \delta(2k + 1)$  for  $\delta = 0$ ; and we have  $k - c$  pairs of  $(2c + i, 2k - i + 1)$  generated from  $2c + 1, 2c + 2, 2c + 3, \dots, 2k - 1, 2k$  such that  $(2c + i) + (2k - i + 1) = 2c + \delta(2k + 1)$  for  $\delta = 1$ . Since both  $|V_s|, |V_t|$  are odd, we could not put two numbers of a certain pair  $(i - 1, 2c - i + 1)$  (or  $(2c + j, 2k - j + 1)$ ) for  $j \in [1, 2k]$  in  $V_s$  (or  $V_t$ ) for  $i \in [1, 2c + 1]$ . The proof of the Claim 2 is finished.  $\blacksquare$

Combining Claim 1 and Claim 2 together, this theorem is covered.  $\blacksquare$

A ditree  $H$  with diameter three is called a *directed bistar*. For the sake of convenience, we have the following description about a class of directed bistars  $T(s, t)$ :

Dibistar (I): The vertex set and arc set of a directed bistar  $T(s, t)$  are defined as  $V(T(s, t)) = \{u_i, u, v, v_j : i \in [1, s - 1], j \in [1, t]\}$  and  $A(T(s, t)) = \{\overrightarrow{uu_i}, \overrightarrow{uv}, \overrightarrow{vv_j} : i \in [1, s - 1], j \in [1, t]\}$ , respectively, where  $u$  is the *root* of  $T(s, t)$ . Clearly, the in-degrees  $d_{T(s, t)}^-(u) = 0$  and  $d_{T(s, t)}^-(v) = 1$ , and the out-degrees  $d_{T(s, t)}^+(u) = s$  and  $d_{T(s, t)}^+(v) = t$ . Let  $m = |V(T(s, t))| = s + t + 1$ .

**Lemma 2.3.** *Let  $f$  be a digraceful labelling of a directed bistar  $T(s, t)$  defined by Dibistar (I), where  $s - 1, t \geq 1$ . Then*

- (i) *For  $\delta = 0, 1$ , we have  $f(u) + f(v_j) \neq \delta m + 2f(v)$ ,  $f(u) + f(v_j) \neq \delta m + f(v) + f(u_i)$  and  $\delta m + f(u) + f(v_j) \neq f(v) + f(u_i)$ .*
- (ii) *There are no the following cases: (a)  $f(u) > f(v) > f(x)$  (or  $f(v) < f(u) < f(x)$ ) for  $x \in V(T(s, t)) \setminus \{u, v\}$ ; and (b)  $f(u) = m - 1$  and  $f(v) = 0$ .*
- (iii) *Each of two integers  $s - 1$  and  $t$  must be even.*

*Proof.* It is not hard to verify that  $T(2, 1)$ ,  $T(2, 2)$  and  $T(3, 1)$  are not digraceful, and  $T(3, 2)$  is digraceful. Hence, we consider directed bistars  $T(s, t)$  for  $s, t \geq 3$  in the following.

Clearly,  $d_{T(s, t)}^-(u) = 0$ ,  $d_{T(s, t)}^+(u) = s \geq 3$ ,  $d_{T(s, t)}^-(v) = 1$  and  $d_{T(s, t)}^+(v) = t \geq 3$  in the directed bistar  $T(s, t)$ . We, without loss of generalization, may assume that  $f(u) = a > b = f(v)$  (otherwise,  $f^*(u) > f^*(v)$  where  $f^*$  is the dual labelling of  $f$ ),  $f(u_i) < f(u_{i+1})$  for  $i \in [1, s - 2]$  and  $f(v_j) < f(v_{j+1})$  for  $j \in [1, t - 1]$ .



(i) There are the following possible cases.

(1) The case  $f(\overrightarrow{uv}) \neq f(\overrightarrow{vv_j})$  may include  $a - b \neq b - f(v_j)$  and  $a - b \neq m + (b - f(v_j))$  that yield  $a + f(v_j) \neq \delta m + 2b$  for  $\delta = 0, 1$ .

(2) The case  $f(\overrightarrow{uv_i}) \neq f(\overrightarrow{vv_j})$  may contain  $a - f(u_i) \neq b - f(v_j)$  and  $a - f(u_i) \neq m + (b - f(v_j))$ , thus,  $a + f(v_j) \neq \delta m + b + f(u_i)$  for  $\delta = 0, 1$ .

