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C-Characteristically Simple Groups

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Abstract. Let G be a group and let $\mathrm{Aut}_c(G)$ be the group of central automorphisms of G. We say that a subgroup H of a group G is c-characteristic if $\alpha(H) = H$ for all $\alpha \in \mathrm{Aut}_c(G)$. We say that a group G is c-characteristically simple group if it has no non-trivial c-characteristic subgroup. If every subgroup of G is c-characteristic then G is called co-Dedekindian group. In this paper we characterize c-characteristically simple groups. Also if G is a direct product of two groups A and B we study the relationship between the co-Dedekindianness of G and the co-Dedekindianness of A and B. We prove that if G is a co-Dedekindian finite non-abelian group, then G is Dedekindian if and only if G is isomorphic to Q_8 where Q_8 is the quaternion group of order 8.

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1. Introduction and results

Let G be a group, and let G' and $Z(G)$ denote the commutator subgroup and the centre of G respectively. An automorphism α of G is called central if $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$. The set of all central automorphisms of G, denoted by $\mathrm{Aut}_c(G)$, is a normal subgroup of the full automorphism group of G. A subgroup H of G is called characteristic if H is invariant under all automorphisms of G. A group G is called characteristically simple group if it has no non-trivial characteristic subgroup. The structure of the finite characteristically simple groups are well known. They are a direct product of finitely many isomorphic copies of a simple group (see [5, Theorem 8.10]). The definition of characteristically simple groups suggests the consideration of a new class of groups, called c-characteristically simple groups which are defined as following.

We say that a subgroup H of a group G is c-characteristic if $\alpha(H) = H$ for each $\alpha \in \mathrm{Aut}_c(G)$. We say that a group G is c-characteristically simple group if it has

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no non-trivial c-characteristic subgroup. In Section 2, we give some results on the c-characteristic subgroups and characterize the c-characteristically simple groups.

If every subgroup of G is c-characteristic then G is called co-Dedekindian group. In 1994 Deaconescu and Silberberg in [2] studied the finite co-Dedekindian groups. They gave a Dedekind-like structure theorem for the non-nilpotent co-Dedekindian groups with trivial Frattini subgroup and by reducing the finite nilpotent co-Dedekindian groups to the case of p-groups they obtained the following theorem.

Theorem 1.1. *Let G be a co-Dedekindian finite non-abelian p-group. If $Z_2(G)$ is non-abelian, then $G \simeq Q_8$. If $Z_2(G)$ is cyclic, then $G \simeq Q_{2^n}$ $(n \geq 4)$ where Q_{2^n} is the generalized quaternion group of order 2^n.*

In 2002, Jamali and Mousavi in [3] gave some necessary conditions for certain finite p-groups with non-cyclic abelian second centre to be co-Dedekindian. Section 3 contains some properties of co-Dedekindian groups and if G is a direct product of two groups A and B we study the relationship between the co-Dedekindianness of G and the co-Dedekindianness of A and B. Also we give a characterization for certain p-groups which are co-Dedekindian groups. At the end of this section, we prove that if G is a co-Dedekindian finite non-abelian group, then G is Dedekindian if and only if G is isomorphic to Q_8.

2. C-characteristically simple groups

Definition 2.1. *We say that a subgroup H of a group G is c-characteristic if $\alpha(H) = H$ for each $\alpha \in Aut_c(G)$. We note that if $Z(G) = 1$ or $G' = G$ then $Aut_c(G) = 1$ and hence every subgroup of G is c-characteristic subgroup.*

Remark 2.1. It is clear that every characteristic subgroup of a group G is c-characteristic, but a c-characteristic subgroup of a group G is not necessary characteristic. For example, let n be an integer greater than equal 5, and A_n be the alternating group of degree n. Since $Z(A_n) = 1$, every subgroup of A_n is c-characteristic but the only characteristic subgroups of A_n are 1 and A_n. Also let $G \simeq D_8 = \langle x, y | x^4 = y^2 = 1, y^{-1}xy = x^{-1} \rangle$. Then we have $\mathrm{Aut}(D_8) = \{1, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7\}$ where $\varphi_1 : x \longmapsto x$ and $y \longmapsto x^2y$, $\varphi_2 : x \longmapsto x^3$ and $y \longmapsto y$, $\varphi_3 : x \longmapsto x^3$ and $y \longmapsto x^2y$, $\varphi_4 : x \longmapsto x^3$ and $y \longmapsto xy$, $\varphi_5 : x \longmapsto x$ and $y \longmapsto xy$, $\varphi_6 : x \longmapsto x$ and $y \longmapsto x^3y$ and $\varphi_7 : x \longmapsto x^3$ and $y \longmapsto x^3y$. Hence $\mathrm{Aut}_c(G) = \{1, \varphi_1, \varphi_2, \varphi_3\}$. Now it is easy to check that $H = \{1, x^2, y, x^2y\}$ and $K = \{1, x^2, xy, x^3y\}$ are c-characteristic subgroups of D_8 but are not characteristic subgroups.

In the following we give some information about c-characteristic subgroups of a group.

Proposition 2.1. *Let G be a group. Then*

(1) *If H is a c-characteristic subgroup of G, then $Z_2(G) \leq N_G(H)$.*
(2) *If H is a c-characteristic subgroup of K and K is normal in $Z_2(G)$, then H is normal in $Z_2(G)$.*
(3) *If H is a c-characteristic subgroup of K and K is a c-characteristic subgroup of G, then H is a c-characteristic subgroup of G.*

Proof. (1) Let $x \in Z_2(G)$. Then the inner automorphism θ_x induced by x defines a central automorphism of G. Since H is c-characteristic subgroup of G, $H^x = H$.
(2) Let $x \in Z_2(G)$ and θ_x be the inner automorphism induced by x. Then the restriction θ_x on K is an automorphism of K since, K is normal in G. Since $k^{-1}\theta_x(k) = k^{-1}x^{-1}kx = [k, x] \in [K, Z_2(G)] \leq K \cap Z(G) \leq Z(K)$ for all $k \in K$, θ_x is a central automorphism of K and so $H^x = H$ since, H is a c-characteristic subgroup of K.
(3) Let $\alpha \in \mathrm{Aut}_c(G)$. Then the restriction α on K defines a central automorphism of K since, K is a c-characteristic subgroup of G and $k^{-1}\alpha(k) \leq K \cap Z(G) \leq Z(K)$ for all $k \in K$. Therefore $H^\alpha = H$ since, H is a c-characteristic subgroup of K. ∎

Proposition 2.2. *Let G be a group. Suppose that $G \times G$ has a c-characteristic subgroup K, and write $G_1 = G \times 1$ and $G_2 = 1 \times G$. Then*

(1) *$K \cap G_1$ and $K \cap G_2$ are c-characteristic subgroups of G_1 and G_2 respectively.*
(2) *If π_1 and π_2 are the natural projections on $G \times G$. Then $\pi_1(K)$ and $\pi_2(K)$ are c-characteristic subgroups of G.*

Proof. (1) Let K be a c-characteristic subgroup of $G \times G$ and let $\theta_1 \in \mathrm{Aut}_c(G_1)$. Then $\theta : G \times G \longrightarrow G \times G$ defined by $\theta(g_1, g_2) = (1, g_2)\theta_1(g_1, 1)$ defines a central automorphism of $G \times G$. Let $(g, 1) \in K \cap G_1$. Then $\theta(g, 1) = (1, 1)\theta_1(g, 1) = \theta_1(g, 1) \in K \cap G_1$ since, K is a c-characteristic subgroup of $G \times G$. Similarly $K \cap G_2$ is a c-characteristic subgroup of G_2.
(2) Let $\alpha \in \mathrm{Aut}_c(G)$. Then $\beta : G \times G \longrightarrow G \times G$ defined by $\beta(g_1, g_2) = (\alpha(g_1), g_2)$ defines a central automorphism of $G \times G$. Since K is c-characteristic, for each $(g_1, g_2) \in K$ we have $\alpha(g_1) = \pi_1(\alpha(g_1), g_2) = \pi_1(\beta(g_1, g_2)) \in \pi_1(K)$ whence $\alpha(\pi_1(g_1, g_2)) = \alpha(g_1) \in \pi_1(K)$. Similarly $\pi_2(K)$ is a c-characteristic subgroup of G. ∎