Again,  $f(\overrightarrow{uv_i}) \neq f(\overrightarrow{vv_j})$  may lead to cases  $m + (a - f(u_i)) \neq b - f(v_j)$  and  $m + (a - f(u_i)) \neq m + (b - f(v_j))$ , immediately,  $\delta m + a + f(v_j) \neq b + f(u_i)$  for  $\delta = 0, 1$ .

(ii) We have the following two cases in discussion.

**Case A.** If  $a > f(u_{s-1})$  and  $b > f(v_t)$ , thus,  $a = m - 1$ . And it follows the arc label  $m - 1$  that  $f(u_1) = 0$ . Again, the arc label  $m - 2$  leads  $f(u_2) = 1$ , go on in this way, we have that the arc labels  $m - k$  yield  $f(u_k) = k - 1$  for  $k \in [1, s - 1]$ . Therefore,  $b = m - 2$  by the above deduction on the arc labels of  $\overrightarrow{uu_i}$  with  $i \in [1, s - 1]$ . But  $f(\overrightarrow{uv}) = 1 = f(\overrightarrow{vv_i})$ , where  $f(v_t) = m - 3$ , a contradiction.

If  $b < a < f(v_1)$ , it follows  $m + (b - f(v_t)) = m + (a - (f(v_t) + a - b))$  that no vertex  $u_j$  satisfies  $f(u_j) = f(v_t) + a - b$ . Hence, there is a vertex  $v_k$  that holds  $f(v_k) = f(v_t) + a - b$ , and furthermore  $f(v_t) > f(v_k) = f(v_t) + a - b$ , which is impossible since  $a > b$ .

**Case B.** If  $a = m - 1$  and  $b = 0$ , we have

$$f(\overrightarrow{vv_1}) = m + (f(v) - f(v_1)) = m + (0 - f(v_1)) = (m - 1) - (f(v_1) - 1),$$

it means that no vertex  $u_i$  is assigned as  $f(u_i) = f(v_1) - 1$ . Therefore, there exists a vertex  $v_j$  that holds  $f(v_j) = f(v_1) - 1$ , and furthermore  $f(v_1) < f(v_j) = f(v_1) - 1$ , an absurd result.

(iii) For  $a > f(u_i)$ , we have  $f(\overrightarrow{uu_i}) = a - f(u_i) = b - (f(u_i) + b - a)$ , so no vertex  $v_j$  is labelled as  $f(v_j) = f(u_i) + b - a$ , in turn, there is a vertex  $u'_i$  such that  $f(u'_i) = f(u_i) + b - a + \delta m$  for  $\delta = 0$  or  $1$ . If  $a < f(u_i)$ , we have  $f(\overrightarrow{uu_i}) = m + (a - f(u_i)) = m + (b - (f(u_i) + b - a))$ , similarly, there exists  $f(u'_i) = f(u_i) + b - a + \delta m$  for  $\delta = 0$  or  $1$ . Notice that  $f(u'_i) \neq f(u'_k)$  as if  $f(u_i) \neq f(u_k)$ . Therefore, it must be that  $s - 1$  is even by the upper deduction.

Analogously, each number  $f(v_j)$  is corresponding to a number  $f(v'_j)$  such that  $f(v'_j) = f(v_j) + a - b + \delta m$  for  $\delta = 0$  or  $1$ , and furthermore for  $f(v_j) \neq f(v_k)$  there is  $f(v'_j) \neq f(v'_k)$ . Thereby, we can claim that  $t$  is even. ■

**Lemma 2.4.** *Every directed bistar  $T(2l + 1, 2k - 2l)$  defined by Dibistar (I) is digraceful for integers  $k > l \geq 0$ .*

*Proof.* For  $l = 0$ , the directed bistar  $T(1, 2k)$  is a directed star  $\overrightarrow{K}_{1, 2k+1}$  on  $(2k + 2)$  vertices, so it admits a digraceful labelling  $h$  described in Theorem 2.2. We rewrite the vertex set and arc set of  $T(1, 2k)$  as  $V(T(1, 2k)) = \{u, v, x_i, y_i : i \in [1, k]\}$  and  $A(T(1, 2k)) = \{\overrightarrow{uv}, \overrightarrow{vx_i}, \overrightarrow{vy_i} : i \in [1, k]\}$ , respectively. Then we have  $h(u) = k + 1$ ,  $h(v) = 0$ ,  $h(x_i) = i$  and  $h(y_i) = k + 1 + i$  for each  $i \in [1, k]$ .