Proposition 2.3. *Let G be a finite non-abelian p-group. Then every non-trivial abelian direct factor of G is not c-characteristic.*

Proof. Suppose that A is a non-trivial abelian direct factor of G. Then there exists a non-abelian subgroup B of G such that $G = A \times B$. Choose an element z of order p in $Z(B)$. Write A as a direct product $C \times D$, where C is a cyclic p-group. Let the map α be defined by $\alpha(b) = b$ for all b in B, $\alpha(c) = zc$ where c is a generator of C and $\alpha(d) = d$ for all d in D. Then the map α extends to a central automorphism of G. But A is not invariant under α For, otherwise $zc \in \langle c \rangle$ and so $z \in \langle c \rangle \leq A \cap B$ which is a contradiction. ∎

Definition 2.2. *We say that a non-trivial group G is c-characteristically simple group if it has no non-trivial c-characteristic subgroup.*

In the following we give relationship between c-characteristically simple groups and characteristically simple groups, and characterize finite c-characteristically simple groups.

Proposition 2.4. *Let G be a non-trivial group. Then G is a c-characteristically simple group if and only if G is abelian and characteristically simple group.*

Proof. Let G be a c-characteristically simple group and H is a characteristic subgroup of G. Then by Remark 2.1 H is c-characteristic and so $H = 1$ or $H = G$.

Now we show that G is abelian. Since $Z(G)$ is a characteristic subgroup of G and G is characteristically simple group, we have $Z(G) = 1$ or $Z(G) = G$. If $Z(G) = 1$ then $\mathrm{Aut}_c(G) = 1$ and so every subgroup of G is c-characteristic. Since G is c-characteristically simple group, G has no non-trivial subgroup and hence is cyclic of prime order which is a contradiction. Therefore $Z(G) = G$ which shows that G is abelian. Conversely let G be abelian and characteristically simple group. Also let K be a c-characteristic subgroup of G. Since G is abelian, $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$. This shows K is a characteristic subgroup of G and so $K = 1$ or $K = G$. ∎

Remark 2.2. Simple groups are certainly characteristically simple, but are not c-characteristically simple. For example A_n for $n \geq 5$ is simple but is not c-characteristically simple. Also a c-characteristically simple group is not necessary simple. For example $\mathbb{Z}_p \times \mathbb{Z}_p$ is c-characteristically simple but is not simple.

As a consequent of Proposition 2.4, we have

Corollary 2.1. *Let G be a finite group. Then G is a c-characteristically simple group if and only if G is an elementary abelian p-group for some prime p.*

Proof. Let G be a c-characteristically simple finite group. Then by Proposition 2.4, G is abelian and characteristically simple. Since G is a finite group, by [5, 7.41], G is elementary abelian p-group.

Suppose conversely that G is elementary abelian p-group. Since G is characteristically simple, by Proposition 2.4, G is c-characteristically simple group. ∎

Remark 2.3. A group G is said to be completely reducible if either $G = 1$ or G is the direct product of a finite number of simple groups. By Corollary 2.1 every finite c-characteristically simple group is completely reducible. Without the condition that G is finite, the result is false. For instance by Proposition 2.4 the additive group of the rational numbers is an example of a c-characteristically simple group which is not completely reducible.

Corollary 2.2. *If G is a c-characteristically simple group, then $G \times G$ is a c-characteristically simple group.*

Proof. By Proposition 2.4 G is abelian and characteristically simple. Also by [5, page 144, 362] $G \times G$ is characteristically simple. Now Proposition 2.4 implies that $G \times G$ is a c-characteristically simple group, since G is abelian. ∎

3. Some properties of co-Dedekindian groups

In this section we shall obtain further information about co-Dedekindian groups. First we give some results that will be used in the sequel. We recall that a direct decomposition of a group G into direct product of finitely many non-trivial indecomposable subgroups is said to be a Remak decomposition. Let G be a group. For $\alpha \in \mathrm{Aut}_c(G)$, let $F_\alpha = \{x \in G \mid \alpha(x) = x\}$, $K_\alpha = \langle x^{-1}\alpha(x) \mid x \in G\rangle$, $F = \bigcap_{\alpha \in \mathrm{Aut}_c(G)} F_\alpha$ and $K = \langle K_\alpha \mid \alpha \in \mathrm{Aut}_c(G)\rangle$. It is clear that $G' \leq F$ and $K \leq Z(G)$, so in particular F, K, F_α, K_α are normal subgroups in G. We now collect some information of [2] about the subgroups F_α and K_α. Let G be a finite co-Dedekindian group and $\alpha \in \mathrm{Aut}_c(G)$, $H \leq G$. Then

(1) $H \cap F_\alpha = 1 \Rightarrow H \leq K_\alpha$.
(2) $H \cap K_\alpha = 1 \Rightarrow H \leq F_\alpha$.
(3) $G = HF_\alpha \Rightarrow K_\alpha \leq H$.
(4) $G = HK_\alpha \Rightarrow F_\alpha \leq H$.

Theorem 3.1. *Let G be a co-Dedekindian group. Then*

(1) *Every direct summand of G is co-Dedekindian.*
(2) *If $G = A \times B = A \times C$ are two decomposition for G, then $B = C$.*
(3) *If G is a finite group, then G has a unique Remak decomposition (up to the orders of the direct factors).*
(4) *Let G be a finite group and let N be a minimal normal subgroup of G. Then $N \leq Z(G)$ or $N \leq F$.*
(5) *Let G be a finite group such that $Aut_c(G) \neq 1$. If M is a maximal subgroup of G, then $G^{'} \leq M$ or $M \cap Z(G) \neq 1$.*

Proof. (1) Let $G = A \times B$, $\gamma \in \mathrm{Aut}_c(A)$ and $L \leq A$. Then there exists a central automorphism $\theta : ab \longmapsto \gamma(a)b$ of G for each $a \in A$ and $b \in B$. Since G is co-Dedekindian, $L = \theta(L) = \gamma(L)$.
(2) There is an isomorphism $\theta : B \longmapsto C$ such that for each $b \in B$, $b^{-1}\theta(b) \in A$ and hence $b^{-1}\theta(b) \in Z(G)$. Now the map $\gamma : ab \longmapsto a\theta(b)$ defines a central automorphism which maps B onto C. Since G is co-Dedekindian, $C = \gamma(B) = B$.
(3) Let $G = H_1 \times \cdots \times H_r = K_1 \times \cdots \times K_s$ be two Remak decompositions of G. Then by [4, 3.3.8] we have $r = s$ and there is a central automorphism α of G such that, after suitable relabelling of the $K_j's$ if necessary, $H_i{}^\alpha = K_i$ whence $H_i = K_i$ since G is co-Dedekindian.
(4) If there exists $\alpha \in \mathrm{Aut}_c(G)$ such that $N \cap F_\alpha = 1$ then $N \leq K_\alpha$ and so $N \leq Z(G)$.
Now let for each $\alpha \in \mathrm{Aut}_c(G)$, $N \cap F_\alpha \neq 1$. Since N is minimal normal, for each $\alpha \in \mathrm{Aut}_c(G)$ we have $N \cap F_\alpha = N$ and so $N \leq F$.
(5) Let α be a non-trivial central automorphism of G. If $MF_\alpha = G$, then $K_\alpha \leq M$. On the other hand $K_\alpha \leq Z(G)$ and so $1 \neq K_\alpha \leq M \cap Z(G)$. If $MF_\alpha = M$ then $G^{'} \leq F_\alpha \leq M$. ∎

We recall [2] that a co-Dedekindian group is said to be trivial co-Dedekindian group if $\mathrm{Aut}_c(G) = 1$. In the following we show that the restriction $\mathrm{Aut}_c(G) \neq 1$ on the group G in Theorem 3.1(5) is necessary. First we need the following lemma.