For  $i \in [1, k]$ , the arc labels  $h(\overrightarrow{vx_i}) = (2k + 2) - (0 - i) = 2k + 2 - i$  distribute the set  $[k + 2, 2k + 1]$ ; and the arc labels  $h(\overrightarrow{vy_i}) = (2k + 2) - (0 - (k + 1 + i)) = k + 1 - i$  distribute the set  $[1, k]$ . Notice that  $h(\overrightarrow{uv}) = k + 1$ . Furthermore, each  $h(x_i)$  is corresponding

to  $h(y_i)$ , since  $h(y_i) = h(x_i) + h(u) - h(v)$  (see the proof of the assertion (iii) in Lemma 2.3).

For  $l = 1$ , the directed bistar  $T(3, 2k - 2)$  can be obtained by deleting two arcs  $\overrightarrow{vx_i}, \overrightarrow{vy_i}$  from  $T(1, 2k)$  and then adding two arcs  $\overrightarrow{ux_i}, \overrightarrow{uy_i}$  to the remainder  $T(1, 2k) - \{\overrightarrow{vx_i}, \overrightarrow{vy_i}\}$ . And we retain the labelling  $h$  for  $T(3, 2k - 2)$ . It is easy to see that  $h(\overrightarrow{ux_i}) = k + 1 - i$  and  $h(\overrightarrow{uy_i}) = (2k + 2) + ((k + 1) - (k + 1 + i)) = 2k + 2 - i$ . So,  $h$  still is a digraceful labelling of  $T(3, 2k - 2)$ .

Notice that  $i$  is arbitrarily selected in the upper discussion, thus, we can delete  $l$  pairs of arcs  $\overrightarrow{vx_i}$  and  $\overrightarrow{vy_i}$  and then add  $l$  pairs of arcs  $\overrightarrow{ux_i}$  and  $\overrightarrow{uy_i}$  such that the resulting ditree is  $T(2l + 1, 2k - 2l)$  with the digraceful labelling  $h$  defined above. ■

Combining Lemma 2.3 with Lemma 2.4, we have

**Theorem 2.3.** *Let  $T(s, t)$  be a directed bistar defined by Dibistar (I) with  $s, t \geq 1$ . Then  $T(s, t)$  is digraceful if and only if both integers  $s - 1$  and  $t$  are even.*

We define another class of directed bistars in the following:

Dibistar (II): The vertex set  $V(H(s, t; m, n))$  of a directed bistar  $H = H(s, t; m, n)$  contains vertices  $u_i, x_k, u, v, v_j$  and  $y_l$ , and the arc set  $A(H(s, t; m, n))$  contains arcs  $\overrightarrow{uu_i}, \overrightarrow{ux_k}, \overrightarrow{uv}, \overrightarrow{vv_j}, \overrightarrow{vy_l}$  for  $i \in [1, s - 1], k \in [1, m], j \in [1, t]$  and  $l \in [1, n]$ . Clearly, the in-degrees  $d_H^-(u) = m$  and  $d_H^-(v) = n + 1$ , and the out-degrees  $d_H^+(u) = s$  and  $d_H^+(v) = t$ . So  $|V(H)| = s + m + t + n + 1$ .

**Lemma 2.5.** *If both integers  $s - 1 + n$  and  $t + n$  are even in a directed bistar  $H(s, t; n, n)$  defined by Dibistar (II) with  $s - 1, t, n \geq 1$ , then  $H(s, t; n, n)$  is digraceful.*

*Proof.* In the proof of Lemma 2.4, the directed bistar  $T(1, 2k)$  has a digraceful labelling  $h$ , and each pair of arcs  $\overrightarrow{vx_i}$  and  $\overrightarrow{vy_i}$  with  $h(x_i) = i$  and  $h(y_i) = k + 1 + i$  for each  $i \in [1, k]$  is said to be *arc-label matchable*. And each pair of arcs  $\overrightarrow{vx_i}, \overrightarrow{vy_i}$  is corresponding to a pair of arcs  $\overrightarrow{vx_{k+1-i}}$  and  $\overrightarrow{vy_{k+1-i}}$ . Furthermore, it is not hard to see

$$(2.1) \quad \begin{aligned} h(x_{k+1-i}) &= k + 1 - i = h(\overrightarrow{vy_i}), & h(y_{k+1-i}) &= 2k + 2 - i = h(\overrightarrow{vx_i}); \\ h(\overrightarrow{vx_{k+1-i}}) &= k + 1 + i = h(y_i), & h(\overrightarrow{vy_{k+1-i}}) &= i = h(x_i). \end{aligned}$$