Lemma 3.1. *If G is a perfect group, then $G/Z(G)$ is a trivial co-Dedekindian group.*

Proof. We show that $Z(G/Z(G)) = Z_2(G)/Z(G) = 1$. Let $x \in Z_2(G)$. Then the map θ_x defined by $\theta_x(g) = [x, g]$ for all $g \in G$ defines an homomorphism of G since, $[G, Z_2(G)] \leq Z(G)$. Since $G/ker(\theta_x) \simeq Im(\theta_x) \leq Z(G)$, we have $G' \leq \ker(\theta_x)$. Since G is perfect, for all $g \in G$ we have $[x, g] = 1$ whence $x \in Z(G)$. This completes the proof. ∎

Remark 3.1. If G is a perfect group, then by Lemma 3.1 the factor group $\bar{G} = G/Z(G)$ is a trivial co-Dedekindian group. But for every maximal subgroup $\bar{M}$ of $\bar{G}$ neither conditions $\bar{G}' \leq \bar{M}$ nor $\bar{M} \cap Z(\bar{G}) \neq 1$. Therefore the restriction $\mathrm{Aut}_c(G) \neq 1$ on the group G in Theorem 3.1(5) is necessary.

Remark 3.2. By Theorem 3.1(1) if the direct product of two groups A and B is co-Dedekindian, then A and B are co-Dedekindian. But the converse is not in general true. For example let $A = \mathbb{Z}_2$ and $B = Q_8$. Then the groups A and B are co-Dedekindian but their direct product $G = \mathbb{Z}_2 \times Q_8$ is not. Suppose, for a contradiction, that G is co-Dedekindian. Since $Z_2(G) = G$ is non-abelian, by Theorem 1.1 we have $G \simeq Q_8$ which is a contradiction. However as we will see below, under certain conditions on co-Dedekindian groups A and B, $A \times B$ is co-Dedekindian. First we need the following theorem.

Theorem 3.2. [7] *The Remak decomposition of a finite group is uniquely determined if and only if the order of the factor commutator group of each of its factors is relatively prime to the order of the centre of each of its other factors.*

Theorem 3.3. *Let G be the direct product of two groups A and B. Then*

(1) *If G is a co-Dedekindian group, then A and B are co-Dedekindian,* Hom$(A, Z(B)) = 1$ *and Hom$(B, Z(A)) = 1$.*
(2) *If A and B are finite co-Dedekindian indecomposable group with $(|A|, |B|) = 1$, then G is co-Dedekindian.*

Proof. (1) By Theorem 3.1, A and B are co-Dedekindian. Let $\theta \in \mathrm{Hom}(A, Z(B))$. We define the mapping $\gamma : G \longrightarrow G$ by

$$\gamma(ab) = a\theta(a)b \ for \ all \ a \in A, \ b \in B.$$

Then the mapping γ is a homomorphism. Clearly γ is one to one. Let $a \in A$, $b \in B$ then $\gamma(a\theta(a^{-1})b) = a\theta(a)\theta(a^{-1})b = ab$. So γ is onto. γ is a central automorphism because, $(ab)^{-1}\gamma(ab) = b^{-1}a^{-1}a\theta(a)b = b^{-1}\theta(a)b = \theta(a)b^{-1}b = \theta(a) \in Z(B) \leq Z(G)$. Since G is co-Dedekindian, for each $a \in A$ we have $\gamma(a) = a\theta(a) \in A$ and hence $\theta(a) = 1$ for each $a \in A$. Similarly $\mathrm{Hom}(B, Z(A)) = 1$.
(2) Let $\alpha \in \mathrm{Aut}_c(G)$ and $L \leq G$. Since $(|A|, |B|) = 1$, by [5, Corollary 8.20] we have $L = A_1 \times B_1$ for some subgroups A_1 and A_2 of G. If $A \simeq B$ then $G = 1$ which is co-Dedekindian.