Therefore, each pair of arcs  $\overrightarrow{vx_{k+1-i}}$  and  $\overrightarrow{vy_{k+1-i}}$  is called the *dually arc-label matchable pair* of the pair of arcs  $\overrightarrow{vx_i}$  and  $\overrightarrow{vy_i}$ .

Notice that  $H(s + n, t + n; 0, 0) = T(s + n, t + n)$  defined by Dibistar (I). Suppose that each of  $s - 1 + n$  and  $t + n$  is of even. We apply the directed bistar  $T(1, 2k)$  with a digraceful labelling  $h$  defined in the proof of Lemma 2.4, where  $2k = s - 1 + t + 2n$ . We delete arcs  $\overrightarrow{vx_j}, \overrightarrow{vy_j}$  for  $1 \leq j \leq \frac{1}{2}(n + s - 1)$  from  $T(1, 2k)$ , and then add arcs  $\overrightarrow{ux_j}, \overrightarrow{uy_j}$  to the remainder, so that we obtain  $H(s + n, t + n; 0, 0)$  with the digraceful labelling  $h$ . By the equation (2.1), the dually arc-label matchable pair for each pair of arcs  $\overrightarrow{ux_j}$  and  $\overrightarrow{uy_j}$  is the pair of arcs  $\overrightarrow{vx_{k+1-j}}$  and  $\overrightarrow{vy_{k+1-j}}$ ,  $j \in [1, n]$ . We, now, converse arcs  $\overrightarrow{ux_j}$  and  $\overrightarrow{vy_{k+1-j}}$  so that the resulting ditree is  $H(s + n - 1, t + n - 1; 1, 1)$  that admits the digraceful labelling  $h$ . Next, we converse arcs  $\overrightarrow{uy_j}$  and  $\overrightarrow{vx_{k+1-j}}$  such that the resulting ditree is  $H(s + n - 2, t + n - 2; 2, 2)$  that still admits the above labelling  $h$  as a digraceful labelling. Go on in this way, we obtain  $H(s, t; n, n)$  with the digraceful labelling  $h$ . ■

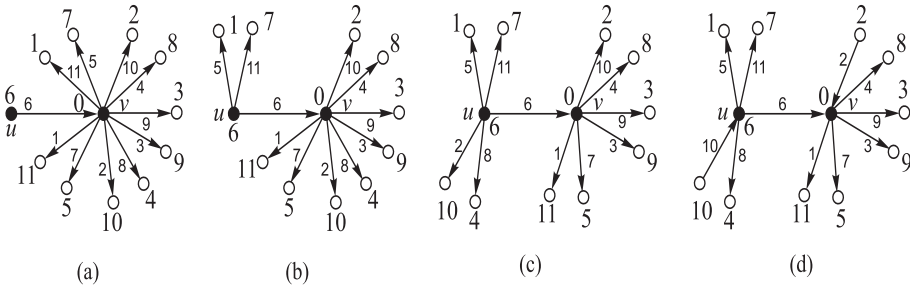


Figure 3. (a) A directed bistar  $T(1, 10)$  in which each pair of arcs  $(v, i)$  and  $(v, 6+i)$  is an arc-label matchable pair, and each pair of arcs  $(v, 12-i)$  and  $(v, 6-i)$  is the dually arc-label matchable pair of the pair of arcs  $(v, i)$  and  $(v, 6+i)$ ; (b) A directed bistar  $T(3, 8)$ ; (c) A directed bistar  $T(5, 6)$ ; (d) A directed bistar  $H(4, 5; 1, 1)$  generated from  $T(5, 6)$ .