So let $A \not\simeq B$. We have $G = A^\alpha \times B^\alpha$ and clearly this decomposition is a Remak decomposition. Since $(|A/A'|, |Z(B)|) = 1$, $(|B/B'|, |Z(A)|) = 1$, by Theorem 3.2 the finite group G has only one Remak decomposition and hence $A^\alpha = A$ and $B^\alpha = B$. This means that the automorphism α induces an automorphism $\alpha_1 \in \mathrm{Aut}_c(A)$ and an automorphism $\alpha_2 \in \mathrm{Aut}_c(B)$. Since the groups A and B are co-Dedekindian, ${A_1}^\alpha = {A_1}^{\alpha_1} = A_1$ and ${B_1}^\alpha = {B_1}^{\alpha_2} = B_1$ whence $L^\alpha = L$. ∎

We recall that a group G is called purely non-abelian if it has no non-trivial abelian direct factor.

Corollary 3.1. *Let G be a co-Dedekindian finite non-abelian p-group, then G is purely non-abelian.*

Proof. Suppose, for a contradiction, that $G = A \times B$ where A is a non-trivial abelian group. Then by Theorem 3.3, $\mathrm{Hom}(A, Z(B)) = 1$ which is a contradiction. ∎

Corollary 3.2. *Let G be a co-Dedekindian finite non-abelian p-group. Then $Aut_c(G)$ is a p-group.*

Proof. By Corollary 3.1, G is purely non-abelian and hence by [1, Theorem 1] $\mathrm{Aut}_c(G)$ is a p-group. ∎

In the following we characterize certain p-groups which are co-Dedekindian groups.

Theorem 3.4. *Let G be a finite non-abelian p-group having a cyclic maximal subgroup. Then G is co-Dedekindian if and only if $G \simeq Q_{2^n}$ for some $n \geq 3$.*

Proof. If G has a cyclic maximal subgroup then by [4, 5.3.4] if p is odd then $G \simeq M_{p^n} = \langle x, y | x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{1+p^{n-2}} \rangle$ $(n \geq 3)$.

Since $cl(G) = 2$ and G is co-Dedekindian, by Theorem 1.1 we have $G \simeq Q_8$ which is a contradiction. Therefore $p = 2$ and G is isomorphic to one of the following groups:

(1) $G \simeq D_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1} \rangle$
(2) $G \simeq S_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \rangle$
(3) $G \simeq Q_{2^n} = \langle x, y | x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, y^{-1}xy = x^{-1} \rangle$ where $n \geq 3$.

(1) If $G \simeq D_{2^n}$, then the map $\theta : x \longmapsto x$ and $y \longmapsto x^{2^{n-2}}y$ extends to a central automorphism on G. Since G is co-Dedekindian, $\theta(y) \in \langle y \rangle$ whence $y \in Z(G)$ which is a contradiction.
(2) If $G \simeq S_{2^n}$, then the map $\alpha : x \longmapsto x$ and $y \longmapsto x^{2^{n-2}}y$ extends to a central automorphism on G. Since G is co-Dedekindian, $\alpha(y) \in \langle y \rangle$ whence $y \in Z(G)$ which is a contradiction.

So $G \simeq Q_{2^n}$ for some $n \geq 3$. ∎

Theorem 3.5. *Let G be a finite non-abelian p-group such that $Z(G) = G^p \simeq \mathbb{Z}_p$. Then G is co-Dedekindian if and only if $\Omega_1(G) \leq \Phi(G)$.*

Proof. Let $x \in G \setminus \Phi(G)$. We show that $|x| > p$. Let z be a generator of $Z(G)$. Since $x \notin \Phi(G)$, there exists a maximal subgroup M such that $x \notin M$. The map $\phi : G \longmapsto G$ by $\alpha(x^i m) = x^i m z^i$ where $0 \leq i < p$ defines a central automorphism of G. Since G is co-Dedekindian, $\alpha(x) \in \langle x \rangle$ whence $Z(G) \subset \langle x \rangle$ for every $x \in G \setminus \Phi(G)$ and hence $|x| > p$ for every $x \in G \setminus \Phi(G)$. Conversely let $\Omega_1(G) \subseteq \Phi(G)$ and $1 \neq \alpha \in \mathrm{Aut}_c(G)$. Then $1 \neq K_\alpha \leq Z(G)$ and so $K_\alpha = Z(G)$. Since $|G| = |K_\alpha||F_\alpha|$, F_α is a maximal subgroup of G. This implies that $\alpha(x) = x$ for every $x \in \Phi(G)$. Let $x \in G \setminus \Phi(G)$, then $|x| > p$. Since $Z(G) = G^p \simeq \mathbb{Z}_p$, $x^p \in Z(G)$ and so $x^{p^2} = 1$. As $|x| > p$ so we have $|x| = p^2$ whence $Z(G) = \langle x^p \rangle$. Now $x^{-1}\alpha(x) \in Z(G) = \langle x^p \rangle \subset \langle x \rangle$ and so G is co-Dedekindian. ∎