**Lemma 2.6.** *Given even integers  $t$  and  $s - 1$ , then  $H(s, t; 4r, 0)$  defined by Dibistar (II) is digraceful, where  $s - 1, t, r \geq 1$ .*

*Proof.* We consider  $H(s + 4r, t; 0, 0) = T(s + 4r, t)$  defined by Dibistar (I) (the same way for  $H(s, t + 4r; 0, 0)$ ). We employ the directed bistar  $T(1, 2k)$  that has a digraceful labelling  $h$  defined in the proof of Lemma 2.4, where  $2k = s - 1 + t + 4r$ . We delete two pairs of  $\overrightarrow{vx_1}, \overrightarrow{vy_1}$  and  $\overrightarrow{vx_k}, \overrightarrow{vy_k}$  and arcs  $\overrightarrow{vx_j}, \overrightarrow{vy_j}$  for  $2 \leq j \leq \frac{1}{2}(s - 1)$  from  $T(1, 2k)$ , and then add arcs  $\overrightarrow{ux_1}, \overrightarrow{uy_1}, \overrightarrow{ux_k}, \overrightarrow{uy_k}$ , and arcs  $\overrightarrow{ux_j}, \overrightarrow{uy_j}$  for  $2 \leq j \leq \frac{1}{2}(s - 1)$  to the remainder, so that we obtain  $H(s + 4r, t; 0, 0)$  with the digraceful labelling  $h$ . Thereby, we have  $H(s, t; 4r, 0)$  by conversing four arcs  $\overrightarrow{ux_1}, \overrightarrow{uy_1}, \overrightarrow{ux_k}$  and  $\overrightarrow{uy_k}$ . Now, it is to verify that the digraceful labelling  $h$  still is available for  $H(s, t; 4r, 0)$  in the following. Observe that  $h(\overrightarrow{x_1\hat{u}}) = 2k + 2 + (1 - (k + 1)) = k + 2 = h(y_1)$ ,  $h(\overrightarrow{y_1\hat{u}}) = (k + 2) - (k + 1) = 1 = h(x_1)$ ,  $h(\overrightarrow{x_k\hat{u}}) = 2k + 2 + (k - (k + 1)) = 2k + 1 = h(y_k)$  and  $h(\overrightarrow{y_k\hat{u}}) = (2k + 1) - (k + 1) = k = h(x_k)$ , we are done. ■

**Lemma 2.7.** *Let each of  $s - 1$  and  $t$  be even, and let  $s - 1 + t = 2k$ . If  $k$  is odd, we then have two digraceful directed bisters  $H(s, t - 1; 1, 0)$  and  $H(s + 1, t - 1; 0, 1)$  defined by Dibistar (II).*

*Proof.* Let  $2\ell = k + 1$ . Notice that two arcs  $\overrightarrow{vx_\ell}, \overrightarrow{vy_\ell}$  in the directed bistar  $T(1, 2k)$  ( $2k = s - 1 + t$ ) with a digraceful labelling  $h$  defined in the proof of Lemma 2.4 is not only an arc-label matchable pair but also the dually arc-label matchable pair about itself, since

$$h(\overrightarrow{vx_\ell}) = 2k + 2 + (0 - \ell) = 2k + 2 - \ell = 3\ell = k + 1 + \ell = h(y_\ell), \text{ and}$$

$$h(\overrightarrow{vy_\ell}) = 2k + 2 + (0 - (k + 1 + \ell)) = k + 1 - \ell = \ell = h(x_\ell).$$

Delete the arc  $\overrightarrow{vx_\ell}$  from  $T(1, 2k)$ , and follows to add an arc  $\overrightarrow{x_\ell\hat{u}}$  to the remainder, thus, we obtain  $H(s, t - 1; 1, 0)$  with the labelling  $h$  defined above. Notice that  $h(\overrightarrow{x_\ell\hat{u}}) = 2k + 2 + (\ell - (k + 1)) = k + 1 + \ell = h(\overrightarrow{vx_\ell})$ , it shows that  $h$ , also, is a digraceful labelling of  $H(s, t - 1; 1, 0)$ . Analogously, we, first, delete the arc  $\overrightarrow{vy_\ell}$  from  $T(1, 2k)$ . Next, we add an arc  $\overrightarrow{uy_\ell}$  and converse the arc  $\overrightarrow{vx_\ell}$  in the remainder in order to obtain  $H(s + 1, t - 1; 0, 1)$  with the digraceful labelling  $h$ . ■

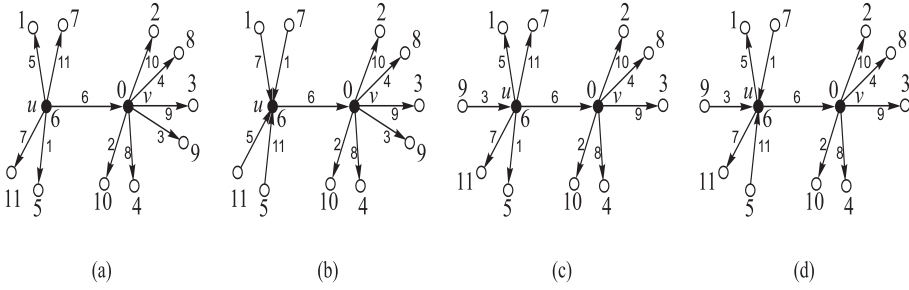


Figure 4. (a) A directed bistar  $T(5, 6)$ ; (b) A directed bistar  $H(1, 6; 4, 0)$  generated from  $T(5, 6)$ ; (c) A directed bistar  $H(5, 5; 1, 0)$ ; (d) A directed bistar  $H(3, 5; 3, 0)$ .

By Lemma 2.5, 2.6 and 2.7 we have

**Theorem 2.4.** *If each of  $s-1+n$  and  $t+n$  is even in a directed bistar  $H(s, t; n+4r, n)$  defined by Dibistar (II), where integers  $s-1, t \geq 1$  and  $n, r \geq 0$ , then we have*

- (1)  $H(s, t; n+4r, n)$  is digraceful.
- (2) Both  $H(s, t-1; n+4r+1, n)$  and  $H(s+1, t-1; n+4r, n+1)$  are digraceful as if  $s-1+t+2n+4r = 2k$  for odd  $k$ .

**2.4. An application of digraceful ditrees**

We replace each edge  $uv$  of a complete graph  $K_{2m+1}$  by two arcs  $\overrightarrow{uv}$  and  $\overleftarrow{vu}$ , the resulting graph is called the *biorientation* of  $K_{2m+1}$ , denoted by  $\overleftrightarrow{K}_{2m+1}$ . Recall the famous Ringel-Kotzig Decomposition Conjecture [10]: A complete graph  $K_{2m+1}$  can be decomposed into  $2m+1$  subgraphs that are all isomorphic with a given tree of  $m$  edges.

**Theorem 2.5.** *Let  $\overleftrightarrow{K}_{2m+1}$  be the biorientation of a complete graph  $K_{2m+1}$ . If a ditree  $T$  with size  $m$  is digraceful, then  $\overleftrightarrow{K}_{2m+1}$  can be decomposed into  $2m+1$  arc-disjoint copies of  $T$  and  $2m+1$  arc-disjoint copies of converse of  $T$ .*

*Proof.* Let  $V(T) = \{u_1, u_2, \dots, u_{m+1}\}$  and let  $f$  be a digraceful labelling of a ditree  $T$  such that  $f(u_i) = i-1$  for  $i \in [1, m+1]$ . We label each vertex  $v_j$  of  $\overleftrightarrow{K}_{2m+1}$  with the color  $j$ , write this labelling by  $g$ , so  $g(v_j) = j-1$  for  $j \in [1, 2m+1]$ . We take first copy  $H_1$  of the ditree  $T$  such that  $g(v_i) = f(u_i)$  for  $i \in [1, m+1]$  (notice that this is available since  $\overleftrightarrow{K}_{2m+1}$  is biorientated). And the second copy  $H_2$  of  $T$  satisfies  $g(v_{i+1}) = f(u_i) + 1$  for  $i \in [1, m+1]$ . Go on in this way, we have the  $k$ th copy  $H_k$  of  $T$  satisfies  $g(v_{i+k-1}) = f(u_i) + k-1 \pmod{2m}$  for  $i \in [1, m+1]$  and  $k \in [1, 2m+1]$ . For each copy  $H_k$ , we define the label of an arc  $\overrightarrow{uv}$  of  $H_k$  as the following:

$$\begin{aligned}
 g(\overrightarrow{uv}) &= g(u) - g(v) \text{ if } 1 \leq g(u) - g(v) \leq m; \\
 g(\overrightarrow{uv}) &= (g(u) - g(v)) - m \text{ if } g(u) - g(v) > m; \\
 g(\overrightarrow{uv}) &= m + 1 + (g(u) - g(v)) \text{ if } -m \leq g(u) - g(v) < 0; \text{ and} \\
 g(\overrightarrow{uv}) &= 2m + 1 + (g(u) - g(v)) \text{ if } g(u) - g(v) < -m.
 \end{aligned}$$