Proposition 3.1. *Let G be a finite p-group and $N \trianglelefteq G$ such that $N \subseteq \phi(G)$. Then N is co-Dedekindian if and only if N is cyclic.*

Proof. Let N be co-Dedekindian. Then $Z(N)$ is cyclic by [3, Proposition 2.2]. Since $N \subseteq \phi(G)$ and $Z(N)$ is cyclic, N is cyclic by [6, 4.21]. ∎

Corollary 3.3. *Let G be a finite p-group, then $\phi(G)$, $G^{'}$, G^p are co-Dedekindian groups if and only if $\phi(G)$, $G^{'}$, G^p are cyclic.*

Recall that a group G is called Dedekindian if every subgroup of G is invariant under all inner automorphisms of G. The structure of the finite Dedekindian groups is well-known . They are either abelian or the direct product $Q_8 \times F \times E$, where Q_8

is the quaternion group of order 8, F is an abelian group of odd order and E is an elementary abelian 2-group (See [4, 5.3.7]). In the following we state relationship between Dedekindian groups and co-Dedekindian groups.

Proposition 3.2. *Let G be a co-Dedekindian group. Then*

(1) *If N is a subgroup of G such that $[G, N] \leq Z(G)$, then N is a Dedekindian group.*
(2) *$Z_2(G)$ is Dedekindian.*
(3) *If G is a nilpotent group of class at most 2, then G is a Dedekindian group.*

Proof. (1) Each element of N induces by conjugation a central automorphism of G. Now let $H \leq N$. Since G is co-Dedekindian, it follows that $H \trianglelefteq N$, so N is Dedekindian.
(2) Since $[G, Z_2(G)] \leq Z(G)$, by (1) $Z_2(G)$ is Dedekindian.
(3) If G is abelian, then it is clear that G is Dedekindian. If $cl(G) = 2$ then by (2) G is Dedekindian. ∎

Theorem 3.6. *Let G be a non-abelian finite group such that G is co-Dedekindian. Then G is Dedekindian if and only if $G \simeq Q_8$.*

Proof. We first prove that if G is a finite nilpotent co-Dedekindian group, then $Z(G)$ is cyclic. Let $G = P_1 \times \cdots \times P_n$ where P_i is the Sylow p_i-subgroup of G. By Theorem 3.1 each P_i is co-Dedekindian. Suppose that $P_1, \ldots, P_k$ are abelian and $P_{k+1}, \ldots, P_n$ are non-abelian. Since $P_1, \ldots, P_k$ are co-Dedekindian, $\mathrm{Aut}_c(P_1), \ldots, \mathrm{Aut}_c(P_k)$ are abelian and hence $P_1, \ldots, P_k$ are cyclic. On the other hand for each $k+1 \leq i \leq n$, $Z(P_i)$ is cyclic group by [3, Proposition 2.2]. Therefore $Z(G)$ is cyclic.

Let G be a Dedekindian group. Then by [4, 5.3.7] we have $G \simeq Q_8 \times F \times E$ where Q_8 is the quaternion group of order 8, F is abelian of odd order and E is an elementary abelian 2-group. By Theorem 3.1(1), F and E are co-Dedekindian. Now since F and E are abelian, F and E are cyclic and hence $G \simeq Q_8 \times \mathbb{Z}_{2m}$ where m is an odd number. Since G is a finite nilpotent co-Dedekindian group, $Z(G)$ is cyclic and so $G \simeq Q_8$. ∎

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