Notice that  $2|A(T)| = 2m = \sum_{u \in V(T)} (d_T^+(u) + d_T^-(u))$ , so we have  $\sum_{k=1}^{2m+1} (d_{H_k}^+(w) + d_{H_k}^-(w)) = 2m$  for each vertex  $w \in V(\overleftrightarrow{K}_{2m+1})$ . Clearly, the labelling  $g$  shows that two copies  $H_k$  and  $H_s$  share no arc in common for  $k, s \in [1, 2m + 1]$ .

Let  $H_k^*$  be the converse of  $H_k$  for  $k \in [1, 2m + 1]$ . Therefore, the biorientation  $\overleftrightarrow{K}_{2m+1}$  can be decomposed into arc-disjoint ditrees  $H_k$  and  $H_s^*$  for  $k, s \in [1, 2m + 1]$ . ■

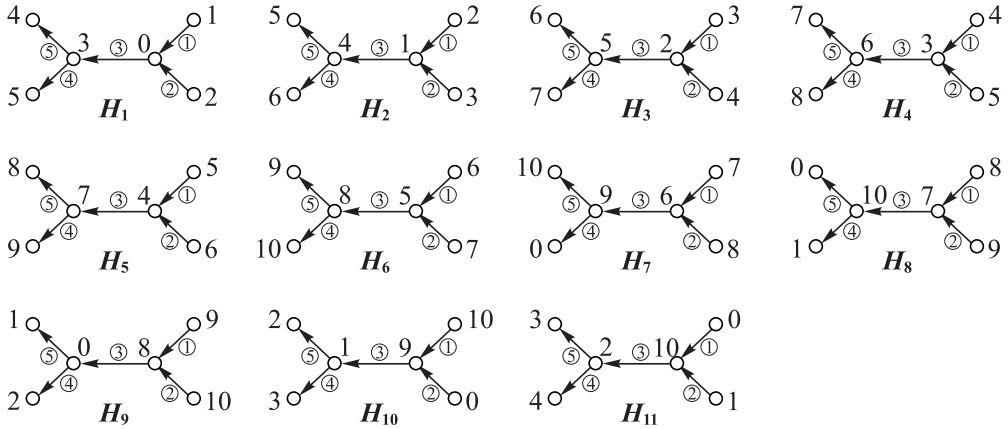


Figure 5. A digraceful ditree  $H_1$  and its copies for illustrating the labels of arcs in the proof of Theorem 2.5.

### 3. Further works

In [9] a graceful labelling  $f$  of an undirected tree  $T$  is *ordered* if either  $f(x) < f(u)$  (or  $f(x) > f(u)$ ) for all  $x \in N(u)$  and  $u \in V(T)$ , where  $N(x)$  is the neighbor set of a vertex  $x$  in a graph.

**Conjecture 3.1.** [6] *Each tree is ordered graceful.*

We orientate a tree  $T$  with an ordered graceful labelling  $f$  in the way: If  $f(x) < f(u)$  for all  $x \in N(u)$ , then we replace by an arc  $\overrightarrow{ux}$  each edge  $ux$ . For  $f(x) > f(u)$  as if  $x \in N(u)$ , we replace by an arc  $\overrightarrow{xu}$  each edge  $ux$ . The resulting digraph is just an in-zero-out ditree. Thereby, we propose

**Conjecture 3.2.** *Each in-zero-out ditree with diameter not less than five is digraceful.*

For diameter  $D(T) = 2$  or  $3$ , Theorem 2.1 answers Conjecture 3.2.

We discover that a digraceful ditree may have an arc  $\overrightarrow{uv}$  such that the ditree obtained by conversing the arc  $\overrightarrow{uv}$  is not digraceful. Thereby, the following problems may be interesting.

**Problem 1.** Characterize a digraceful ditree  $T$  such that the ditree obtained by conversing a certain arc of  $T$  is not digraceful.

**Problem 2.** Let  $D$  be a simple-connected digraph. Is the number of in-leaf vertices equal to that of out-leaf vertices if  $\sum_{u \in V(D)} (d_D^+(u) - d_D^-(u))f(u) = 0$  for a certain digraceful labelling  $f$  of  $D$ ?

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